Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 20 (1979), No. 3, 539--546

Persistent URL: http://dml.cz/dmlcz/105950

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20,3 (1979)

SMALL-FIBRED SEMITOPOLOGICAL FUNCTORS WITHOUT SMALL-FIBRED INITIAL COMPLETIONS

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Abstract: The first example of a semitopological functor $U:\underline{A} \longrightarrow \underline{X}$ which is small-fibred but has no small fibred initial completion appeared in Herrlich [3] (as a negative solution of a problem of J. Adámek); there $\underline{X}$ was an artificial category (in fact, a preordered class) and the question whether there exists such a $U:\underline{A} \longrightarrow \underline{X}$, say, for $\underline{X}$ = Set, remained open. In this note, we construct such an example for any category $\underline{X}$ which is cocomplete, has a terminal object and is not a preordered class.

Key words: Semitopological functor, initial completion, Mac Neille completion, small-fibred functor, strongly small-fibred functor.

Classification: 18D30, 18A35

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0. Introduction

0.1. The present note has three parts. In the first one we prove that natural functors between comma-categories over a cocomplete category are semitopological. In the second part we present a general construction which yields a semitopological functor starting from a chain of semitopological functors. The third part contains the counterexample in question.

0.2. Recall that a functor $U:\underline{A} \longrightarrow \underline{X}$ is semitopological if any U-sink $\{UA \xrightarrow{f_i} X;\ i \in I\}$ in $\underline{X}$ has a semifinal solution;

by a solution is meant any $\underline{X}$-map $X \xrightarrow{g} UA$ such that for every $i \in I$ there is $g_i: A_i \longrightarrow A$ in $\underline{A}$ with $Ug_i = g\, f_i$. The solution is semifinal if for any other solution $X \xrightarrow{g'} UA'$ there is a unique $h: A \longrightarrow A'$ in $\underline{A}$ with $Uh\, g = g'$.

0.3. Further, recall from [1] that a functor $U: \underline{A} \longrightarrow \underline{X}$ admits a small-fibred initial completion iff it has a small-fibred Mac Neille completion; this is the case iff $U: \underline{A} \longrightarrow \underline{X}$ is strongly small-fibred, that is, iff there is no proper class $\{X \xrightarrow{g_\delta} UA_\delta \,;\, \delta \in I\}$ of $\underline{X}$-maps such that the following holds: if $\delta, \varepsilon \in I$, $\delta \neq \varepsilon$, then there exists an $\underline{X}$-map $UA \xrightarrow{f} X$ such that $g_\delta f = Uh$ for some $h: A \longrightarrow A_\delta$ in $\underline{A}$ while $g_\varepsilon f \neq Uh$ for all $h: A \longrightarrow A_\varepsilon$ in $\underline{A}$, or conversely.

1. Comma-categories

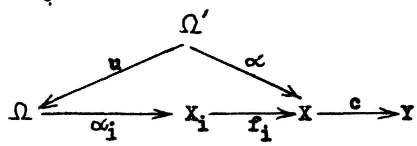
1.1. If Ω is a fixed object of a category $\underline{X}$, consider the comma-category $(\Omega \downarrow \underline{X})$; objects of $(\Omega \downarrow \underline{X})$ are $\underline{X}$-maps $\Omega \xrightarrow{\alpha} X$; maps from $\Omega \xrightarrow{\alpha} X$ to $\Omega \xrightarrow{\alpha'} X'$ in $(\Omega \downarrow \underline{X})$ are those $f: X \longrightarrow X'$ in $\underline{X}$ with $f\alpha = \alpha'$. The category $(\Omega \downarrow \underline{X})$ will be considered as a category over $\underline{X}$: the underlying functor $U: (\Omega \downarrow \underline{X}) \longrightarrow \underline{X}$ is defined by $U(\Omega \xrightarrow{\alpha} X) = X$, $Uf = f$.

Each $\underline{X}$-map $u: \Omega' \longrightarrow \Omega$ in $\underline{X}$ induces a functor $(u \downarrow \underline{X})$: $(\Omega \downarrow \underline{X}) \longrightarrow (\Omega' \downarrow \underline{X})$ over $\underline{X}$ defined by $(u \downarrow \underline{X})(\Omega \xrightarrow{\alpha} X) = \Omega' \xrightarrow{u} \Omega \xrightarrow{\alpha} X$.

