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DEGREES OF INTERPRETABILITY

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Abstract: T is a fixed theory containing arithmetic. For sentences φ, ψ in the language of T, $\varphi \preceq_T \psi$ means that T with the additional axiom φ is relatively interpretable in T with the additional axiom ψ. The structure V_T of degrees induced by $\preceq_T$ is considered and various algebraic properties of V_T are exhibited. For example, if T is essentially reflexive, then V_T is a distributive lattice with 0 and 1 and no element except 0 and 1 has a complement.

Key words: Interpretability, axiomatic theory, preorder on theories.

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1. Introduction. In this paper we consider formal axiomatic theories. Intuitively, some of these theories are stronger than others. This is certainly related to the question of consistency. As is well known, all the famous results concerning the consistency of the axiom of choice, continuum hypothesis and their negations were reduced to finding some interpretations. In this work we use interpretations as a mean to explicate the notion that a theory S is stronger or more complex than a theory T: it is just in the case that T is interpretable in S. In this way we have defined a (partial) preorder on theories and we may ask what properties this preorder has. In particular, is it den-

se?, are there incomparable elements?, etc.

First of all, let us restrict ourselves to theories of the form (T,φ) arising by adding one new axiom to a fixed theory T. Hence we define the ordering only for sentences of T: $\varphi \leq_T \psi$ iff (T,φ) is interpretable in (T,ψ). The restriction to theories of this form is convenient because we may consider only one fixed language, and it is also natural because it corresponds to the situation that we work in some theory and we are interested in the strength of additional axioms. Sentences φ and ψ have the same degree (notation $\varphi \equiv_T \psi$) if both $\varphi \leq_T \psi$ and $\psi \leq_T \varphi$. V_T is the set of all degrees. V is a partially ordered set with greatest and lowest element and it is a lower semilattice where meet is the disjunction of sentences.

Now there are two kinds of questions we have to solve. Firstly, questions concerning algebraic properties of the semilattice V_T: are there incomparable elements in V_T, is V_T a lattice?, are there complements in V_T?, etc. Secondly, the questions on syntactical complexity: what is the simplest sentence in a given degree?

As to the first kind of questions, it follows from the results of R.G. Jeroslow [J] that for reasonable theories the ordering on V_T is dense and that there are many incomparable elements. We shall further show that for every degree $d \neq 0,1$ there are degrees incomparable with d. If T is essentially reflexive then V is a distributive lattice. No element in V_T distinct from 0 and 1 has a complement.

If the theory T is essentially reflexive then, furthermore, in every degree in V_T there is an arithmetical Π_2 and a Σ_2 sentence. There are degrees containing neither Π_1 sentences nor Σ_1 sentences, but Π_1 sentences are in V_T cofinal whereas Σ_1 sentences are not.

J. Mycielski's work [M] is motivated similarly as the present paper but the author makes no restriction on theories. In his structure every degree contains with each theory T many "copies" of T with different language and the l.u.b. of two degrees is simply the union of sets of representatives with disjoint languages. If the theory T is essentially reflexive then V_T is a substructure of Mycielski's lattice according to $\leq_T$, but I was unable to decide whether also l.u.b.'s coincide.

This paper uses the method of arithmetization described in the fundamental Feferman's paper [F]. It is a continuation of papers of R.G. Jeroslow, M. Hájková and P. Hájek. It was written under supervision of P. Hájek. I would like to thank P. Hájek for the time he spent with me during many valuable discussions and for the help with translation of the work into English.

2. Preliminaries. We shall use the logical system described in [VH 1] Chapt. I, Sect. 2. The reader may omit the following part concerning logic but he is supposed to understand the statement "the theory T contains arithmetic". For example, in the set theory we may use the arithmetical operation symbols +, $\cdot$, $'$, $\overline{0}$ and form arithmetical formulas.

The language L of a theory can contain variables of various sorts which are distinguished by indices (x^i , x^j where i, j are numbers of sorts in L). Every theory has one universal sort i such that for every term in L, $T \vdash \exists x^i (t = x^i)$. We suppose to have fixed one sort as the arithmetical sort. Variables without indices will usually be variables of the arithmetical sort.

The language of Robinson and Peano arithmetic has only the arithmetical sort and operation symbols $+$, $\cdot$, $'$, $\overline{0}$. For the axioms see [F].

We restrict ourselves to theories T satisfying the following:

(a) T has a finite language, i.e. finitely many predicates, functions and sorts (we have of course at our disposal infinitely many variables $x_1^i, x_2^i, \ldots$ of every sort i)

(b) T has a recursively enumerable set of axioms

(c) T contains Robinson arithmetic, i.e. its language has the arithmetical sort and the arithmetical operation symbols and all the axioms of Robinson arithmetic are provable in T

(d) T is consistent.

The notion of interpretation is an obvious modification of the corresponding notion for one sorted systems.

The knowledge of Feferman's paper [F] is assumed. The predicates Tm(n) (number n is a term), Fm(n) (n is a formula), $Prf_T(n,d)$ (n is a formula, d is a sequence of formulas and it is a proof of n in T) are primitive recursive. The predicate $Pr_T(\varphi)$ (φ is provable in T) is recursively enumerable and the relation "(T,φ) is interpretable in (S,ψ)" is recursively enumerable whenever T is finitely axiomatizable, see Lemma 5 in [HH]. The definitions of Π_m and Σ_m formulas can be found e.g. in [G] and PR-formulas are defined in [F]. The sets Π_n and Σ_n are closed under conjunction, disjunction and bounded quantification; in addition, Π_n and Σ_n is closed under universal and existential quantification respectively. The negation of a Π_n formula is a Σ_n formula and vice versa. The set PR is included in Σ_1 and the conjunction, disjunction, negation and bounded quantification of PR-formulas is always P-equivalent to a PR-formula, where P is the Peano arithmetic. All formulas without unbounded quantifiers are PR.

The definition of numeration and binumeration are known (see [F]). A relation is primitie recursive iff it is binumerable by a PR-formula (in any theory). For every theory T, a relation is recursively enumerable iff it is numerable in T (by a Σ_1-formula). Every finite set $A = \{a_1,\dots,a_n\}$ has a natural PR-binumeration $x = \bar{a}_1 \vee \dots \vee x = \bar{a}_n$ which is denoted by [A].

We shall use the Feferman's formulas $\underset{\bullet}{\mathrm{Tm}}(x)$, $\underset{\bullet}{\mathrm{Fm}}_{\iota}(x)$, $\underset{\bullet}{\mathrm{St}}_{\iota}(x)$, $\underset{\bullet}{\mathrm{Prf}}_{\iota,\alpha}(x,y)$, $\underset{\bullet}{\mathrm{Pr}}_{\iota,\alpha}(x)$, $\underset{\bullet}{\mathrm{Con}}_{\alpha}$ which are read " x is a (formal) term of L ", " x is a formula", "x is a sentence", " y is a proof of the formula x", "the formula x is provable" and "the theory described by α is consistent". These formulas are formalizations of the related meta-mathematical notions. First four of them are PR and binumerate the sets of all terms, formulas etc., the formula $\underset{\bullet}{\mathrm{Pr}}_{\alpha}$ is Σ_1 and the formula $\underset{\bullet}{\mathrm{Con}}_{\alpha}$ is Π_1 whenever α is a Σ_1-formula.

