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HIGHER ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN UNBOUNDED DOMAINS OF R^n

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Abstract: The Dirichlet problem for a certain nonlinear partial differential equation on an unbounded domain is studied. The existence of a weak solution is proved by means of the theory of monotone operators.

Key words: Nonlinear differential equation, unbounded domain.

Classification: 35J60, 47H05

Introduction. Our purpose in the present short paper is to describe an application of some general techniques in nonlinear functional analysis to the study of a class of higher order nonlinear boundary value problems.

We consider the problem in Ω

$$(1)\quad \begin{cases} Au = \sum_{|\alpha|\leq m} (-1)^{|\alpha|}\, \partial^{\alpha}\, a_{\alpha}(x,u,\dots,\partial^{m}u)+f(u) = 0, \\ u \quad \mathring{U}^{m,p}_{s}(\Omega), \end{cases}$$

where:

(i) Ω is an unbounded open set in R^n with the cone property;

(ii) for each $\alpha\in N_0^n$, $|\alpha|\leq m$

$$a_{\alpha}(x,u,\dots,\partial^{m}u)=a_{\alpha}^{(1)}(x,u,\dots,\partial^{m}u)+a_{\alpha}^{(2)}(x,u,\dots,\partial^{m-1}u),$$

satisfies the Carathéodory conditions and some Nemytskii hy-

potheses on polynomial growth (assumption K_1 in Sec. 2);
(iii) $f(u)$ is a nonlinear perturbation whose behaviour is described by a suitable hypothesis (assumption K_5 in Sec.2);
(iv) $\overset{\circ}{U}{}^{m,p}_{s}(\Omega)$ are weighted Sobolev spaces defined as the completion of $\mathcal{D}(\Omega)$ with respect to the norm

$$\|u\| = \Big(\sum_{|\alpha| \leq m} \int_{\Omega} \rho^{sp}_{(x)} \, |\partial^{\alpha} u(x)|^{p} \, dx \Big)^{1/p},$$

where $\rho(x)$ is a continuous function such that $\inf \rho(x) > 0$ and $\rho(x) \to +\infty$ as $|x| \longrightarrow +\infty$, satisfying assumption H_1 in Sec. 1.

Many authors (see for instance [1],[4],[7],[13]) have studied similar problems, some in bounded open subsets of R^n, others in unbounded ones. In both cases their existence theorems have been proved either by assuming coercivity (see, for instance,[1],[13]) or by giving a coercivity condition which involves all derivatives (assumption A in [7]). In order to get free from the hypothesis on the lower order derivatives, F. Browder, for example, imposed conditions upon the boundedness of the domain and the smoothness of its boundary to make the application of the Sobolev imbedding theorems possible.

Here, by assuming a coercivity condition depending only on the highest order derivatives (assumption K_3 in Sec. 2), we prove that there exists at least one solution of the problem (1) in $\overset{\circ}{U}{}^{m,p}_{s}(\Omega)$ with $s < -\frac{n}{p}$, $p \geq 2$.

The use of these spaces allows us to apply some continuous and compact imbedding theorems for unbounded domains which are proved in [1],[2],[15], and to specify the asymptotic behaviour of the solution.

The proof of the existence theorem is obtained by using a well-known result of H. Brezis (see [5]) in the framework of the theory of monotone operators.

In Sec. 1 we recall the spaces $\overset{\circ}{U}{}^{m,p}_{s}(\Omega)$ and some related results.

In Sec. 2 we formulate the hypotheses of the existence theorem and state the theorem itself, which is proved in Sec. 3.

Finally, Sec. 4 is devoted to an application of the above theorem to a problem of the following type:

$$\begin{cases} \Delta^2 u - \Delta u + f_1(u) + f_2(x,u,\text{grad } u) = 0 \text{ in } \Omega, \\ u \in \overset{\circ}{U}{}^{2,2}_{s}(\Omega). \end{cases}$$

1. <u>Notations and preliminaries</u>. Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, with boundary $\partial\Omega$ and $\alpha=(\alpha_1,\dots,\alpha_n)$ an ordered n-tuple of non negative integers; we set:

$$|\alpha| = \alpha_1+\dots+\alpha_n, \quad \partial^\alpha = \partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n}, \quad x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n} \text{ for } x \in \Omega$$

and, if $|\alpha| = m$, $\partial^m = \partial^\alpha$.

