Karel Svoboda

On surfaces in E^3 with constant Gauss curvature

Commentationes Mathematicae Universitatis Carolinae, Vol. 19 (1978), No. 4, 755--761

Persistent URL: http://dml.cz/dmlcz/105890

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

19,4 (1978)

ON SURFACES IN E^3 WITH CONSTANT GAUSS CURVATURE

Karel SVOBODA, Brno

Abstract: A global characterization of surfaces in E^3 with constant Gauss curvature.

Key words: Surface, Gauss and mean curvatures, integral formula.

AMS: 53C45

H. Fath el Bab introduced in [1] the conditions implying H = const on a surface M in E^3. In what follows, we apply the method used in [1] and prove an analogous theorem for the Gauss curvature K of M.

Let M be a surface in the 3-dimensional Euclidean space E^3 and ∂M its boundary. On M, consider fields of orthonormal frames $\{M; v_1, v_2, v_3\}$ with $v_1, v_2 \in T(M)$, $T(M)$ being the tangent bundle of M. Then we have

$$(1) \qquad \begin{aligned} dM &= \omega^1 v_1 + \omega^2 v_2, \\ dv_1 &= \qquad\qquad \omega_1^2 v_2 + \omega_1^3 v_3, \\ dv_2 &= -\omega_1^2 v_1 \qquad\qquad + \omega_2^3 v_3, \\ dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 \end{aligned}$$

and (see [2], p. 8)

(2) $\omega_1^3 = a\omega^1 + b\omega^2, \ \omega_2^3 = b\omega^1 + c\omega^2;$

(3) $$\begin{aligned} \Delta a &= da - 2b\omega_1^2 = \alpha\omega^1 + \beta\omega^2, \\ \Delta b &= db + (a-c)\omega_1^2 = \beta\omega^1 + \gamma\omega^2, \\ \Delta c &= dc + 2b\omega_1^2 = \gamma\omega^1 + \delta\omega^2; \end{aligned}$$

(4) $$\begin{aligned} \Delta\alpha &= d\alpha - 3\beta\omega_1^2 = A\omega^1 + (B-bK)\omega^2, \\ \Delta\beta &= d\beta + (\alpha - 2\gamma)\omega_1^2 = (B+bK)\omega^1 + (C+aK)\omega^2, \\ \Delta\gamma &= d\gamma + (2\beta - \delta)\omega_1^2 = (C+cK)\omega^1 + (D+bK)\omega^2, \\ \Delta\delta &= d\delta + 3\gamma\omega_1^2 = (D-bK)\omega^1 + E\omega^2 \end{aligned}$$

where

(5) $$K = ac - b^2$$

is the Gauss curvature of M.

The covariant derivatives K_i, K_{ij} $(i,j = 1,2)$ of K, defined by

(6) $$dK = K_1\omega^1 + K_2\omega^2,$$

$$dK_1 - K_2\omega_1^2 = K_{11}\omega^1 + K_{12}\omega^2, \ dK_2 + K_1\omega_1^2 = K_{12}\omega^1 + + K_{22}\omega^2$$

are given, according to (3) and (4), by

(7) $$K_1 = a\gamma - 2b\beta + c\alpha, \ K_2 = a\delta - 2b\gamma + c\beta;$$

(8) $$\begin{aligned} K_{11} &= aC - 2bB + cA + 2(\alpha\gamma - \beta^2) + (ac - 2b^2)K, \\ K_{12} &= aD - 2bC + cB + (\alpha\delta - \beta\gamma) - b(a+c)K, \\ K_{22} &= aE - 2bD + cC + 2(\beta\delta - \gamma^2) + (ac - 2b^2)K. \end{aligned}$$

Now, we formulate the

Theorem 1. *Let M be a surface in E^3 with $K > 0$ and ∂M its boundary. Let $V_1, V_2 \in T(M)$ be orthonormal vector fields on M such that*

$$(9) \qquad V_1 K = 0, \; V_2 K = 0$$

on ∂M and

$$(10) \qquad V_1 V_1 K = 0, \; V_2 K = 0$$

on M. Then $K = \text{const}$ on M.

Proof. Consider a 1-form

$$\varphi = R_1 \omega^1 + R_2 \omega^2$$

on M. The covariant derivatives of R_i $(i = 1,2)$ being defined by

$$dR_1 - R_2 \omega_1^2 = R_{11} \omega^1 + R_{12} \omega^2,$$

$$dR_2 + R_1 \omega_1^2 = R_{21} \omega^1 + R_{22} \omega^2$$

we have, according to [1], p. 247-250, the integral formula

$$(11) \qquad \int_{\partial M} [(R_1 R_{21} - R_2 R_{11}) \omega^1 + (R_1 R_{22} - R_2 R_{12}) \omega^2] =$$

$$= \int_M [2(R_{11} R_{22} - R_{12} R_{21}) - (R_1^2 + R_2^2) K] \, \omega^1 \wedge \omega^2.$$

Now, let us choose the tangent frames associated to M in such a way that $v_1 = V_1$, $v_2 = V_2$. Then it follows from (6)

$$V_1 K = K_1, \; V_2 K = K_2$$

and

$$V_1 V_1 K = K_{11} + K_2 \cdot \omega_1^2(V_1).$$

Thus we have, using (9),(10),

$$K_1 = 0,\ K_2 = 0$$

on ∂M,

$$K_{11} = 0,\ K_2 = 0$$

on M and hence the integral formula (11), re-written for the 1-form $K_1\omega^1 + K_2\omega^2$, yields

$$\int_M (2K_{12}^2 + K_1^2 K)\,\omega^1 \wedge \omega^2 = 0.$$

Thus especially

$$K_1 = V_1 K = 0$$

on M, i.e. K = const on M.