Further we shall extensively use the Feferman's diagonal lemma: for every theory T and for every T-formula $\psi(x)$ there is a sentence φ such that $T \vdash \varphi \equiv \psi(\bar{\varphi})$.

3. The semilattice of degrees of interpretability and its basic properties.

In this section we shall give the basic definition and collect the most obvious facts. I include also some nontrivial results of general character.

3.1. Definition. Let T be a theory, let φ, ψ be sentences in the language of T. φ is said to be T-below ψ if the theory (T,φ) is interpretable in (T,ψ). This relation is denoted by $\varphi \trianglelefteq_T \psi$.

3.2. Lemma. (a) $\trianglelefteq_T$ is reflexive and transitive.

(b) If $T \vdash \psi \to \varphi$ then $\varphi \trianglelefteq_T \psi$.

3.3. Theorem. If both $\varphi \trianglelefteq_T \psi_1$ and $\varphi \trianglelefteq_T \psi_2$ then $\varphi \trianglelefteq_T \psi_1 \vee \psi_2$.

Proof. For simplicity, let us restrict ourselves to the case that the language of T consists only of one sort and of one binary predicate $\in$. We have two interpretations $*$ and $\square$ of (T,φ) in (T,ψ_1) and (T,ψ_2) respectively and we have to determine a new interpretation $\perp$ of (T,φ) in $(T,\psi_1 \vee \psi_2)$. Let $\delta_1(x)$ be the definition of the sort x^* in (T,ψ_1), $\delta_2(x)$ be the definition of the sort $x^\square$ in (T,ψ_2) (the ranges of interpretations $\square,*$). Let $E_1(x,y)$ and $E_2(x,y)$ be definitions of $\in^*$ and $\in^\square$ in (T,ψ_1) and (T,ψ_2) respectively. Let us define a new sort $x^\perp$ and new $\in$ (in $(T,\psi_1 \vee \psi_2)$) as follows:

$$\exists x^\perp (x = x^\perp) \equiv (\psi_1 \mathbin{\&} \delta_1(x)) \vee (\neg\psi_1 \mathbin{\&} \psi_2 \mathbin{\&} \delta_2(x))$$

$$x^\perp \in^\perp y^\perp \equiv (\psi_1 \mathbin{\&} E_1(x^\perp, y^\perp)) \vee (\neg\psi_1 \mathbin{\&} \psi_2 \mathbin{\&} E_2(x^\perp, y^\perp)).$$

Now it is easy to check that for any formula χ,

$$T, \psi_1 \vdash \chi^\perp \equiv \chi^*$$

$$T, \neg\psi_1 \mathbin{\&} \psi_2 \vdash \chi^\perp \equiv \chi^\square$$

and that $\perp$ is indeed an interpretation of (T,φ) in $(T,\psi_1 \vee \psi_2)$. $\dashv$

The last theorem shows how $\preceq_T$ is related to the Lindenbaum algebra of sentences (with contradiction as the greatest element).

3.4. <u>Definition</u>. (a) We say that a sentence φ <u>has the same degree</u> as ψ (notation: $\varphi \equiv_T \psi$) iff both $\varphi \preceq_T \psi$ and $\psi \preceq_T \varphi$.

(b) The <u>degree</u> $[\varphi]$ of a sentence φ is the set $\{\psi;\ \varphi \equiv_T \psi\}$. The set of all degrees is denoted by V_T.

(c) $[\varphi] \preceq_T [\psi]$ iff $\varphi \preceq_T \psi$.

3.5. <u>Lemma</u>. (a) $(V_T, \preceq_T)$ is a lowěr semilattice and $[\varphi] \wedge [\psi] = [\varphi \vee \psi]$.

(b) $1_T = \{\varphi;\ T \vdash \neg\varphi\}$ is its greatest element and $0_T = \{\varphi;\ (T,\varphi)$ is interpretable in $T\}$ is its least element.

This is a consequence of Theorem 3.3 abd the fact that if (T,ψ) is consistent and $\varphi \preceq_T \psi$ then (T,φ) is also consistent. The following lemma follows from Theorem 3.3 by elementary logic.

3.6. <u>Lemma</u>. (a) Let $\varphi \preceq_T \psi$. Then there is a sentence φ' such that $\varphi \equiv_T \varphi'$ and $T, \psi \vdash \varphi'$.

(b) If $\varphi \preceq_T \psi \mathbin{\&} \neg\varphi$ then $\varphi \preceq_T \psi$.

(c) If $\varphi \preceq_T \psi$ then $[\psi \rightarrow \varphi] = 0_T$.

Proof. (a) It suffices to choose $\varphi' \equiv \varphi \vee \psi$ and use 3.3 and 3.2 (b).

(b) Let $\varphi \preceq_T \psi \mathbin{\&} \neg\varphi$; furthermore, we have $\varphi \preceq_T \psi \mathbin{\&} \mathbin{\&} \varphi$. By 3.3, we have $\varphi \preceq_T (\psi \mathbin{\&} \neg\varphi) \vee (\psi \mathbin{\&} \varphi)$ and the last formula is equivalent to ψ.

(c) $\psi \rightarrow \varphi \preceq_T \varphi$ by 3.2 and $\varphi \preceq_T \psi$ by assumption. Obviously $\psi \rightarrow \varphi \preceq_T \neg\psi$, thus by 3.2 (a) and 3.3 we have $\psi \rightarrow \varphi \leq_T \psi \vee \neg \psi$ and the last sentence is of degree zero. $\dashv$

Observe that the converse of 3.6 (c) does not hold. Choose a refutable sentence for φ and let ψ be independent and such that $(T, \neg\psi)$ is interpretable in T. Then $[\psi] <_T [\varphi] = 1_T$ by 3.5 (b), moreover $T \vdash \psi \rightarrow \varphi \equiv \neg \psi$ and the sentence $\neg\psi$ is of degree zero by 3.5 (b).

The following two theorems were stated in the Feferman's paper [F]. Recall that we assume all theories to contain Robinson arithmetic.

3.7. Theorem. Let τ be arbitrary numeration of a theory T in some theory K. Then there is a finite subtheory F of Peano arithmetic such that T is interpretable in $K \cup F \{ \dot{Con}_\tau \}$.

3.8. Theorem. Let K be a theory and let T be interpretable in S. Then to every numeration σ of S in K there is a numeration τ of T in K such that

$$P \vdash \dot{Con}_\sigma \rightarrow \dot{Con}_\tau .$$

Moreover, τ is a Σ_1-formula whenever σ is. If T is finitely axiomatized we may choose $\tau \equiv [T]$.