Let $\varrho(x)$ be a continuous function on Ω with $\inf\limits_{\Omega} \varrho(x) > 0$ and such that $\varrho(x) \to +\infty$ as $|x| \to +\infty$.

<u>Definition 1.1</u>. Let $k \in N$, $p \in [1,+\infty]$, $s \in \mathbb{R}$. We denote by $U^{k,p}_s(\Omega)$ the space of distributions u on Ω such that

$$\sum_{|\alpha|\leq k} \int_\Omega \varrho^{sp}(x)\, |\partial^\alpha u(x)|^p\, dx < +\infty$$

normed by

$$(1.1) \qquad \|u\|_{k,s,p} = [\sum_{|\alpha|\leq k} \int_\Omega \varrho^{sp}(x)\, |\partial^\alpha u(x)|^p\, dx]^{1/p}.$$

As usual, we set $U^{0,p}_s(\Omega) = L^p_s(\Omega)$ and $U^{k,2}_s(\Omega) = U^k_s(\Omega)$.

Definition 1.2. Let $k\in N$, $p\in [1,+\infty[$, $s\in \mathbb{R}$. We denote by $\overset{\circ}{U}{}^{k,p}_{s}(\Omega)$ the completion of $\mathcal{D}(\Omega)$ with respect to the norm (1.1).

Definition 1.3. Let $k\in N$, $p\in]1,+\infty[$, $s\in \mathbb{R}$. We denote by $U^{-k,p}_{s}(\Omega)$ the space of distributions u on Ω which are equal to a finite sum of derivatives or order $\leq k$ of functions belonging to $U^{0,p}_{s}(\Omega)$ and normed by

$$\| u \|_{-k,s,p}=\inf\Big(\sum_{|\alpha|\leq k}\| g_\alpha\|^p_{0,s,p}\Big)^{1/p},$$

where the infimum is taken over all representations of u of the form $u=\sum_{|\alpha|\leq k}\partial^\alpha g_\alpha$, $g_\alpha\in U^{0,p}_{s}(\Omega)$.

We assume that:

(H_1) $\rho\in C^\infty(\Omega)$ and for every $r\in\mathbb{R}$ and $\alpha\in N^n_0$ there exists a $c\in\mathbb{R}_+$ such that

$$|\partial^\alpha\rho^r(x)|\leq c(\rho(x))^r \text{ for every } x\in\Omega.$$

It is not difficult to prove that the function $\rho(x)=(1+|x|^2)^{1/2}$ satisfies property H_1.

Under assumption H_1 and if Ω has the cone property, continuous and compact imbedding theorems have been proved in [1],[2],[15]; it is also proved that there is a topological isomorphism of $U^{-k,p'}_{-s}(\Omega)$ onto the topological dual $(\overset{\circ}{U}{}^{k,p}_{s}(\Omega))'$ of the space $\overset{\circ}{U}{}^{k,p}_{s}(\Omega)$.

To write nonlinear partial differential operators in a convenient form, we introduce the vector space $\mathbb{R}^{1_k}$ whose elements are $\xi_k=\{\xi_\alpha/|\alpha|\leq k\}$ and divide such ξ_k into two parts $\xi_k=(\zeta,\eta)$ where $\eta=\{\eta_\beta/|\beta|\leq k-1\}$ is the lower order part of ξ_k and $\zeta=\{\xi_\alpha/|\alpha|=k\}$ is the part of ξ_k corresponding to the k-th derivatives.

Let us now recall some definitions which will be useful in the sequel. Let U and V be two Banach spaces; a map $f: U \longrightarrow V$ is called compact if it is continuous and maps bounded sets of U into relatively compact sets of V; f is called (sequentially) completely continuous if it maps weakly convergent sequences into strong convergent ones. If f is linear and U is reflexive, compactness and complete continuity are equivalent properties. If X is a topological vector space, X' denotes its topological dual and $\langle \cdot , \cdot \rangle$ the canonical pairing.

