Ladislav Bican

Factor-splitting Abelian groups of arbitrary rank

Commentationes Mathematicae Universitatis Carolinae, Vol. 19 (1978), No. 4, 653--672

Persistent URL: http://dml.cz/dmlcz/105882

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

19,4 (1978)

FACTOR-SPLITTING ABELIAN GROUPS OF ARBITRARY RANK

Ladislav BICAN, Praha

Abstract: A structural description of factor splitting torsion-free abelian groups of arbitrary rank is presented. This criterion enables us to show that a torsion-free abelian group G, every element of which is p-divisible for all but a finite number of primes p, is factor-splitting if and only if G/pG is finite for each prime p. Two examples are included. The first one shows the existence of a non-factor-splitting group whose every pure subgroup of finite rank is factor-splitting and the other shows that the class of factor-splitting torsionfree abelian groups is not closed under finite sums.

Key words: Factor-splitting group, almost divisible group, basis, p-independent set, increasing p-height ordering of a basis.

AMS: Primary 20K15
Secondary 20K25, 20K99

Throughout this paper by a group it is always meant an additively written abelian group. A torsionfree group G is called factor-splitting if any of its factor-group G/H is splitting (see [9]). We shall use the following notations: If M is a subset of a group G then $\langle M \rangle$ denotes the subgroup of G generated by M. If g is an element of infinite order of a mixed group G then $h_p^G(g)$, ($\tau^G(g)$) denotes the p-height (the characteristic) of g in the group G. If $\alpha \neq 0$ is an integer, $\alpha = p^k\alpha'$, $(\alpha', p) = 1$, then we write $h_p(\alpha) = k$. We also put $h_p(0) = \infty$ for all primes p. The symbol π will denote the set of all primes. If $\pi' \subseteq \pi$ and M is a subset of a torsionfree group G then $\langle M\rangle^G_{\pi'}$ is the π'-pure closure of M in G, i.e. the largest subgroup of G such that $\langle M\rangle^G_{\pi'}/\langle M\rangle$ is π'-primary.

Every maximal linearly independent set of elements of a torsionfree group G is called a basis of G. A set $M = \{a_\lambda \mid \lambda \in \Lambda\}$ of elements of a mixed group G with the torsion part T is said to be a basis of G if the subset $\overline{M} = \{a_\lambda + T \mid \lambda \in \Lambda\}$ is a basis of the torsionfree group $\overline{G} = G/T$. A linearly independent subset $M = \{a_\lambda \mid \lambda \in \Lambda\}$ of a mixed group G consisting of elements of infinite order is said to be p-independent if for every finite subset $\{a_1, a_2, \dots, a_n\} \subseteq M$ the relation $px = \sum_{i=1}^{n} \lambda_i a_i$ implies $p \mid \lambda_i$, $i = 1, 2, \dots, n$.

A sequence $g_0, g_1, \dots$ of elements of a mixed group G is said to be a p-sequence of g_0 if $pg_{i+1} = g_i$, $i = 0, 1, \dots$. Let U be any torsionfree subgroup of a mixed group G and let $g \in G \setminus U$ be an element of infinite order. If $h_p^{G/U}(g + U) = \infty$ then every sequence $g = g_0, g_1, \dots$ of elements of G such that $p(g_{i+1} + U) = g_i + U$, $i = 0, 1, \dots$, is called a generalized p-sequence of G with respect to U.

Let $M = \{a_\alpha \mid \alpha < \mu\}$ (μ is an ordinal number) be a well-ordered basis of a mixed group G. We define the generalized p-height $H_p^G(a_\alpha)$ of the element a_α as the p-height of $a_\alpha + \sum_{\beta<\alpha} \langle a_\beta\rangle$ in $G/\sum_{\beta<\alpha}\langle a_\beta\rangle$. The well-ordering on M is said to be an increasing p-height ordering if $H_p^G(a_\alpha) \le H_p^G(a_\beta)$ whenever $\alpha \le \beta < \mu$.

The following assertion has been proved in [5].