3.9. Definition - lemma. Let $\tau(x)$ be an arithmetical formula. Then (τ, z) is an abbreviation for the formula $\tau(x) \vee x = z$. This formula has the following properties:

(a) $P \vdash \dot{St}(z) \,\&\, \dot{Fm}(y) \rightarrow (\dot{Pr}_\tau (z \dot{\rightarrow} y) \equiv \dot{Pr}_{(\tau, z)}(y))^{\mathcal{M}}$

(formalized deduction theorem)

(b) If τ (bi)numerates T in K then $(\tau, \overline{\varphi})$ (bi)numerates (T, φ) in K.

3.10. Definition. A theory T is Σ_1-sound iff each Σ_1-sentence provable in T is true (in the structure N of natural numbers).

3.11. Theorem. Let $T \supseteq P$ and let τ be a Σ_1-numeration of T in T. Then

(a) If φ is consistent (i.e. if (T,φ) is consistent) then $\varphi \prec_T Con_{(\tau,\bar{\varphi})}$.

(b) If T is Σ_1-sound and both φ and ψ is consistent then $Con_{(\tau,\bar{\varphi})}$ & $Con_{(\tau,\bar{\psi})}$ is a consistent upper bound of the set $\{\varphi, \psi\}$.

(c) $[Con_\tau] \neq 0_T$, $[\neg Con_\tau] = 0_T$.

(d) $\varphi \equiv_T (\varphi \,\&\, \neg Con_{(\tau,\bar{\varphi})})$.

(e) If T is finitely axiomatized and $\varphi \preceq_T \psi$ then $T \vdash Con_{[\tau,\psi]} \rightarrow Con_{[\tau,\varphi]}$.

Proof. (a) By 3.9 and 3.7 (T,φ) is interpretable in a certain theory $T \cup F \cup \{Con_{(\tau,\bar{\varphi})}$ which is equivalent to $(T, Con_{(\tau,\bar{\varphi})})$ because $F \subseteq P \subseteq T$. So we have $\varphi \preceq_T Con_{(\tau,\bar{\varphi})}$ and it remains to prove $Con_{(\tau,\bar{\varphi})} \not\preceq_T \varphi$. Assume $Con_{(\tau,\bar{\varphi})} \preceq_T \varphi$. Then $Con_{(\tau,\bar{\varphi})}$ is consistent because φ is, and by 3.8 (applied to $(\tau, \bar{\varphi})$) there is a Σ_1-numeration σ of $(T, Con_{(\tau,\bar{\varphi})})$ such that T proves $Con_{(\tau,\bar{\varphi})} \rightarrow Con_\sigma$. This is just the situation excluded by the second Gödel's theorem (see [F]): no consistent theory $S \supseteq P$ can prove the formula Con_σ whenever σ is a Σ_1-numeration of S in any $F \subseteq S$.

(b) By (a), $Con_{(\tau,\bar{\varphi})}$ & $Con_{(\tau,\bar{\psi})}$ is an upper bound of φ, ψ. We show that $(T, Con_{(\tau,\bar{\varphi})} \,\&\, Con_{(\tau,\bar{\psi})})$ is consistent. Assume the contrary. Then $T \vdash Pr_\tau(\overline{\neg\varphi}) \vee Pr_\tau(\overline{\neg\psi})$; since the last formula is Σ_1, we have $\models Pr_\tau(\overline{\neg\varphi}) \vee Pr\ (\overline{\neg\psi})$ by Σ_1-soundness. Then $\models Pr_\tau(\overline{\neg\varphi})$ or $\models Pr_\tau(\overline{\neg\psi})$, for example, let $\models Pr_\tau(\overline{\neg\varphi})$. Let $T_0 = \{\chi; \models \tau(\bar{\chi})\}$. Then $T_0 \vdash \neg\varphi$ and $T_0 \subseteq T$ (since each true Σ_1-sentence is provable in Q). Thus $T \vdash \neg\varphi$ which contradicts the assumption that (T,φ) is consistent.

(c) We know $T \vdash Con_{\tau} \equiv Con_{(\tau, \overline{\neg Con_{\tau}})}$ see [F]. By (a) and 3.2 (b) we have

$$\neg Con_{\tau} <_T Con_{(\tau, \overline{\neg Con_{\tau}})} \equiv_T Con_{\tau}$$

so indeed $0_T <_T [Con_{\tau}]$. Moreover from $\neg Con_{\tau} \not\leq_T Con_{\tau}$ and $\neg Con_{\tau} <_T \neg Con_{\tau}$ we get $[\neg Con_{\tau}] = 0_T$ using 3.3 and 3.5 (b).

(d) is a direct application of (c) to the theory (T, φ) and

(e) is immediate from 3.8. ⊣

Theorem 3.11 (b) shows that the greatest degree 1_T is not a l.u.b. of any two smaller degrees; hence there are no "upper exact pairs". The existence of lower exact pairs is an easy consequence of the next theorem 3.12. Another consequence of Theorem 3.12 is the existence of (infinitely many) incomparable elements in V_T. Theorems 3.12 and 3.13 were proved by R.G. Jeroslow in [J], the latter had to be slightly reworked for our purpose. Theorem 3.14 is my contribution to the subject.

Theorem 3.12 requires some preliminaries. Let B be the set of all propositional formulas built up from infinitely many atomic formulas $A_1, A_2, \ldots$ by Boolean operations $\vee$, & and $\neg$. The set B can be ordered by "$\varphi \leq_B \psi$ iff φ is a tautological consequence of ψ". By a natural factorization similar as in 3.4 B becomes an infinite countable atomless Boolean algebra. By a positive element of B we shall mean a (equivalence class determined by) propositional formula not containing the negation sign

3.12. <u>Theorem</u>. (a) If T is a consistent theory then the countable atomless Boolean algebra can be embedded into V_T.

More precisely, there is a one-one function f from B to V_T preserving greatest lower bounds. In particular, for $x, y \in B$ $x \leq_B y$ iff $f(x) \leq_T f(y)$.

(b) If, moreover, $T \supseteq P$ and if τ is a Σ_1-numeration of T in T then f maps all positive members T-below the formula Con_τ.

For the proof see [J].

3.13. Theorem. Let a theory T be essentially reflexive or finitely axiomatized. Then for every $a <_T b$ there is a $c \in V_T$ such that $a <_T c <_T b$.