2. Assumptions and main result. We consider the following problem on Ω

$$(2.1) \qquad \begin{cases} Eu = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \partial^{\alpha} a_{\alpha}(x,u,\ldots,\partial^m u) + f(u) = 0, \\ u \in \overset{\circ}{U}{}^{m,p}_{s}(\Omega), \end{cases}$$

where the functions $a_{\alpha}(x,\xi) = a_{\alpha}^{(1)}(x,\xi) + a_{\alpha}^{(2)}(x,\eta)$ satisfy the Carathéodory conditions and the following properties:

$$(K_1) \qquad |a_{\alpha}^{(1)}(x,\xi_m)| \le g_{\alpha}(x) + c_1 \sum_{|\gamma| \le m} |\xi_{\gamma}|^{p/p'},$$

$$|a_{\alpha}^{(2)}(x,\eta)| \le h_{\alpha}(x) + c_2 \sum_{|\beta| \le m-1} |\eta_{\beta}|^{p/p'},$$

where $g_{\alpha}, h_{\alpha} \in U_s^{0,p'}(\Omega)$ and $c_1, c_2 \in \mathbb{R}_+$, $p \ge 2$.
We also assume that

(K_2) $s < -n/p$;

(K_3) there exists a positive constant $c_0 > 0$ and $h(x) \in U_s^{0,p}(\Omega)$ such that for all $x \in \Omega$ and for $\xi_m = (\zeta, \eta) \in \mathbb{R}^{s_m}$

$$\sum_{|\alpha| \le m} a_{\alpha}^{(1)}(x,\xi_m)\xi_{\alpha} \ge c_0 |\zeta|^p - h(x);$$

(K_4) for each x in Ω and each pair $(\xi_m, \xi'_m) \in \mathbb{R}^{s_m} \times \mathbb{R}^{s_m}$

the following inequality holds:

$$\sum_{|\alpha|\leq m} [a_{\alpha}^{(1)}(x,\xi_m)-a_{\alpha}^{(1)}(x,\xi'_m)] \quad [\xi_{\alpha}-\xi'_{\alpha}]\geq 0;$$

(K_5) f is an increasing function and $f(o)=0$; moreover

$|f(\lambda t)|\leq\gamma(\lambda)|f(t)|$, for every $t,\lambda\in\mathbb{R}$ with $\gamma:$ $:\mathbb{R}\longrightarrow\mathbb{R}_+$,

$|f(t)|\geq c_3|t|^{p-1+\mu}$, $\mu>0$.

Definition 2.1. We say that u is a solution of the problem (2.1) if u $\overset{\circ}{U}{}^{m,p}_{s}(\Omega)$ and $\langle Eu,v\rangle=0$ for every $v\in\overset{\circ}{U}{}^{m,p}_{s}(\Omega)$.

We are now able to state the following theorem:

Theorem 2.1. *Under assumptions* H_1,K_1-K_5, *the problem* (2.1) *has at least one solution.*

3. Proof of Theorem 2.1. The problem (2.1) is equivalent to the following one:

$$(3.1)\quad \begin{cases} E_s u=\rho^{sp}(x)\ Eu=0, \\ u\in\overset{\circ}{U}{}^{m,p}_{s}(\Omega). \end{cases}$$

It is not difficult to prove that

$$E_s u=A_s^{(1)}+B_s u+A_s^{(2)}u+\rho^{sp}f(u),$$

with

$$A_s^{(1)}u=\sum_{|\alpha|\leq m}(-1)^{|\alpha|}\,\partial^{\alpha}[\rho^{sp}a^{(\alpha)}(x,\dots,\partial^{m}u)],$$

$$B_s u=-\sum_{|\alpha|\leq m}(-1)^{|\alpha|}\sum_{\beta<\alpha}c_{\alpha\beta}\,\partial^{\alpha-\beta}\rho^{sp}\,\partial^{\beta}a_{\alpha}^{(1)}(x,\dots,\partial^{m}u),$$

$$A_s^{(2)}u=\sum_{|\alpha|\leq m}(-1)^{|\alpha|}\ \rho^{sp}\,\partial^{\alpha}a^{(2)}(x,\dots,\partial^{m-1}u).$$

Theorem 3.1. *Under assumptions* K_1,K_2,K_4 *and* H_1, *the operator*

$$u\in\overset{\circ}{U}{}^{m,p}_{s}(\Omega)\longrightarrow A_s^{(1)}u+B_s u+A_s^{(2)}u\in U^{-m,p'}_{-s}(\Omega)$$

is pseudomonotone.