1.2. Proposition. If $\underline{X}$ is cocomplete then the functor $(u \downarrow \underline{X}): (\Omega \downarrow \underline{X}) \longrightarrow (\Omega' \downarrow \underline{X})$ is semitopological for every $\underline{X}$-map $u: \Omega' \longrightarrow \Omega$.

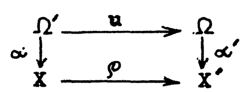
Proof. Let $\{(u \downarrow \underline{X})A_i \xrightarrow{f_i} A;\ bi \in I\}$ be a non-void $(u \downarrow \underline{X})$-sink in $(\Omega' \downarrow \underline{X})$, $A_i = (\Omega \xrightarrow{\alpha_i} X_i)$, $A = (\Omega' \xrightarrow{\alpha} X)$.

Thus $f_i : X_i \longrightarrow X$ and the triangle in the following diagram commutes for every $i \in I$:



Let $c : X \longrightarrow Y$ be a coequalizer of the set $\{f_i \alpha_i : \Omega \longrightarrow X;\ i \in I\}$ in $\underline{X}$. Put $B = (\Omega \xrightarrow{\beta} Y)$ where β is the common value of the composites $c\, f_i\, \alpha_i$. Then $c : A \longrightarrow (u \downarrow \underline{X})B$ in $(\Omega' \downarrow \underline{X})$ because $c\alpha = c\, f_i\, \alpha_i\, u = \beta u$. It is routine to prove, using the universal property of the pushout, that it is a semifinal solution of the sink in question.

The existence of a semifinal solution for the void sinks in $(\Omega' \downarrow \underline{X})$ is equivalent to the existence of free objects over $(\Omega' \downarrow \underline{X})$-objects w.r.t. $(u \downarrow \underline{X})$. So, if $A = (\Omega' \xrightarrow{\alpha} X)$ is a $(\Omega' \downarrow \underline{X})$-object, consider the pushout



It is easy to see that $B = (\Omega \xrightarrow{\alpha'} X')$ together with $\rho : A \longrightarrow (u \downarrow \underline{X})B$ is a free object over A.

1.3. <u>Corollary</u>. $(u \downarrow \underline{X})$ has a left adjoint.

1.4. <u>Corollary</u>. The forgetful functor $U : (\Omega \downarrow \underline{X}) \longrightarrow \underline{X}$ is semitopological.

<u>Proof</u>. $U = (\emptyset \downarrow \underline{X})$ where $\emptyset$ is the map from the initial object of $\underline{X}$ to Ω.

2. <u>A general construction</u>

2.1. Let L be a large lattice such that 1) each bounded subset of L has a least upper bound, 2) for every $\varepsilon \in L$, the class $\{\delta \in L; \delta < \varepsilon\}$ is a set.

Let $\underline{A}_\alpha$ ($\alpha \in L$) be categories over a category $\underline{X}$ and U_α : $:\underline{A}_\alpha \longrightarrow \underline{X}$ their underlying functors. Let $U_{\alpha\beta} :\underline{A}_\beta \longrightarrow \underline{A}_\alpha$ ($\alpha, \beta \in L$, $\alpha \leq \beta$) be functors over $\underline{X}$ such that $U_{\alpha\alpha}$ is identical for every α and $U_{\alpha\beta} U_{\beta\gamma} = U_{\alpha\gamma}$ whenever $\alpha \leq \beta \leq \gamma$.

Define a category $\underline{A} = \sum_L \underline{A}_\alpha$ with an underlying functor $U = \sum_L U_\alpha :\underline{A} \longrightarrow \underline{X}$ as follows:

(i) obj $\underline{A}$ is a disjoint union of obj $\underline{A}_\alpha$ ($\alpha \in L$);

(ii) if $A \in$ obj $\underline{A}_\alpha$, $B \in$ obj $\underline{A}_\beta$ then $\underline{A}(A,B) = \underline{A}_\alpha(A, U_{\alpha\beta} B)$ if $\alpha \leq \beta$ and $\underline{A}(A,B) = \emptyset$ otherwise;

(iii) the composition is defined in an obvious way;

(iv) $UA = U_\alpha A$ for A in $\underline{A}_\alpha$, $Uf = U_\alpha f$ for $f \in \underline{A}(A,B)$, $A \in$ obj $\underline{A}_\alpha$.