Proof. By 3.6 (a) we can choose $\varphi_1 \in a$, $\varphi_2 \in b$ such that $T \vdash \varphi_2 \rightarrow \varphi_1$. There is a finitely axiomatized theory $F \subseteq T$ such that (F, φ_2) is not interpretable in (T, φ_1). Indeed, if T is finitely axiomatized, then we may choose $F = T$ and if T is essentially reflexive then F exists by Theorem 6.9 in [F] and by the reflexivity of (T, φ_1). Recall that the set of all $\langle \vartheta, \lambda \rangle$ such that (F, ϑ) is interpretable in (T, λ) is recursively enumerable. By the Feferman's diagonal lemma we can construct a self-referring sentence ψ saying "if $(F, \varphi_2 \vee (\varphi_1 \,\&\, \psi))$ is interpretable in (T, φ_1) then (F, φ_2) is interpretable in $(T, \varphi_2 \vee (\varphi_1 \,\&\, \psi))$". Then $\chi = \varphi_2 \vee (\varphi_1 \,\&\, \psi)$ is our required formula. Obviously $\varphi_1 \leq_T \chi \leq_T \varphi_2$, because $\varphi_2 \vdash \chi \vdash \varphi_1$. For the proof of $\chi \not\leq_T \varphi_1$ and $\varphi_2 \not\leq_T \chi$ see the analogous proof in [J] Theorem 3.2. Alternatively, if the reader has [J] not at his disposal, he may extract some information from the proof of our next theorem. $\dashv$

3.14. Theorem. Let T be essentially reflexive or finitely axiomatized. Let $a, b \in V_T$ be such that $a \neq 1_T$, $b \neq 0_T$. Then there is a $c \in V_T$ such that $c \not\leq_T a$ and $b \not\leq_T c$.

Proof. Let us choose $\gamma_1 \in a$, $\gamma_2 \in b$. By the same reason as in the proof of 3.13 there is a finitely axiomatized theory $F \subseteq T$ such that (F, γ_2) is not interpretable in T. Similarly as in 3.13, there are primitive recursive relations $R_1(\varphi, n)$ and $R_2(\varphi, n)$ such that

$R_1(\varphi, n) \vee R_2(\varphi, n)$ implies φ is a formula

$\exists n\, R_1(\varphi, n)$ iff (F, φ) is interpretable in (T, γ_1)

$\exists n\, R_2(\varphi, n)$ iff (F, γ_2) is interpretable in (T, φ)

Let the formulas $\alpha(x,y)$ and $\beta(x,y)$ binumerate R_1 and R_2 in Q. Let us define a diagonal sentence φ by

(1) $Q \vdash \varphi \equiv \forall y\, (\alpha(\overline{\varphi}, y) \rightarrow \exists z \leq y\ \beta(\overline{\varphi}, z))$

We shall prove that φ determines the required degree c. We have to prove $\varphi \not\leq_T \gamma_1$. We shall even prove that (F, φ) is not interpretable in (T, γ_1). Assume that it is interpretable by some interpretation $*$. Then

$$T, \gamma_1 \vdash \varphi^*$$

hence

(2) $T, \gamma_1 \vdash \forall y^* (\alpha^*(\overline{\varphi}^*, y^*) \rightarrow \exists z^* \leq^* y^*\ \beta^*(\overline{\varphi}^*, z^*)$

and, furthermore, $R_1(\varphi, p)$ for some p. Let m be the least such p. Since α binumerates R , we have

$$Q \vdash \alpha(\overline{\varphi}, \overline{m})\ \&\ \forall u < \overline{m}\ \neg\alpha(\overline{\varphi}, u)$$

Since interpretations preserve provability, we have

(3) $T, \gamma_1 \vdash \alpha^*(\overline{\varphi}^*, \overline{m}^*)$

From (2) and (3) we obtain

$$T, \gamma_1 \vdash (\exists z \leq \overline{m}\ \beta(\overline{\varphi}, z))^*$$

We have proved that the sentence $\exists z \leq \overline{m}\, \beta(\overline{\varphi}, z)$ is consistent with the theory (F, φ), hence it is consistent with Q. But such a simple sentence is decided in Q (according to whether $\exists n \leq m\ R_2(\varphi, n)$ or not). So it is decided positively, hence

(4) $\exists n \leq m\ R_2(\varphi, n)$ and

(5) $Q \vdash \exists x \leq \overline{m}\ \beta(\overline{\varphi}, x)$.

By (4), (F, γ_2) is interpretable in (T, φ), but from (5) and (1) we can prove φ in Q. This is a contradiction because F was such that (F, γ_2) is not interpretable in T. So we have proved that (F, φ) is not interpretable in (T, γ_1), hence $R_1(\varphi, n)$ does not hold for any n, hence for each n

(6) $Q \vdash \neg\alpha(\overline{\varphi}, \overline{n})$.

It remains to prove that $\gamma_2 \not\leq_T \varphi$. We shall again show that even (F, γ_2) is not interpretable in (T, φ). If it were interpretable, i.e. if $R_2(\varphi, m)$ for some m, then for this m,

(7) $Q \vdash \beta(\overline{\varphi}, \overline{m})$.

From (6) and (7) we can prove φ in Q, which is impossible by the same reasons as above. $\dashv$

If we choose a = b in Theorem 3.14 we see that to every degree different from 0_T and 1_T there is an incomparable degree.

4. <u>The lattice of degrees of interpretability given by an essentially reflexive theory</u>. All results of this section concern only essentially reflexive theories. Analogous problems e.g. for finitely axiomatizable theories remain open. As is known, both Peano arithmetic and Zermelo-Fraenkel set theory is essentially reflexive.

4.1. <u>Definition</u>. We say that a theory T is <u>reflexive</u> if for every n $T \vdash Con_{[T \restriction n]}$. T is <u>essentially reflexive</u> if every extension of T with the same language is reflexive.

The following lemma utilizes the fact that if $\tau(x)$ is a binumeration of a set T in K then for every n

$K \vdash \tau(x) \mathbin{\&} x \leq \bar{n} \equiv [T \restriction n](x)$,

see [F], Lemma 4.14.

4.2. Lemma. Let $T \supseteq P$ be a recursively axiomatized theory and let τ be arbitrary binumeration of T in T. Then

(a) T is reflexive iff

$T \vdash \mathrm{Con}_{T \restriction \bar{n}}$ for each n.

(b) T is essentially reflexive iff for every T-sentence φ and for each n,

$T, \varphi \vdash \mathrm{Con}_{(\tau, \bar{\varphi}) \restriction \bar{n}}$.

In the remaining part of this paper we assume that $T \supseteq P$, T is essentially reflexive and recursively axiomatized and τ is a binumeration of T in T.

4.3. Lemma. For arbitrary sentences φ, ψ $\varphi \leq_T \psi$ iff $T, \psi \vdash \mathrm{Con}_{(\tau, \bar{\varphi}) \restriction \bar{n}}$ for each n.

This is a form of Orey's arithmetical compactness theorem, see [F] and [HH].