Proof. We first prove that the operator $u \in \overset{\circ}{U}{}^{m,p}_{s}(\Omega) \longrightarrow$ $\longrightarrow A^{(1)}_{s} u \in U^{-m,p'}_{-s}(\Omega)$ is continuous and monotone. Assumption K_4 implies the monotonicity property. Notice that, for each $\alpha \in N^n_o$ such that $|\alpha| \leq m$, we have

$$u \in \overset{\circ}{U}{}^{m,p}_{s}(\Omega) \xrightarrow{d} (u,\dots,\partial^m u) \in \pi\, U^{o,p}_{s}(\Omega) \xrightarrow{A^{(1)}_{\alpha}} a^{(1)}(x,\dots,\partial^m u) \in$$

$$\in U^{o,p'}_{s}(\Omega) \xrightarrow{f} \wp^{sp}(x)\ a^{(1)}(x,\dots,\partial^m u) \in U^{o,p'}_{s-sp}(\Omega) \overset{i}{\hookrightarrow} U^{o,p'}_{-s}(\Omega)$$

$$\xrightarrow{\partial^{\alpha}} \partial^{\alpha}[\wp^{sp}(x)\ a^{(1)}_{\alpha}(x,\dots,\partial^m u)] \in U^{-m,p'}_{-s}(\Omega),$$

where d, f, i and ∂^{α} are continuous by some of the imbedding theorems proved in [1],[10] and[14]. The continuity and the boundedness of $A^{(1)}_{\alpha}$, under assumption K_1, follow from the standard theorems on Nemytskii operators (see [1],[15]). Now the operator $u \in \overset{\circ}{U}{}^{m,p}_{s}(\Omega) \longrightarrow B_s u \in U^{-m,p'}_{s}(\Omega)$ is compact; indeed we have

$$u \in \overset{\circ}{U}{}^{m,p}_{s}(\Omega) \xrightarrow{d} (u,\dots,\partial^m u) \in \pi\, U^{o,p}_{s}(\Omega) \xrightarrow{A^{(1)}_{\alpha}} a^{(1)}_{\alpha}(x,\dots,\partial^m u) \in$$

$$\in U^{o,p'}_{s}(\Omega) \xrightarrow{\partial^{\beta}} \partial^{\beta} a^{(1)}_{\alpha}(x,\dots,\partial^m u) \in U^{-m,p'}_{-s+sp}(\Omega)$$

$$\xrightarrow{c} \partial^{\alpha-\beta} \wp^{sp}(x)\ \partial^{\beta} a^{(1)}_{\alpha}(x,\dots,\partial^m u) \in U^{-m,p'}_{-s}(\Omega),$$

where, since $|\beta| \leq m-1$, ∂^{β} is compact (see Theorem 5.2 in [1]); d and $A^{(1)}_{\alpha}$ are continuous as before, while c is continuous as it is the conjugate of the mapping

$$v \in \overset{\circ}{U}{}^{m,p}_{s}(\Omega) \longrightarrow (\partial^{\alpha-\beta}\wp^{sp}) v \in \overset{\circ}{U}{}^{m,p}_{s-sp}(\Omega),$$

which turns out to be continuous (see imbedding theorems in [1] and [14]). Finally the operator $u \in \overset{\circ}{U}{}^{m,p}_{s}(\Omega) \longrightarrow A^{(2)} u \in$ $\in U^{-m,p'}_{-s}(\Omega)$ is completely continuous since for every $|\alpha| \leq m$

we have

$$u \in \overset{\circ}{U}{}^{m,p}_{s}(\Omega) \xrightarrow{d} (x,\dots,\partial^{m-1}u) \in \pi\, U^{o,p}_{sp-s}(\Omega) \xrightarrow{A^{(2)}_{\alpha}} a^{(2)}_{\alpha}(x,\dots$$

$$\dots,\partial^{m-1}u) \in U^{o,p'}_{sp-s}(\Omega) \xrightarrow{\partial^{\alpha}} \partial^{\alpha} a^{(2)}_{\alpha}(x,\dots,\partial^{m-1}u) \in U^{-m,p'}_{sp-s}(\Omega)$$

$$\xrightarrow{f} \wp^{sp}(x)\ \partial^{\alpha} a_{\alpha}(x,\dots,\partial^{m-1}u) \in U^{-m,p'}_{-s}(\Omega)$$

where d is a compact mapping, $A^{(2)}_{\alpha}$, ∂^{α}, f are continuous as before.