2.2. <u>Proposition</u>. Let $\underline{X}$ be cocomplete. Let all $U_\alpha :A \longrightarrow$ $\longrightarrow \underline{X}$ ($\alpha \in L$) be semitopological and let each $U_{\alpha\beta} :\underline{A}_\beta \longrightarrow \underline{A}_\alpha$ have a left adjoint $F_{\alpha\beta} :\underline{A}_\alpha \longrightarrow \underline{A}_\beta$. Then $U:\underline{A} \longrightarrow \underline{X}$ is almost semitopological in the sense that if a U-sink has a solution then it has a semifinal solution.

2.3. <u>Corollary</u>. Suppose that $\underline{X}$ has, in addition, a terminal object $\underline{1}$. Let $\underline{A}^*$ be the category obtained from $\underline{A}$ by adding a formal terminal object ∞ with maps $A \longrightarrow \infty$ ($A \in$ obj $\underline{A}^*$). Then the extension $U^* :\underline{A}^* \longrightarrow \underline{X}$ of $U:\underline{A} \longrightarrow \underline{X}$ defined by $U^*\infty = \underline{1}$ is semitopological.

<u>Proof of 2.2</u>. Let $\{UA_i \xrightarrow{f_i} X;\ i \in I\}$ be a U-sink in $\underline{X}$. For every $i \in I$, let $A_i \in$ obj $\underline{A}_{\lambda(i)}$. If the sink has a solution then the class $J = \{\lambda(i);\ i \in I\}$ is bounded; so it is a set; denote σ its least upper bound. For each $\iota \leq \sigma$, let $X \xrightarrow{g_\iota}$ $\xrightarrow{g_\iota} U_\iota B_\iota$ be a semifinal solution of the U_ι-sink

$\{UA_i \xrightarrow{f_i} X; \lambda(i) = \iota\}$ in $\underline{A}_\iota$. Consider the natural maps $B_\iota \xrightarrow{\eta_\iota} U_{\iota\sigma} F_{\iota\sigma} B_\iota$. As $U_\iota U_{\iota\sigma} F_{\iota\sigma} B_\iota = UF_{\iota\sigma} B_\iota$ and $U_\iota B_\iota = UB_\iota$, we can form a multiple pushout $\{UF_{\iota\sigma} B_\iota \xrightarrow{\varphi_\iota} C;\ \iota \in J\}$ of the set $\{X \xrightarrow{g_\iota} UB \xrightarrow{U\eta_\iota} UF_{\iota\sigma} B_\iota\ ;\ \iota \in J\}$. As U coincides with U_σ on $\underline{A}_\sigma$, maps φ_ι form a U_σ-sink $\{U_\sigma F_{\iota\sigma} B_\iota \xrightarrow{\varphi_\iota} C;\ \iota \in J\}$ in $\underline{A}_\sigma$ which has a semifinal solution $C \xrightarrow{\psi} U_\sigma B$. Now it is easy to see that $C \xrightarrow{\psi} UB$ serves as a semifinal solution of the original U-sink in $\underline{X}$.

3. <u>The counterexample</u>

3.1. Let $\underline{X}$ be cocomplete with a terminal object $\underline{1}$ and let $\underline{X}$ be not a preordered class; the latter means that there is $\Omega \in \mathrm{obj}\ \underline{X}$ such that card $\underline{X}(\Omega,\Omega') > 1$ for some Ω'; the object Ω will be fixed in what follows.

Let $\underline{A}$ be the category over $\underline{X}$ whose objects are of the form $A = (X,(h_{ij})_{i \leq n,\, j \leq \sigma})$, for various ordinals σ and various finite ordinals n, where $X \in \mathrm{obj}\ \underline{X}$, $h_{ij}: \Omega \longrightarrow X$ in $\underline{X}$. Morphisms in $\underline{A}$ from A to $B = (X',(h'_{ij})_{i \leq n',\, j \leq \sigma'})$ are those $f: X \longrightarrow X'$ in $\underline{X}$ with $fh_{ij} = h'_{ij}$ if $n \leq n'$ and $\sigma \leq \sigma'$; otherwise we put $\underline{A}(A,B) = \emptyset$. The underlying functor $U: \underline{A} \longrightarrow \underline{X}$ is defined by $U(X,(h_{ij})) = X$, $Uf = f$.