4.4. Theorem. Every pair of degrees in V_T has a l.u.b., i.e. V_T is a lattice.

Proof. Let a, b be a given pair of degrees and choose $\varphi_1 \in$ a and $\varphi_2 \in$ b. By the diagonal lemma there is a sentence ψ such that

(1) $T \vdash \psi \equiv \forall y\, (\mathrm{Con}_{(\tau, \bar{\psi}) \restriction y} \rightarrow (\mathrm{Con}_{(\tau, \bar{\varphi}_1) \restriction y} \mathbin{\&} \mathrm{Con}_{(\tau, \bar{\varphi}_2) \restriction y}))$

We shall prove that ψ determines the required degree, i.e. that $[\psi] = \sup\{a, b\}$. By the essential reflexivity of T (see 4.2 (b)) we have

(2) $T, \psi \vdash \mathrm{Con}_{(\tau, \bar{\psi}) \restriction \bar{n}}$ for each n .

The formula $\mathrm{Con}_{(\tau, \bar{\psi}) \restriction \bar{n}}$ is the antecedent in the formula ψ ;

hence from (1) and (2) we have for each n

$$T, \psi \vdash Con_{(\tau,\bar{\varphi}_1)\restriction \bar{n}} \ \& \ Con_{(\tau,\bar{\varphi}_2)\restriction \bar{n}} \ .$$

Now $\varphi_1 \leq_T \psi$ and $\varphi_2 \leq_T \psi$ by 4.3, hence ψ is an upper bound. Let χ be arbitrary upper bound. By 3.6 (b) it suffices to prove $\psi \leq_T \chi \ \& \ \neg\psi$. Let n be arbitrary. As χ is an upper bound we have (by 4.3)

(3) $T, \chi \vdash Con_{(\tau,\bar{\varphi}_1)\restriction \bar{n}} \ \& \ Con_{(\tau,\bar{\varphi}_2)\restriction \bar{n}}$.

Moreover, by (1),

(4) $T, \neg\psi \vdash \exists y\, (Con_{(\tau,\bar{\psi})\restriction y} \ \& \ \neg(Con_{(\tau,\bar{\varphi}_1)\restriction y} \ \& \ Con_{(\tau,\bar{\varphi}_2)\restriction y}))$.

From (3) and (4) we can prove

$$T, \chi \ \& \ \neg\psi \vdash \exists y\, (\bar{n} < y \ \& \ Con_{(\tau,\bar{\psi})\restriction y})$$

hence

$$T, \chi \ \& \ \neg\psi \vdash Con_{(\tau,\bar{\psi})\restriction \bar{n}}$$

and we get $\psi \leq_T \chi \ \& \ \neg\psi$ by 4.3. This completes the proof.⊣

From 4.3 we can prove that $[\varphi] = 0_T$ iff for every n T proves $Con_{(\tau,\bar{\varphi})\restriction \bar{n}}$. This will be used in the proof of the following lemma.

4.5. Lemma. For every theory T, there is a sentence φ such that $[\varphi] = [\neg\varphi] = 0_T$.

Proof. Let neg (x,z) be a formula that functionally binumerates negation in Q, i.e. for arbitrary formula φ ,

(1) $Q \vdash neg(\bar{\varphi}, x) \equiv x = \overline{\neg\varphi}$.

Let us define a diagonal sentence φ by

$$T \vdash \varphi \equiv \forall y\, (Con_{(\tau,\bar{\varphi})\restriction y} \rightarrow \forall x\, (neg(\bar{\varphi}, x) \rightarrow Con_{(\tau,x)\restriction y}))\, .$$

By (1) we have

(2) $T \vdash \varphi \equiv \forall y\, (Con_{(\tau,\bar{\varphi})\restriction y} \rightarrow Con_{(\tau,\overline{\neg\varphi})\restriction y})$.

By the reflexivity of the theory (T, φ) we have

(3) $T, \varphi \vdash Con_{(\tau,\bar{\varphi})\restriction \bar{n}}$ for each n .

From (2) we get

(4) $T, \varphi \vdash \mathrm{Con}_{(\tau, \overline{\neg\varphi})} \restriction \bar{m}$ for each n.

By the reflexivity of $(T, \neg\varphi)$ we have

(5) $T, \neg\varphi \vdash \mathrm{Con}_{(\tau, \overline{\neg\varphi})} \restriction \bar{m}$.

By (4) and (5)

$$T \vdash \mathrm{Con}_{(\tau, \overline{\neg\varphi})} \restriction \bar{m}$$

and indeed $[\neg\varphi] = 0_T$. Furthermore, by (2) we have

(6) $T, \neg\varphi \vdash \exists y\, (\mathrm{Con}_{(\tau, \bar{\varphi})} \restriction y \;\&\; \neg\, \mathrm{Con}_{(\tau, \overline{\neg\varphi})} \restriction y)$.

From (5) and (6) (using the fact that $x_1 < x_2 \;\&\; \mathrm{Con}_{\sigma} \restriction x_2 \rightarrow \mathrm{Con}_{\sigma} \restriction x_1$ we get

(7) $T, \neg\varphi \vdash \mathrm{Con}_{(\tau, \bar{\varphi})} \restriction \bar{m}$.

And again by (3) and (7)

$$T \vdash \mathrm{Con}_{(\tau, \bar{\varphi})} \restriction \bar{m} \quad \text{for each n,}$$

i.e. $[\varphi] = 0_T$. ⊣

If we apply Lemma 4.5 to the theory (T, ψ) we get the following

Corollary. In every degree $[\psi]$ there are mutually contradictory sentences of the form $\psi \;\&\; \varphi$ and $\psi \;\&\; \neg\varphi$.

4.6. Lemma. For arbitrary sentences φ, ψ

$$T \vdash \mathrm{Con}_{(\tau, \overline{\varphi \vee \psi})} \equiv \mathrm{Con}_{(\tau, \bar{\varphi})} \vee \mathrm{Con}_{(\tau, \bar{\psi})} .$$

Proof. We know that for arbitrary sentences χ, χ_1, χ_2, $P \vdash \mathrm{Pr}_{\tau}(\overline{\neg\chi}) \equiv \neg\, \mathrm{Con}_{(\tau, \bar{\chi})}$ and $P \vdash \mathrm{Pr}_{\tau}(\overline{\chi_1 \& \chi_2}) \equiv \mathrm{Pr}_{\tau}(\bar{\chi}_1) \,\&\, \mathrm{Pr}_{\tau}(\bar{\chi}_2)$.

Lemma 4.6 is an easy consequence of these facts. ⊣

Having Theorem 4.4 in mind we can use in V_T the lattice operations $\vee$ (least upper bound, join) and $\wedge$ (meet). Recall that if $a, b \in V_T$ and $\varphi \in a$, $\psi \in b$ then $\varphi \vee \psi \in a \wedge b$ (see 3.5 (a)) and $[\varphi \;\&\; \psi] \geq_T a \vee b$ by 3.2 (b).

4.7. Theorem. The lattice V_T is distributive.

Proof. It suffices to prove that $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ because the dual distributivity law follows from this one.