Now the theorem follows directly by the definition of the pseudomonotone operator (see [5]).

Now let us set

$$F(t)= \int_0^t f_1(\tau)d\tau \quad \text{and} \quad D(\varphi)=\{u \in \overset{\circ}{U}{}^{m,p}_{s}(\Omega): \int_{\Omega} F(u(x))dx < +\infty\}.$$

By means of K_5 it is easy to show that the functional

$$\varphi(u) = \begin{cases} +\infty & \text{if } u \in \overset{\circ}{U}{}^{m,p}_{s}(\Omega) \setminus D(\varphi), \\ \int_{\Omega} F(u(x))dx, & \text{fu} \in D(\varphi), \end{cases}$$

is convex and $D(\varphi)$ is a linear subspace of $\overset{\circ}{U}{}^{m,p}_{s}(\Omega)$.

<u>Theorem 3.2</u>. <u>Under assumptions</u> H_1,K_1,K_2,K_3 <u>and</u> K_5 <u>we have</u>

$$\frac{\langle A^{(1)}_{s}u+B_s u+A^{(2)}_{s}u,u\rangle + \varphi(u)}{\|u\|_{m,s,p}} \longrightarrow +\infty$$

as $\|u\|_{m,s,p} \longrightarrow +\infty$.

Proof: For each $u \in \overset{\circ}{U}{}^{m,p}_{s}(\Omega)$, from the hypothesis K_3 and the Ehrling inequality (see [15]) we get

$$(3.3) \qquad \langle A^{(1)}_{s}u,u\rangle = \sum_{|\alpha|\leq m} \int_{\Omega} \wp^{sp}(x) a^{(1)}_{\alpha}(x,u,\dots,\partial^{m}u)\, \partial^{\alpha}u\, dx$$

$$\geq c_0 \|u\|^{p}_{m,s,p} - c_0 \sum_{|\alpha|<m} \|\partial^{\alpha}u\|^{p}_{o,s,p} - \|h\|_{o,s,p}$$

$$\geq c_0 \| u \|^p_{m,s,p} - \varepsilon \| u \|^p_{m,s,p} - c(\varepsilon) \| u \|^p_{o,s,p} - \| h \|_{o,s,p}$$

$$=(c_0 - \varepsilon) \| u \|^p_{m,s,p} - c(\varepsilon) \| u \|^p_{o,s,p} - c_4 .$$

Furthermore, assumptions H_1, K_1 and the Schwarz-Hölder and Ehrling inequalities imply that

$$(3.4) \quad |\langle B_s u,u \rangle| \leq \sum_{|\alpha| \leq m} \sum_{\beta<\alpha} \sum_{\delta \leq \beta} c_{\alpha\beta\delta} \int_\Omega \varrho^{sp}(x) |a^{(1)}_\alpha(x, u,\dots,\partial^m u)| \, |\partial^\delta u| \, dx \leq \sum_{|\alpha| \leq m} \sum_{\beta<\alpha} \sum_{\delta \leq \beta} c_{\alpha\beta\delta} \int_\Omega \varrho^{sp}(x) [|g_\alpha(x)| + \sum_{|\gamma| \leq m} |\partial^\gamma u|^{\frac{p}{p'}}] \, |\partial^\delta u| dx$$

$$\leq \sum_{|\alpha| \leq m} \sum_{\beta<\alpha} \sum_{\delta \leq \beta} c_{\alpha\beta\delta} [\| g_\alpha \|_{o,s,p'} \, \| \partial^\delta u \|_{o,s,p} + \sum_{|\gamma| \leq m} \| \partial^\gamma u \|^{\frac{p}{p'}}_{o,s,p} \, \| \partial^\delta u \|_{o,s,p}] \leq c_5 \| u \|_{m,s,p} + c_6 \| u \|^{p-1}_{m,s,p} [\varepsilon \| u \|_{m,s,p} + c(\varepsilon) \| u \|_{o,s,p}] .$$