Let $U^*: \underline{A}^* \longrightarrow \underline{X}$ be obtained from $U: \underline{A} \longrightarrow \underline{X}$ by adding a formal terminal object ∞ as in 2.3.

3.2. $U^*: \underline{A}^* \longrightarrow \underline{X}$ is semitopological.

<u>Proof</u>. Each $\underline{A}$-object $(X,(h_{ij})_{i \leq n,\, j \leq \sigma})$ can be naturally identified with $\Omega_{n\sigma} \xrightarrow{h} X$ where h is the $\underline{X}$-map from the coproduct $\Omega_{n\sigma}$ of $(n+1) \times (\sigma+1)$ copies of Ω defined by $h\nu^n_{ij} = h_{ij}$, $i \leq n$, $j \leq \sigma$; here $\nu^n_{ij}: \Omega \longrightarrow \Omega_{n\sigma}$ are the coproduct

injections. Thus $\underline{A} = \sum\limits_{L} A_{\alpha}$ (see 2.1), where

(i) $L = \omega \times \mathrm{Ord}$;

(ii) for $\alpha \in L$, $\alpha = (n, \sigma)$, $\underline{A}_{\alpha} = (\Omega_{n\sigma} \downarrow \underline{X})$;

(iii) for $\alpha \in L$, the functor $U_{\alpha}: \underline{A}_{\alpha} \longrightarrow \underline{X}$ is defined to be the restriction of $U: \underline{A} \longrightarrow \underline{X}$; it is semitopological by 1.4;

(iv) for $\alpha, \beta \in L$, $\alpha = (n, \sigma)$, $\beta = (n', \sigma')$, the functor $U_{\alpha\beta}: \underline{A}_{\beta} \longrightarrow \underline{A}_{\alpha}$ is defined to be $(u_{\alpha\beta} \downarrow \underline{X})$ where $u_{\alpha\beta}:$ $: \Omega_{n\sigma} \longrightarrow \Omega_{n'\sigma'}$ is induced by the inclusion $(n+1) \times (\sigma+1) \hookrightarrow$ $\hookrightarrow (n'+1) \times (\sigma'+1)$; the functor $U_{\alpha\beta}$ has a left adjoint by 1.3.

So U^* is semitopological by 2.3.

3.3. Let $\underline{B}$ be the full subcategory of $\underline{A}$ such that obj $\underline{B}$ consists of ∞ and of those $(X, (h_{ij})_{i \leq n, j \leq \sigma})$ such that all $h_{nj} (j \leq \sigma)$ are pairwise distinct. Let $V: \underline{B} \longrightarrow \underline{X}$ be the restriction of U. Clearly,

3.4. $V: \underline{B} \longrightarrow \underline{X}$ is small-fibred.

3.5. $\underline{B}$ is reflective in $\underline{A}$. Thus, $V: \underline{B} \longrightarrow \underline{X}$ is semitopological, too.

Proof. Let $A \in$ obj $\underline{A}$ be not in obj $\underline{B}$.

a) If the only $\underline{B}$-object B such that there exists a map $A \longrightarrow B$ is $B = \infty$ then the $\underline{B}$-reflection of A is ∞.

b) Let $A = (X, (h_{ij})_{i \leq n, j \leq \sigma})$ admit a morphism $f: A \longrightarrow B$, $B \in$ obj $\underline{B}$, $B = (X', (h'_{ij})_{i \leq m, j \leq \vartheta})$. Then $m \geq n$, $\sigma \geq \vartheta$. Even $m > n$ for all these f; indeed, as $A \notin$ obj $\underline{B}$, $h_{ns} = h_{nr}$ for some $r \neq s$; then $h'_{nr} = fh_{nr} = fh_{ns} = h'_{ns}$; so the h'_{nj}'s are not pairwise distinct; it follows $m \neq n$.