Remark that the surfaces with K = const depend on 4 functions of 1 variable.

Following [1], we are going to prove that there are, locally, surfaces M in E^3 possessing two orthonormal tangent vector fields V_1, V_2 such that $V_2K = 0$, $V_1V_1K = 0$ and with K not constant on M. For this purpose, we shall prove that the surfaces satisfying the preceding conditions depend on 4 functions of 1 variable.

The considered surfaces are defined by the system (4) and

$$(12)\qquad V_2K = a\sigma - 2b\gamma + c\beta = 0,$$
$$V_1V_1K = aC - 2bB + cA + 2(\alpha\gamma - \beta^2) + (ac - 2b^2)K = 0.$$

Because of $K \neq$ const , we have $V_1K = K_1 \neq 0$. By exterior differentiation of (4) we obtain

$$(13)\qquad \Delta A \wedge \omega^1 + \Delta B \wedge \omega^2 = (4\beta K + bK_1)\,\omega^1 \wedge \omega^2,$$
$$\Delta B \wedge \omega^1 + \Delta C \wedge \omega^2 = [(3\gamma - 2\alpha)K - aK_1 + bK_2]\,\omega^1 \wedge \omega^2,$$

$$\Delta C \wedge \omega^1 + \Delta D \wedge \omega^2 = [(2\sigma - 3\beta)K - bK_1 + cK_2]\omega^1 \wedge \omega^2,$$

$$\Delta D \wedge \omega^1 + \Delta E \wedge \omega^2 = -(4\gamma K + bK_2)\omega^1 \wedge \omega^2$$

where

$$(14) \qquad \Delta A = dA - 2(2B + bK)\omega_1^2,$$

$$\Delta B = dB + [A - 3C - (2a+c)K]\,\omega_1^2,$$

$$\Delta C = dC + 2(B-D)\omega_1^2,$$

$$\Delta D = dD + [3C - E + (a+2c)K]\,\omega_1^2,$$

$$\Delta E = dE + 2(2D + bK)\omega_1^2.$$

Differentiating (12) and applying (3),(4),(14) we get

$$(15) \qquad a\Delta\sigma - 2b\Delta\gamma + c\Delta\beta + \sigma\Delta a - 2\gamma\Delta b + \beta\Delta c - K_1\omega_1^2 = 0,$$

$$a\Delta C - 2b\Delta B + c\Delta A +$$

$$+ [C + c(2K - b^2)]\,\Delta a - 2[B + b(3K - b^2)]\,\Delta b +$$

$$+ [A + a(2K - b^2)]\,\Delta c + 2(\alpha\Delta\gamma - 2\beta\Delta\beta + \gamma\Delta\alpha) +$$

$$+ 2K_{12}\omega_1^2 = 0.$$

With regard to the second equation (15), the closure (13), (15) of the system (4),(12) contains $q = 4$ linearly independent forms and $s_1 = 4$ linearly independent exterior equations, so that $s_2 = 0$ and $Q = 4$. Applying the Cartan's lemma we obtain from (13)

$$\Delta A = F_1\omega^1 + F_2\omega^2,$$

$$\Delta B = (F_2 + 4\beta K + bK_1)\omega^1 + F_3\omega^2,$$

$$\Delta C = [F_3 + (3\gamma - 2\alpha)K - aK_1 + bK_2]\omega^1 + F_4\omega^2,$$

$$\Delta D = [F_4 + (2\sigma - 3\beta)K - bK_1 + cK_2]\omega^1 + F_5\omega^2,$$

$$\Delta E = (F_5 - 4\gamma K - bK_2)\omega^1 + F_6\omega^2,$$

the functions $F_1,\dots,F_6$ satisfying two independent relations obtained from (15) by elimination of ω_1^2. Thus, $N = 4$ and the general solution of the considered system depends on 4 functions of 1 variable.

Finally notice that the heorem 1 and that one due to H. Fath el Bab can be generalized to this form:

Theorem 2. Let M be a surface in E^3 with $K > 0$ and ∂M its boundary. Let $F(H,K)$ be a non-zero function defined on M. Let $V_1, V_2 \in T(M)$ be orthonormal vector fields such that

$$V_1F(H,K) = 0, \quad V_2F(H,K) = 0$$

on ∂M and

$$V_1V_1F(H,K) = 0, \quad V_2F(H,K) = 0$$

on M. Then $F(H,K) = \text{const}$ on M.

The proof of this assertion is analogous to the above mentioned one.

References

[1] H. FATH EL BAB: On surfaces with constant mean curvature, Comment. Math. Univ. Carolinae 16(1975), 245-254

[2] A. ŠVEC: Contributions to the global differential geometry of surfaces, Rozpravy ČSAV 1, 87, 1977, 1 - 94

Katedra matematiky FS VUT
Gorkého 13
60200 Brno
Československo

(Oblatum 6.6. 1978)