Likewise it can be proved that

$$(3.5) \quad |\langle A^{(2)}_s u,u \rangle| \leq c_7 \| u \|_{m,s,p} + c_8 \| u \|_{m,s,p} [\varepsilon \| u \|^{p-1}_{m,s,p} + c(\varepsilon) \| u \|^{p-1}_{o,s,p}] ,$$

where $c_4, \dots, c_8$ are positive constants independent of u. Finally, by virtue of the Schwarz-Hölder inequality and assumption K_2 we have

$$(3.6) \quad \varphi(u) = \int_\Omega \varrho^{sp}(x) \left(\int_0^{u(x)} f(t) dt \right) dx \geq c_3 \int_\Omega \varrho^{sp}(x) \left[\int_0^{u(x)} |t|^{p-1+\mu} dt \right] dx = \frac{c_3}{p+\mu} \int_\Omega \varrho^{sp}(x) |u(x)|^{p+\mu} dx$$

$$\geq c_9 \left(\int_\Omega \varrho(x)^{\frac{p\mu s}{p+\mu} \cdot \alpha'} dx \right)^{\frac{1}{\alpha'} \cdot \frac{p+\mu}{p}} \left(\int_\Omega \varrho^{sp}(x) |u(x)|^p \right)^{\frac{p+\mu}{p}}$$

$$\geq c_{10}\left(\int_\Omega \rho^{sp}(x)|u(x)|^p dx\right)^{\frac{p+\mu}{p}} = c_{10}\|u\|^{p+\mu}_{o,s,p},$$

where $\alpha=(p+\mu)/p$, $\alpha'=\alpha/(\alpha-1)$ and c_9, c_{10} are positive constants. Now (3.2) is an easy consequence of (3.3),(3.4), (3.5) and (3.6).

Proof of Theorem 2.1: in virtue of Theorems 3.1 and 3.2 we may apply Corollary 30 of [5] and state that there exists $u\in D(\varphi)$ such that

$$\langle A_s^{(1)}u+B_s u+A_s^{(2)}u,\ v-u\rangle \geq \varphi(u)-\varphi(v),\quad \forall v\in \mathring{U}_s^{m,p}(\Omega).$$

The proof is completed by means of known procedures (see e.g. Theorem 3.1 of [3]) which allow to show that u is also a solution of the problem (3.1).

<u>Remark</u>. Let us observe that a weaker Nemytskii condition, such as in [7] and in [13], allows us to prove Theorem 3.1 but it fails in the proof of coercivity.

4. <u>Example</u>. We consider the problem on Ω

$$(4.1)\quad \begin{cases} Eu=\Delta^2 u-\Delta u+f(x,u,\operatorname{grad} u)=0,\\ u\in \mathring{U}_s^2(\Omega).\end{cases}$$

It is equivalent to

$$\begin{cases} E_s u=\rho^{2s}(x)\,[\Delta^2 u-\Delta u+f(x,u,\operatorname{grad} u)]=0,\\ u\in \mathring{U}_s^2(\Omega).\end{cases}$$

We set $f(x,u,\operatorname{grad} u)=f_1(u)+f_2(x,u,\operatorname{grad} u)$.

Now E_s may be written in the form

with

$$E_s u=A_s^{(1)}u+B_s u+A_s^{(2)}u+\rho^{2s}f_1(u),$$

with

$$A_s^{(1)}u = \Delta(\rho^{2s}\Delta u) - \sum_{i=1} \partial_{x_i}(\rho^{2s}\partial_{x_i}u),$$

$$B_s u = \sum_{i=1}^{n} (\partial_{x_i}\rho^{2s})(\partial_{x_i}u) - 2(\mathrm{grad}\,\rho^{2s}, \mathrm{grad}\,\Delta u) -$$

$$-(\Delta\rho^{2s})(\Delta u),$$

$$A_s^2 u = \rho^{2s} f_2(x, u, \mathrm{grad}\, u).$$

By supposing that f is a real function defined in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, we shall also assume that the functions f_1 and f_2 satisfy the properties

(α_1) f_1 is increasing, $f_1(0)=0$,

$|f_1(\lambda t)| \leq \gamma(\lambda)|f_1(t)|$ for every $\lambda, t \in \mathbb{R}$, with $\gamma : \mathbb{R} \longrightarrow \mathbb{R}_+$,

$|f_1(t)| \geq c|t|^{1+\mu}$, $\mu > 0$;

(α_2) f_2 satisfies the Carathéodory conditions and

$|f_2(x,t,\xi)| \leq h(x) + b_1|t| + b_2|\xi|$;

with $h \in \mathring{U}_s^{0,2}$ and $b_1, b_2 \in \mathbb{R}_+$.