Moreover, the inequality $\leq$ holds automatically in every lattice. Let us prove $\geq$. Choose $\varphi_1 \in a$, $\varphi_2 \in b$, $\varphi_3 \in c$ and define diagonal formulas

$$\psi_1 \equiv \forall y\,(Con_{(\tau,\overline{\psi_1})}\restriction y \to (Con_{(\tau,\overline{\varphi_1})}\restriction y \;\&\; Con_{(\tau,\overline{\varphi_2})}\restriction y))$$

$$\psi_2 \equiv \forall y\,(Con_{(\tau,\overline{\psi_2})}\restriction y \to (Con_{(\tau,\overline{\varphi_1})}\restriction y \;\&\; Con_{(\tau,\overline{\varphi_3})}\restriction y))$$

$$\chi \equiv \forall y\,(Con_{(\tau,\overline{\chi})}\restriction y \to (Con_{(\tau,\overline{\varphi_1})}\restriction y \;\&\; Con_{(\tau,\overline{\varphi_2 \vee \varphi_3})}\restriction y)).$$

By 3.5 (a) and 4.4 we have $\varphi_2 \vee \varphi_3 \in b \wedge c$, $\psi_1 \in a \vee b$, $\psi_2 \in a \vee c$, $\chi \in a \vee (b \wedge c)$ and $\psi_1 \vee \psi_2 \in (a \vee b) \wedge (a \vee c)$. We have to prove that

$$\psi_1 \vee \psi_2 \leq_T \chi \ .$$

By 3.6 (b) it suffices to prove

$$\psi_1 \vee \psi_2 \leq_T \chi \;\&\; \neg\psi_1 \;\&\; \neg\psi_2 \ .$$

By 4.3 it suffices to prove that, for each n,

$$T, \chi, \neg\psi_1, \neg\psi_2 \vdash Con_{(\tau,\overline{\psi_1 \vee \psi_2})}\restriction \overline{n} \ .$$

We shall prove

$$T, \chi, \neg\psi_1, \neg\psi_2 \vdash Con_{(\tau,\overline{\psi_1})}\restriction \overline{n} \vee Con_{(\tau,\overline{\psi_2})}\restriction \overline{n}$$

and use Lemma 4.6. Let n be given. By the reflexivity of $(T,\chi$ we have

$$T, \chi \vdash Con_{(\tau,\overline{\chi})}\restriction \overline{n} \ .$$

By this and by the definition of χ we have (using Lemma 4.6)

$$T, \chi \vdash Con_{(\tau,\overline{\varphi_1})}\restriction \overline{n} \;\&\; (Con_{(\tau,\overline{\varphi_2})}\restriction \overline{n} \vee Con_{(\tau,\overline{\varphi_3})})$$

hence

$$T, \chi \vdash (Con_{(\tau,\overline{\varphi_1})}\restriction \overline{n} \;\&\; Con_{(\tau,\overline{\varphi_2})}\restriction \overline{n}) \vee (Con_{(\tau,\overline{\varphi_1})}\restriction \overline{n} \vee Con_{(\tau,\overline{\varphi_3})}\restriction \overline{n})$$

From the definition of ψ_1, ψ_2 we get

$$T, \neg\psi_1, Con_{(\tau,\overline{\varphi_1})}\restriction \overline{n} \;\&\; Con_{(\tau,\overline{\varphi_2})}\restriction \overline{n} \vdash Con_{(\tau,\overline{\psi_1})}\restriction \overline{n}$$

$$T, \neg\psi_2, Con_{(\tau,\overline{\varphi_1})}\restriction \overline{n} \;\&\; Con_{(\tau,\overline{\varphi_3})}\restriction \overline{n} \vdash Con_{(\tau,\overline{\psi_2})}\restriction \overline{n} \ .$$

Putting this together we indeed have

$$T, \chi, \neg\psi_1, \neg\psi_2 \vdash Con_{(\tau,\overline{\psi_1})}\restriction \overline{n} \vee Con_{(\tau,\overline{\psi_2})}\restriction \overline{n} \ . \dashv$$

5. <u>Simplest sentences in a degree</u>. The sentence ψ produced in the theorem 4.4 was an arithmetical sentence. If we take in the theorem 4.4 the same sentence for φ_1 and φ_2 we see that in every degree in V_T there is an arithmetical and syntactically simple sentence. This contrasts with the fact that in the Lindenbaum algebra e.g. of ZF there are degrees of arbitrarily high arithmetical complexities and that there are also non-arithmetical degrees, i.e. there are set sentences non-equivalent to any arithmetical sentence. In this section we shall further try to determine for some concrete formulas their position in the lattice V_T .

5.1. <u>Theorem</u>. If $T \supseteq P$ is essentially reflexive and recursively axiomatized then

(a) In every degree in V_T there are Π_2 sentences.

(b) In every degree in V_T there are Σ_2 sentences.

Proof. (a) Let a degree $[\varphi]$ be given and let τ be a Σ_1-binumeration of T in T. Let us define a diagonal sentence ψ by

$$T \vdash \psi \equiv \forall y\, (\mathrm{Con}_{(\tau,\overline{\psi})}\restriction y \rightarrow \mathrm{Con}_{(\tau,\overline{\varphi})}\restriction y+1).$$

The formula ψ is Π_2 and the proof that $\psi \equiv_T \varphi$ is analogous to the proof of the theorem 4.4.

(b) Let φ, ψ be as above and let us take a sentence

$$\sigma \equiv \exists y\, (\mathrm{Con}_{(\tau,\overline{\varphi})}\restriction y \;\&\; \neg \mathrm{Con}_{(\tau,\overline{\varphi})}\restriction y).$$

Obviously σ is a Σ_2 sentence and $T \vdash \sigma \rightarrow \psi$. So we have to prove $\sigma \leq_T \psi$. By 3.11 (d) $\psi \geq_T \psi \;\&\; \mathrm{Con}_{(\tau,\overline{\psi})}$. Furthermore, we have

$$T \vdash \neg \mathrm{Con}_{(\tau,\psi)} \rightarrow \exists y\, \neg \mathrm{Con}_{(\tau,\varphi)}\restriction y \;,$$

$$T, \psi \;\&\; \exists y\, \neg \mathrm{Con}_{(\tau,\varphi)}\restriction y \vdash \sigma$$

and hence $\sigma \leq_T \psi$. $\dashv$

5.2. <u>Theorem</u>. Let T and S be theories containing Peano arithmetic, let the induction for all T-formulas be provable in T and let T enable the coding of finite n-tuples of T-objects. Then to every interpretation $*$ of S in T there is a T-formula $\rho(x,x^*)$ such that

(a) $T \vdash \forall x\, \exists! x^*\, \rho(x,x^*)$

(b) $T \vdash \rho(x_1,x^*)\ \&\ \rho(x_2,x^*) \longrightarrow x_1 = x_2$

(c) $T \vdash \rho(x,x^*)\ \&\ y^* \leq^* x^* \longrightarrow \exists y\, \rho(y,y^*)$

(d) for every arithmetical Σ_1-formula $\varphi(x,\dots)$

$T \vdash \rho(x,x^*)\ \&\ \dots \rightarrow (\varphi(x,\dots) \rightarrow \varphi^*(x^*,\dots))$

For the proof see e.g. [H].