Lemma 1: Let $M = \{a_\lambda \mid \lambda \in \Lambda\}$ be a basis of a mixed group G with the torsion part T. Then G splits if and only if there are non-zero integers m_λ, $\lambda \in \Lambda$, such that

(1) $\tau^G(a) = \tau^{G/T}(a + T)$ for each element $a \in \sum_{\lambda \in \Lambda} \langle m_\lambda a_\lambda \rangle$,

(2) for every prime p there is an increasing p-height ordering $\{m_\alpha a_\alpha \mid \alpha < \mu\}$ on $\tilde{M} = \{m_\lambda a_\lambda \mid \lambda \in \Lambda\}$ such that $H_p^G(m_\alpha a_\alpha) = n_\alpha < \infty$ if and only if $\alpha < \nu$ and for every $\alpha < \nu$ there exists an element $x_\alpha \in G$ such that $p^{n_\alpha}(x_\alpha + \sum_{\beta < \alpha} \langle m_\beta a_\beta \rangle) = m_\alpha a_\alpha + \sum_{\beta < \alpha} \langle m_\beta a_\beta \rangle$ and every element $m_\gamma a_\gamma$, $\nu \leq \gamma < \mu$, has a generalized p-sequence with respect to $U = \langle x_\alpha \mid \alpha < \nu \rangle$.

The systematical study of factor-splitting groups was begun by Procházka [9],[10] and it was continued in my papers [2],[4]. The results obtained here generalize those of the mentioned papers and they have two interesting consequences. The class of factor-splitting almost divisible torsionfree groups is characterized and it is shown that an infinite direct sum of torsionfree groups can be factor-splitting only under very special hypotheses. At the end of the paper an example showing that the class of factor splitting groups is not closed under finite direct sums is presented.

Definition 1: Let p be a prime. We say that an independent subset $\{g_1, g_2, \dots, g_n\}$ of a torsionfree group G satisfies (FSp) if the following holds: if $p^k x = \sum_{i=1}^{n} \alpha_i g_i$ for some $x \in G$ then the equation $p^k y = \sum_{i=1}^{n} \beta_i g_i$ with $h_p(\beta_i) \geq 1$, $i = 1, 2, \dots, n$, and $\beta_i = \alpha_i$ whenever $h_p(\alpha_i) \geq 1$, is solvable in G.

Definition 2: Let p be a prime and M be a basis of a torsionfree group G, $N \subseteq M$. We say that an element $b \in \langle M \setminus N \rangle$ satisfies (FSp) with respect to N if from the solvability of the equations $p^n x = pb + u$, $u \in \langle N \rangle$, in G follows the solvability of the equation $p^n x = p(b + v)$, $v \in \langle N \rangle$, in G.

Lemma 2: Every pure subgroup of a factor-splitting torsionfree group G is factor-splitting.

Proof: Let H be a subgroup of a pure subgroup S of G. Since G/H splits, $G/H = \langle H \rangle_{\pi}^{G} /H \oplus V/H$, the factor-group $S/H = \langle H \rangle_{\pi}^{G} /H \oplus (S \cap V)/H$ splits, too.

Lemma 3: Let H, K be subgroups of a torsionfree group G such that $H \subseteq K \subseteq \langle H \rangle_{\pi}^{G}$. If G/H splits then G/K splits, too.

Proof: By hypothesis, $G/H = \langle H \rangle_{\pi}^{G} /H \oplus V/H$. If $x = v + k$, $x \in \langle H \rangle_{\pi}^{G}$, $v \in V$, $k \in K$, is an arbitrary element of $\langle H \rangle_{\pi}^{G} \cap \langle V \cup K \rangle$ then $v = x - k \in V \cap \langle H \rangle_{\pi}^{G} = H \subseteq K$ and $x = v + k \in K$. Hence $\langle H \rangle_{\pi}^{G} \cap \langle V \cup K \rangle = K$ and $G/K = \langle H \rangle_{\pi}^{G} /K \oplus \langle V \cup K \rangle / /K$ splits.

Corollary 1: A torsionfree group G is factor-splitting if and only if G/F splits for every free subgroup F of G.

Proof: The preceding Lemma proves the sufficiency, the necessity being obvious.

Lemma 4: Let $M = \{a_1, a_2, \ldots\}$ be a p-independent basis of a torsionfree group H. If $K = \langle p^{2i-1}a_i - p^{2i}a_{i+1} \mid i = 1, 2, \ldots \rangle_{\pi \setminus \{p\}}^{H}$ and $L = \langle a_i - pa_{i+1} \mid i = 1, 2, \ldots \rangle_{\pi \setminus \{p\}}^{H}$ then $\langle K \rangle_{\pi}^{H} = L$.