It is easy to verify that the operator E satisfies assumptions K_1, K_3, K_4 and K_5; then Theorem 2.1 implies that the problem (4.1) has at least one solution.

We shall conclude this section by giving some examples of f_1 and f_2: The functions t^{2n+1}, $n \in N$ and $t\,|t|^{\alpha}$, $\alpha \in \mathbb{R}_+$, satisfy condition α_1. The functions $g(x) + \sqrt{|t|^{\alpha}|\xi|^{\beta}}$, $0 < \alpha, \beta < 1$ and $g(x) + \mathrm{arctg}\, t + \mathrm{arctg}\,|\xi|$ where $g \in \mathring{U}_s^{0,2}(\Omega)$, satisfy condition α_2.

R e f e r e n c e s

[1] V. BENCI - D. FORTUNATO: Weighted Sobolev spaces and nonlinear Dirichlet problem in unbounded domains, to apper in Annali di Matematica pure e applicata.

[2] V. BENCI - D. FORTUNATO: Some compact imbedding theorems for weighted Sobolev spaces, Boll. Un. Math. Ital. 14-B(1976), 832-843.

[3] V. BENCI - D. FORTUNATO: Some nonlinear problems with asymptotic conditions, to appear in Nonlinear Analysis theory methods and applications.

[4] M.S. BERGER and M. SCHECHTER: Embedding theorems and quasi-linear elliptic boundary value problems for unbounded domains, Trans. of the Amer. Math. Soc. 172, pp. 261-278(1972).

[5] H. BREZIS: Equations et inéquations nonlinéaires dans les espaces vectoriels en dualité, Ann. Inst. Fourier, Grenoble 18,pp.115-175(1968).

[6] H. BREZIS - G. STAMPACCHIA: Sur la regularité de la solution d'inéquations elliptiques, Bull. Soc. Math. France 96(1968).

[7] F. BROWDER: Existence theorems for nonlinear partial differential equations, Proc. of Symposia in pure Math., Global Analysis vol. XVI, pp. 1-60(1970).

[8] D.E. EDMUNDS - W.D. EVANS: Elliptic and degenerate-elliptic operators in unbounded domains, Ann. Sc. Nor. Sup. Pisa Cl. Sci. 27(1973), 591-640.

[9] D.E. EDMUNDS - J.R.L. WEBB: Quasilinear elliptic problems in unbounded domains, Proc. Royal Soc. London A 334(1973), 397-410.

[10] D. GIACHETTI - E. MASCOLO: Problemi quasi ellittici in spazi di Sobolev con peso, to appear in Rend. Acc. Naz. dei Lincei, Cl. Sc. Fis. Mat. Nat. vol. LXIII, 5(1977).

[11] P. HESS: Nonlinear elliptic problems in unbounded domains,

conference held at Int. Summer School on non-linear Operators Berlin (G.D.R.), Sept. 1975.

[12] P. HESS: Problèmes aux limites nonlinéaires dans des domaines non bornés, C.R. Ac. Sc. Paris, t.281, Série A (Oct. 6, 1975).

[13] J. NEČAS: Les équations elliptiques non linéaires, Czechoslovak Math. Journal 19(94)(1968), 252-274.

[14] R. SCHIANCHI: Spazi dissimetrici e con peso, Rend. Acc. Sc. Fis. Mat. Serie IV, vol. XLII (1975), 1-40.

[15] R. SCHIANCHI: Spazi dissimetrici e con peso e applicazioni ai problemi quasi ellittici non lineari, to appear in Le Matematiche (Catania).

[16] M.M. VAINBERG: Variational methods for the study of non-linear operator, Holden-Day Inc. (1964).

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