If we apply Theorem 5.2 to a Σ_1-sentence φ we get $T \vdash \varphi \longrightarrow \varphi^*$. The dual statement for Π_1-sentence π claims $T \vdash \pi^* \longrightarrow \pi$. This fact has important consequences.

5.3. <u>Corollary</u>. Let T have the properties required in Theorem 5.2. If ψ is a T-sentence and φ is a Π_1-sentence then $\varphi \trianglelefteq_T \psi$ implies $T, \psi \vdash \varphi$.

5.4. <u>Corollary</u>. Let T have the properties from Theorem 5.2 and let φ_1, φ_2 be Π_1-sentences. Then

$[\varphi_1\ \&\ \varphi_2] = [\varphi_1] \vee [\varphi_2]$.

The following definition 5.5 and lemma 5.6 show the connection that interpretability has to partially conservative sentences (studied by D. Guaspari).

5.5. <u>Definition</u> [G]. A sentence φ is said to be Π_1-<u>conservative</u> over T if for every Π_1-sentence π, $T, \varphi \vdash \pi$ implies $T \vdash \pi$.

5.6. <u>Lemma</u> [G]. Let T be reflexive and satisfy the assumptions of 5.2. Then φ is Π_1-conservative iff $[\varphi] = 0_T$.

Proof. T is essentialy reflexive hence $T, \varphi \vdash Con_{(T,\varphi)} \restriction \bar{n}$

for each n. The sentence $\dot{C}on$ ... is Π_1 hence by the Π_1-conservativity of φ we have $T \vdash \dot{C}on_{(\tau,\overline{\varphi})\restriction \overline{n}}$ and by Lemma 4.3 indeed $[\varphi] = 0_T$.

Assume conversely $[\varphi] = 0_T$. Let $T, \varphi \vdash \pi$ and $\pi \in \Pi_1$. We have to prove $T \vdash \pi$. Let $*$ be an interpretation of (T, φ) in T. Then $T, \varphi \vdash \pi$ implies $T \vdash \pi^*$. By Theorem 5.2 or Corollary 5.3 we have $T \vdash \pi$. $\dashv$

5.7. Rosser's sentences. In the rest of the paper assume that T is P or ZF and τ is a PR-binumeration of T in T. Let us define sentences ρ and σ (the former using the diagonal lemma):

$$\rho \equiv \forall y\, (\dot{P}rf_\tau(\overline{\rho}, y) \to \exists x \leq y\; \dot{P}rf_\tau(\overline{\neg\rho}, x))$$

$$\sigma \equiv \forall x\, (\dot{P}rf_\tau(\neg\overline{\rho}, x) \to \exists y < x\; \dot{P}rf_\tau(\overline{\rho}, y)).$$

To be more exact ρ is defined using the formula neg (x,z) similarly as in 4.5. The sentences ρ and σ have the following properties

(a) $[\rho] = [\neg\sigma] \neq 0_T$, $[\neg\rho] = [\sigma] \neq 0_T$

(b) $[0_T] = [\rho] \wedge [\sigma]$

(c) $[\dot{C}on_\tau] = [\rho] \vee [\sigma]$

(d) $[\rho] < [\dot{C}on_\tau]$, $[\sigma] < [\dot{C}on_\tau]$

Proof. It is well known that

(i) The sentence ρ is independent on T. The proof can be formalized in $(T, \dot{C}on_\tau)$ and since $T \vdash \neg\rho \to \sigma$ we have

(ii) $T \vdash \dot{C}on_\tau \to \dot{C}on_{(\tau,\overline{\rho})}$, $T \vdash \dot{C}on_\tau \to \dot{C}on_{(\tau,\overline{\sigma})}$.

(iii) $T \vdash \dot{C}on_\tau \equiv \rho \,\&\, \sigma$. By Corollary 5.4 we have

$[\dot{C}on_\tau] = [\rho] \vee [\sigma]$.

(iv) $T \nvdash \rho \to \dot{C}on_\tau$, $T \nvdash \sigma \to \dot{C}on_\tau$;

otherwise we would reach a contradiction with the second Gödel's theorem (using (ii)).

(v) $T \nvdash \sigma$

otherwise we would have $T \vdash \rho \equiv Con_\tau$ (by (iii)) which contradicts (iv).

(vi) $[\rho] \neq 0_T$, $[\sigma] \neq 0_T$

since ρ and σ are unprovable Π_1-sentences, see 5.3.

(vii) $\neg\rho \leq_T \sigma$, $\neg\sigma \leq_T \rho$

since by 3.11 (d), we have $\sigma \,\&\, \neg Con_{(\tau,\overline{\sigma})} \leq_T \sigma$ and, by (ii), we have $\sigma \,\&\, \neg Con_\tau \leq_T \sigma \,\&\, \neg Con_{(\tau,\overline{\sigma})}$.

In $T, \sigma \,\&\, \neg Con_\tau$ implies $\neg\rho$ by (iii). The proof of $\neg\sigma \leq_T \rho$ is similar. Now it is clear that $[\rho] = [\neg\sigma]$ and $[\neg\rho] = [\sigma]$ since $T \vdash \neg\rho \rightarrow \sigma$.

(viii) The property (d) follows from (a),(b),(c). This completes the proof. ⊣

Let us point out that 5.7 (a) shows that a degree different from 0_T, 1_T can contain both Π_1 and Σ_1 sentence.

5.8. The negation of the Rosser's sentence informally says "there is a proof of my negation such that no my proof is less or equal". Let us slightly change this sentence and define

$$\sigma \equiv \exists x\, (Prf_\tau(\overline{\neg\sigma}, x) \,\&\, \forall y \leq x\, \neg Prf_\tau(\overline{\neg Con_\tau}, y)) .$$

This sentence has the following properties

(a) $\sigma \leq_T Con_{(\tau, \overline{Con_\tau})}$

(b) $\sigma \nleq_T Con_\tau$.

Proof. (i) If $T \vdash \neg\sigma$ then $T \vdash \neg Con_\tau$. By the formalization of this fact we have

(ii) $T \vdash Con_{(\tau, \overline{Con_\tau})} \rightarrow Con_{(\tau, \overline{\sigma})}$,

and by 3.11 (a) we have $\sigma \leq_T Con_{(\tau, \overline{Con_\tau})}$.

(iii) $T \vdash Con_\tau \rightarrow \neg\sigma$

since by Theorem 5.5 in [F] we have $T, \sigma \vdash Pr_\tau(\overline{\sigma})$ and by the definition of σ we have $T, \sigma \vdash \Pr_{\tau}(\overline{\neg\sigma})$, which implies $T, \sigma \vdash \neg Con_{\tau}$.