Proof: If $x \in \langle K \rangle_{\pi}^{H}$ is an arbitrary element then $p^k m x = \sum_{i=1}^{m} r_i (p^{2i-1}a_i - p^{2i}a_{i+1})$ for some non-negative integer k and some non-zero integer m, $(m,p) = 1$. The p-independence of M now yields $p^k \mid pr_1$, $p^k \mid (p^{2i-1}r_i - p^{2i-2}r_{i-1})$, $i = 2,3,\dots,n$, $p^k \mid p^{2n}r_n$, which obviously implies that $p^k \mid p^{2i-1}r_i$, $i = 1,2,\dots,n$. So $p^{2i-1}r_i = p^k r_i'$ and consequently $p^k mx = \sum_{i=1}^{n} p^{2i-1}r_i(a_i - pa_{i+1}) = p^k \sum_{i=1}^{n} r_i'(a_i - pa_{i+1})$. Then $mx = \sum_{i=1}^{n} r_i'(a_i - pa_{i+1})$, H being torsionfree, and $x \in L$. We have proved that $\langle K \rangle_\pi^H \subseteq L$ and hence $\langle K \rangle_\pi^H = L$, the inclusion $L \subseteq \langle K \rangle_\pi^H$ being obvious.

<u>Lemma 5</u>: Let the hypotheses of Lemma 4 be satisfied. Then H/K is a non-splitting mixed group with the torsion part L/K.

Proof: It is easy to see that H/K is of rank one and that the element $a_1 + K$ is of infinite order. Since $a_1 + L = \sum_{i=1}^{n} p^{i-1}(a_i - pa_{i+1}) + p^n a_{n+1} + L = p^n a_{n+1} + L$, the element $a_1 + L$ is of infinite p-height in $H/L \cong {}^{H/K}/_{L/K}$. With respect to [1, Theorem 2] it suffices now to show that no non-zero multiple of $a_1 + K$ has the infinite p-height in H/K. So, let the equation $p^{2k}(x + K) = ma_1 + K$ be solvable in H/K. Then $p^{2k}\alpha x = m\alpha a_1 + \sum_{i=1}^{n} r_i(p^{2i-1}a_i - p^{2i}a_{i+1})$ for some non-zero integer α with $(\alpha,p) = 1$. The p-independence of M yields $p^{2k} \mid (m + pr_1)$, $p^{2k} \mid (p^{2i-1}r_i - p^{2i-2}r_{i-1})$, $i = 1, 2,\dots,n$, $p^{2k} \mid p^{2n}r_n$ and we can obviously assume that $n \geq k$. Now the relation $p^{2k} \mid (p^{2k-1}r_k - p^{2k-2}r_{k-1})$ yields $p \mid r_{k-1}$ so that the relation $p^{2k} \mid (p^{2k-3}r_{k-1} - p^{2k-4}r_{k-2})$ yields $p^2 \mid r_{k-2}$ etc. Continuing in this process we finally obtain $p^{k-2} \mid r_2, p^{k-1} \mid r_1$ and $p^k \mid m\alpha$. Since $(p,\alpha) = 1$ we have $p^k \mid m$ and we are through.

<u>Lemma 6</u>: Let p be a prime and G be a mixed group. If $a_1,a_2,\dots,a_k$ are elements of G with

$h_p^{G/\sum_{j=1}^{i-1}\langle a_j\rangle}(a_i + \sum_{j=1}^{i-1}\langle a_j\rangle) = n_i$, $n_1 \leq n_2 \leq \dots \leq n_k < \infty$ and if x_i are such elements of G that $p^{n_i}x_i = a_i + \sum_{j=1}^{i-1} \lambda_j^{(i)} a_j$ then the set $\{x_1, x_2, \dots, x_k\}$ is p-independent in G.

Proof: Let $px = \sum_{i=1}^{k} \lambda_i x_i$ be solvable in G and let r be the greatest integer between 1 and k for which $(\lambda_r, p) = 1$. Then the equation $py = \sum_{i=1}^{r} \lambda_i x_i$ is obviously solvable in G and if we multiply this equality by p^{n_k} we obtain $p^{n_k+1} y = \sum_{i=1}^{r} p^{n_k-n_i} \lambda_i p^{n_i} x_i = \sum_{i=1}^{r} p^{n_k-n_i} \lambda_i (a_i + \sum_{j=1}^{i-1} \lambda_j^{(i)} a_j) = \lambda_r a_r + \sum_{i=1}^{r-1} \mu_i a_i$ by the hypothesis. Consequently $h_p^{G/\sum_{i=1}^{r}\langle a_j\rangle}(a_r + \sum_{i=1}^{r-1}\langle a_i\rangle) \geq n_r + 1$ - a contradiction proving Lemma 6.