c) Put $\bar{A} = (\bar{X}, (\bar{h}_{ij})_{i \leq n+1, j \leq \sigma})$ where X is a coproduct of X and $\sigma+1$ copies of Ω,

(i) $\bar{h}_{ij} = \nu h_{ij}$ $(i \leq n,\ j \leq \sigma)$,

$\bar{h}_{n+1,j} = \nu_j$ $(j \leq \sigma)$

where $\nu : X \longrightarrow \bar{X}$, $\nu_j : \Omega \longrightarrow \bar{X}$ $(j \leq \sigma)$ are the coproduct injections.

d) We are going to prove that $\bar{A} \in \mathrm{obj}\ \underline{B}$, i.e. that the ν_j's are pairwise distinct $(j \leq \sigma)$.

Consider any f as in b). As ν, ν_j $(j \leq \sigma)$ are coproduct injections, there is $h : \bar{X} \longrightarrow X'$ with

(ii) $h\,\nu_j = h'_{mj}$ $(j \leq \sigma)$,

$h\,\nu = f$.

As $B \in \mathrm{obj}\ \underline{B}$, the h'_{mj}'s $(j \leq \sigma)$ are pairwise distinct. By virtue of (ii), so are the ν_j's.

e) Let us prove that $\nu : A \longrightarrow \bar{A}$ is a $\underline{B}$-reflection of A.

Indeed, ν is a morphism from A to $\bar{A}$ by (i). Let $f : A \longrightarrow B$ be as in b). We are to find $f' : A \longrightarrow B$ with

(iii) $f'\nu = f$.

The map f', being an $\underline{A}$-map from A to B, should satisfy $f'\bar{h}_{ij} = h'_{ij}$ $(i \leq n+1,\ j \leq \sigma)$, that is,

(iv) $f'\nu h_{ij} = h'_{ij}$ $(i \leq n,\ j \leq \sigma)$,

(v) $f'\nu_j = h'_{n+1,j}$ $(j \leq \sigma)$.

As ν, ν_j are coproduct injections, there is a unique f' satisfying (iii),(iv),(v), viz f' determined by (iii),(v).

3.6. $V : \underline{B} \longrightarrow \underline{X}$ is not strongly small-fibred.

Proof. For every ordinal σ, let X_σ be the coproduct of $\sigma+1$ copies of Ω. It follows easily from the fact that card $\underline{X}(\Omega, \Omega') > 1$ for some Ω' that all coproduct injections $\nu_j : \Omega \longrightarrow X_\sigma$ $(j \leq \sigma)$ are pairwise distinct. So $A_\sigma = (X_\sigma, (h_{ij})_{i \leq 0, j \leq \sigma}) \in \mathrm{obj}\ \underline{B}$ where $h_{0j} = \nu_j (j \leq \sigma)$. Let $f_\sigma : X_\sigma \longrightarrow \Omega$

be the codiagonal map. Further, let Y_δ be the coproduct of X_δ and of Ω , and let $h_\delta : X_\delta \longrightarrow Y_\delta$, $g_\delta : \Omega \longrightarrow Y_\delta$ be the coproduct injections. Put $B_\delta = (Y_\delta, (h'_{ij})_{i \leq 1, j \leq \delta})$ where $h'_{oj} = g_\delta$, $h'_{1j} = h_\delta h_{oj}$ $(j \leq \delta)$. Again, $B_\delta \in$ obj $\underline{B}$. Then for $\delta \neq \varepsilon$, say $\delta > \varepsilon$, the map $g_\delta f_{\varepsilon+1}$ is a $\underline{B}$-map from $A_{\varepsilon+1}$ to B_δ , while $g_\varepsilon f_{\varepsilon+1}$ is not a $\underline{B}$-map from $A_{\varepsilon+1}$ to B_ε . The proof is finished.

References

[1] J. ADÁMEK, H. HERRLICH, G.E. STRECKER: Least and largest initial completions I, II, Comment. Math. Univ. Carolinae 20(1979), 43-77.

[2] H. HERRLICH: Initial completions, Math. Z. 150(1976), 101-110.

[3] H. HERRLICH: Reflective Mac Neille completions of fibre-small categories need not be fibre small, Comment. Math. Univ. Carolinae 19(1978), 147-149.

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(Oblatum 30.3. 1979)