(iv) $\sigma \not\leq_T Con_{\tau}$.

Assume $\sigma \leq_T Con_{\tau}$. Let $*$ be an interpretation of (T, σ) in (T, Con_{τ}). The theory (T, Con_{τ}) is consistent and it remains consistent after adding the axiom of formal inconsistency. Thus it will be sufficient to find a contradiction in the theory $(T, Con_{\tau}, Pr_{\tau}(\overline{\neg Con_{\tau}}))$. Let us work in the last theory informally. Let y be least such that $Prf_{\tau}(\overline{\neg Con_{\tau}}, y)$. The formula Prf ... is PR, hence it is Σ_1 and by Theorem 5.2 we have $Prf_{\tau}^{*}(\overline{\neg Con_{\tau}}^{*}, y^{*})$, where y^{*} is such that $\varphi(y, y^{*})$. We know that σ^{*}, hence

$$\exists x^{*}(Prf_{\tau}^{*}(\overline{\neg\sigma}^{*}, x^{*}) \& \forall y^{*} \leq^{*} x^{*} \neg Prf_{\tau}^{*}(\overline{\neg Con}_{\tau}^{*}, y^{*})).$$

Every such x^{*} must be $<^{*} y^{*}$ and by 5.2 (c) there is an x such that $\varphi(x, x^{*})$. By 5.2 (d) $Prf_{\tau}^{*}(\overline{\neg\sigma}^{*}, x^{*})$ implies $Prf_{\tau}(\overline{\neg\sigma}, x)$, since Prf ... is a Π_1-formula in P. By (iii) there is a $y' \leq x$ such that $Prf_{\tau}(\overline{\neg Con_{\tau}}, y')$ and for this y' we have $y' < y$. But y was least such that $Prf_{\tau}(\overline{\neg Con_{\tau}}, y)$. This is a contradiction. $\dashv$

5.9. A truth definition for a theory T is a T-formula $\psi(x)$ such that for every T-sentence φ $T \vdash \varphi \equiv \psi(\overline{\varphi})$. As is known, no consistent theory has such a truth definition. On the other hand, the Peano arithmetic has partial truth definitions. More precisely, for every n there is a Σ_n-formula $Tr_n(x)$ such that for every Σ_n-sentence φ $P \vdash \varphi \equiv Tr_n(\overline{\varphi})$. Let us define the sentences ω_n using the formulas $Tr_n(x)$ and the natural binumeration π of axioms of the Peano arithmetic:

$$\omega_n \equiv \forall x(St_{\Sigma_n}(x) \& Tr_n(x) \rightarrow Con_{(\pi, x)})$$

("every Σ_n-true Σ_n-sentence is consistent with π").

These sentences have the following properties:

(a) $\omega_n \in \Pi_n$

(b) If σ is a Σ_n-sentence then

$P, \omega_n, \sigma \vdash Con_{(\pi, \overline{\sigma})}$

(c) If σ is a Σ_n-sentence then

$P, \omega_n \vdash \sigma$ implies $P, \omega_n \vdash Con_{(\pi, \overline{\sigma})}$.

(d) There is no Σ_n-sentence σ such that $P, \sigma \vdash \omega_n$.

(e) $P \vdash \omega_1 \equiv Con_{\pi}$.

(f) Each ω_n is consistent with P.

Proof. (a) is obvious, (b) follows from the definition and from the fact that $P \vdash \sigma \equiv Tr_n(\overline{\sigma})$. (d) Assume $P, \sigma \vdash \omega_n$. Then, by (b), $P, \sigma \vdash Con_{(\pi, \overline{\sigma})}$ which contradicts the second Gödel's theorem. (e) The interesting direction is $Con_{\pi} \rightarrow \omega_1$. It is a consequence of the fact that $P \vdash St_{\Sigma_1}(x) \& Tr_1(x) \rightarrow \rightarrow Pr_{\pi}(x)$ which is a generalization of the Feferman's theorem 5.5 and is proved by induction on complexity of formulas (in P). (f) It is sufficient to prove $ZF \vdash \omega_n$ for each n. Let us work in ZF informally. Let N be the structure of natural numbers. N is known to be a model of the set $\{x;\ \pi(x)\}$. By induction on complexity of formulas we can prove (all in ZF) that $St_{\Sigma_n}(x) \rightarrow (Tr_n(x) \equiv N \models x)$. We see that every Σ_n-true Σ_n-sentence x holds in N, hence $N \models (\pi, x)$, hence $Con_{(\pi, n)}$. ⊣

We see that every ω_n is a Π_n-sentence which is not Σ_n in P. The ω_1 and ω_2 have analogous properties also in V_P:

5.10. <u>Theorem</u>. (a) There is no Σ_1-sentence σ such that $\omega_1 \leq_P \sigma$. In particular, the degree $[Con_{\pi}]$ contains no Σ_1-sentence.

(b) The degree $[\omega_2]$ contains no Π_1-sentence.

Proof. These are consequences of 5.3 and 5.9 (d). In (a) use the fact that $\omega_1 \in \Pi_1$ and in (b) that $\Pi_1 \subseteq \Sigma_2$. ⊣

Now our picture is almost complete. Every degree contains Π_2 and Σ_2 -sentences. By 5.10 (b) not every degree contains Π_1 -sentences, but by 3.11 (a),(b), Π_1 -setences are cofinal in V_T. On the other hand Σ_1 -sentences are not cofinal in V_P (by 5.10 (a)) and this can be generalized also for V_{ZF}. By 5.8 it is not true that every Σ_1 -sentence is T-below the sentence $\dot{C}on_\tau$. A degree containing a Π_1 -sentence may contain a Σ_1 -sentence (see 5.7) or may not (see 5.10 (a)).

6. Problems. The only question concerning simple formulas in a degree reads: must a degree containing a Σ_1 -sentence contain also a Π_1 -sentence?

We close this paper by collecting some further open problems. The most important question we have left open reads: Is V_T a lattice for finitely axiomatizable T ? In particular, is V_{GB} a lattice? As a consequence of the proof of the theorem 3.4.1 in [VHZ] we have the following fact: If $\xi(x)$ is the natural binumeration of ZF and ZF $\vdash \psi \rightarrow \forall x\, (\dot{C}on_{(\xi,\bar{\psi})\restriction x} \rightarrow$ $\rightarrow \dot{C}on_{(\xi,\bar{\varphi})\restriction x})$ then $\varphi \leq_{GB} \psi$. It follows that the sentence produced in 4.4 is an upper bound also in V_{GB} . Other open problems are: is every $c \in V_T$, $c \neq 1_T$ a l.u.b. of two smaller degrees?, is every $a \neq 0_T, 1_T$ one member of a lower exact pair?

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