<u>Lemma 7</u>: Let p be a prime and G be a mixed group such that the GF(p)-vector space G/pG has finite dimension k. If $\{a_\alpha \mid \alpha < \mu\}$ is an increasingly p-height ordered basis of G then $H_p^G(a_{k+1}) = \infty$.

Proof: Let $H_p^G(a_\alpha) = n_\alpha$, $\alpha < \mu$. If $n_1 \leq n_2 \leq \dots \leq n_k \leq n_{k+1} < \infty$ then Lemma 6 yields the existence of a p-independent subset $\{x_1, x_2, \dots, x_{k+1}\}$ of G. Then the set $\{x_1 + pG, x_2 + pG, \dots, x_{k+1} + pG\}$ is clearly linearly independent over GF(p), which contradicts the hypothesis.

<u>Theorem 1</u>: A torsionfree group G is factor-splitting if and only if

(i) G/pG is finite for each prime p and

(ii) if M is an arbitrary basis of G, $N \subseteq M$, $M \setminus N = \{a_\lambda \mid \lambda \in \Lambda\}$ then there are non-zero integers m_λ , $\lambda \in \Lambda$, such that every element from $\langle m_\lambda a_\lambda \mid \lambda \in \Lambda \rangle$ satisfies (FSp) with respect to N for all primes p.

Proof: Necessity: (i) Let p be a prime such that G/pG is infinite. If $x_1 + pG$, $x_2 + pG, \dots$ are linearly independent elements of the vector space G/pG over GF(p) then the elements $x_1, x_2, \dots$ are obviously p-independent in G and the pure subgroup $H = \langle x_1, x_2, \dots \rangle_\pi^G$ of G is factor-splitting by Lemma 2. Lemma 5 now leads to a contradiction.

(ii) If we denote $L = \langle N \rangle_\pi^G$ then the splitting of $G/\langle N \rangle$ and Lemma 1 yield the existence of non-zero integers m_λ, $\lambda \in \Lambda$, such that $\tau^{G/\langle N \rangle}(b + \langle N \rangle) = \tau^{G/L}(b + L)$ for each element $b \in \langle m_\lambda a_\lambda \mid \lambda \in \Lambda \}$. Let the equation $p^k x = pb + u$, $u \in \langle N \rangle$, $b \in \langle m_\lambda a_\lambda \mid \lambda \in \Lambda \}$, be solvable in G. Then $pu' = u$ for some $u' \in L$ and $h_p^{G/\langle N \rangle}(b + \langle N \rangle) = h^{G/L}(b + L) \geq k - 1$. Thus there is $v \in \langle N \rangle$ such that the equation $p^{k-1}y = b + v$ is solvable in G and the necessity is proved.

Sufficiency: With respect to Corollary 1 it suffices to show that G/F splits for every free subgroup F of G.

Let N be an arbitrary linearly independent subset of G and let M be a basis of G containing N such that $M \setminus N = \{a_\lambda \mid \lambda \in \Lambda\}$. By hypothesis there are non-zero integers m_λ, $\lambda \in \Lambda$, such that every element from $\langle m_\lambda a_\lambda \mid \lambda \in \Lambda \rangle$ satisfies (FSp) with respect to N for all primes p. Let $L = \langle N \rangle_\pi^G$ and let the equation $p^k(x + L) = b + L$, $b \in \langle m_\lambda a_\lambda \mid \lambda \in \Lambda \rangle$ be solvable in G/L. Then, for some $u \in L$, the equation $p^k x = b + u$ is solvable in G. Now since $p^r mu \in \langle N \rangle$ for some non-negative integer r and some non-zero integer m, $(m,p) = 1$, the equation $p^{k+r}mx = p^r mb + p^r mu$ is solvable in

G. Using the condition (FSp) with respect to N r times we see that the equation $p^k y = mb + v$ is solvable in G for some $v \in \langle N \rangle$. Thus $h_p^{G/\langle N \rangle}(b + \langle N \rangle) \geq h_p^{G/L}(b + L) \geq h_p^{G/\langle N \rangle}(b + \langle N \rangle)$ and the condition (1) of Lemma 1 is satisfied.

Now we proceed to the condition (2) of Lemma 1. Let $\{ b_\beta \mid \beta < \nu \}$ be an increasing p-height ordering on the basis N of L. Since the kernel of the mapping $\varphi : L \longrightarrow G/pG$ defined by $\varphi(x) = x + pG$ is equal to $L \cap pG = pL$, φ induces a monomorphism $\tilde{\varphi} : L/pL \longrightarrow G/pG$ and L/pL is finite. By Lemma 7 there is an integer ℓ such that $H_p^L(b_{\ell+1}) = \infty$ and $H_p^L(b_i) = m_i$, $i = 1,2,\dots,\ell$, are finite, $m_1 \leq m_2 \leq \dots \leq m_\ell$. The elements $z_i \in L$ defined by the formula

$$(3) \qquad p^{m_i} z_i = b_i + \sum_{j=1}^{i-1} \nu_j^{(i)} b_j, \; i = 1,2,\dots, \quad ,$$

are p-independent in L by Lemma 6 and

$$(4) \quad p^{m_\ell} \sum_{i=1}^{\ell} \vartheta_i z_i = \sum_{i=1}^{\ell} p^{m_\ell - m_i} \vartheta_i (b_i + \sum_{j=1}^{i-1} \nu_j^{(i)} b_j) \in \langle N \rangle.$$

Moreover, by the definition of the increasing p-height ordering, every element b_β, $\ell < \beta < \nu$, has infinite p-height with respect to $\langle b_1, b_2, \dots, b_\ell \rangle$ so that for each positive integer r there is an element $z_{\beta r} \in L$ such that

$$(5) \qquad p^r z_{\beta r} = b_\beta + \sum_{i=1}^{\ell} \varepsilon_{\beta i} b_i .$$

Further, for the sake of simplicity we can assume that $m_\lambda = 1$, $\lambda \in \Lambda$. The factor-group $G/\langle N \rangle / p(G/\langle N \rangle) = G/\langle N \rangle / \langle pG \cup N \rangle / N \cong G/\langle pG \cup N \rangle$ is finite as a homomorphic image of G/pG. Let $\{ a_\alpha + \langle N \rangle \mid \alpha < \mu \}$ be an increasing p-height ordering on the basis $\{ a_\lambda + \langle N \rangle \mid \lambda \in \Lambda \}$ of $G/\langle N \rangle$. By Lemma 7 there is an integer k such that $H_p^{G/\langle N\rangle}(a_{k+1} + \langle N\rangle) = \infty$ and $H_p^{G/\langle N\rangle}(a_i + \langle N\rangle) = n_i$, $i = 1,2,\dots,k$, are finite, $n_1 \leq \leq n_2 \leq \dots \leq n_k$. The elements $x_i + \langle N\rangle \in G/\langle N\rangle$ defined by

$$(6) \quad p^{n_i}(x_i + \langle N\rangle) = a_i + \sum_{j=1}^{i-1} \lambda_j^{(i)} a_j + \langle N\rangle, \; i = 1,2,\dots,k$$

are p-independent in $G/\langle N\rangle$ by Lemma 6.

Let b be an arbitrary element from the set $\{a_\alpha \mid k < < \alpha < \mu\}$. We are going to show that $b + \langle N\rangle$ has a generalized p-sequence with respect to $\langle x_1 + \langle N\rangle, x_2 + \langle N\rangle,\dots$ $\dots, x_k + \langle N\rangle\rangle / \langle N\rangle$ or, equivalently, that b has a generalized p-sequence with respect to $\langle \{x_1,x_2,\dots,x_k\} \cup N\rangle$.

By the definition of the increasing p-height ordering the element $b + \langle N\rangle$ has the infinite p-height with respect to $\sum_{i=1}^{k} \langle a_i + \langle N\rangle\rangle$. Hence for each $n = 1,2,\dots$ there is an element $y_n' \in G$ such that

$$(7) \quad p^{n+m_\ell} y_n' = b + \sum_{i=1}^{k} \beta_i^{(n)} a_i + u_n, \; u_n \in \langle N\rangle.$$

Since $u_n = \sum_{\beta<\nu} \varphi_\beta b_\beta$ (finite sum), it follows from (5) that (7) can be rewritten in the form

$$(8) \quad p^{n+m_\ell} y_n = b + \sum_{i=1}^{k} \beta_i^{(n)} a_i + \sum_{i=1}^{\ell} \varrho_i^{(n)} b_i.$$

Using (3) and (6) we can write (8) in the form

$$(9) \quad p^{n+m_\ell} y_n = b + \sum_{i=1}^{k} \gamma_i^{(n)} x_i + \sum_{i=1}^{\ell} \sigma_i^{(n)} z_i,$$

where $\sum_{i=1}^{\ell} \sigma_i^{(n)} z_i = \sum_{i=1}^{\ell} \varrho_i^{(n)} b_i \in \langle N\rangle$.

Now $p^{n+m_\ell}(py_{n+1} - y_n) = \sum_{i=1}^{k} (\gamma_i^{(n+1)} - \gamma_i^{(n)}) x_i + + \sum_{i=1}^{\ell} (\sigma_i^{(n+1)} - \sigma_i^{(n)}) z_i$ and Lemma 6 gives $\gamma_i^{(n+1)} - \gamma_i^{(n)} =$

$= p^{n+m_\ell} \sigma_i^{(n)}$, $i = 1,2,\dots,k$, owing to the fact that $\sum_{i=1}^{\ell} (\sigma_i^{(n+1)} - \sigma_i^{(n)}) z_i \in \langle N \rangle$. Consequently, $\sigma_i^{(n+1)} - \sigma_i^{(n)} = p^{n+m_\ell} \vartheta_i^{(n)}$ again by Lemma 6. So $p(p^{m_\ell} y_{n+1}) = p^{m_\ell} y_n + p^{m_\ell} \sum_{i=1}^{k} \sigma_i^{(n)} x_i + \sum_{i=1}^{\ell} \vartheta_i^{(n)} p^{m_\ell} z_i$ and (4) shows that the elements b, $p^{m_\ell} y_1$, $p^{m_\ell} y_2, \dots$ form a generalized p-sequence of b with respect to $\langle \{x_1, x_2, \dots, x_k\} \cup N \rangle$.

Lemma 1 now finishes the proof of Theorem 1.

Remark: It should be noted that the necessity of the condition (i) was observed by Procházka [10].

Now we shall prove a more simple sufficient condition for the factor-splitting.

Corollary 2: Let G be a torsionfree group. If

(a) G/pG is finite for each prime p and

(b) for every basis $M = \{c_\lambda \mid \lambda \in \Lambda\}$ of G there are non-zero integers m_λ, $\lambda \in \Lambda$, such that every finite subset of $\tilde{M} = \{m_\lambda c_\lambda \mid \lambda \in \Lambda\}$ satisfies (FSp) for all primes p, then G is factor splitting.

Proof: Let $\tilde{N} = \{b_\iota \mid \iota \in I\}$ be an arbitrary subset of $\tilde{M}$ and $\tilde{M} \setminus \tilde{N} = \{a_\varkappa \mid \varkappa \in k\}$. If $p^k x = p \sum_{i=1}^{n} \alpha_{\varkappa_i} a_{\varkappa_i} + \sum_{j=1}^{m} \beta_{\iota_j} b_{\iota_j}$ is solvable in G then the condition (FSp) for the subset $\{a_{\varkappa_1}, a_{\varkappa_2}, \dots, a_{\varkappa_n}, b_{\iota_1}, b_{\iota_2}, \dots, b_{\iota_m}\}$ shows that Theorem 1 can be applied.

Corollary 3: Let G be a torsionfree group. If

(a) for every prime p the GF(p)-vector space G/pG has a finite dimension r_p and

(b) for every basis $M = \{c_\tau , \tau \in T\}$ of G there are non-zero integers m_τ, $\tau \in T$, such that for each prime p every subset of $\{m_\tau c_\tau \mid \tau \in T\}$ containing at most r_p elements satisfies (FSp),

then G is factor-splitting.

Proof: For the sake of simplicity we can assume that $m_\tau = 1$, $\tau \in T$. We shall use the notation of the proof of Theorem 1. Let N be an arbitrary subset of M and let the equation $p^s x = p \sum \varphi_\alpha a_\alpha + \sum \sigma_\beta b_\beta$ (finite sum) be solvable in G. It follows easily from (7) and (5) that the equation $p^s y = p \sum_{i=1}^{k} \varphi_i a_i + \sum_{i=1}^{\ell} \sigma_i b_i$ is solvable in G where $\sum \varphi_\alpha a_\alpha + \langle N \rangle = \sum_{i=1}^{k} \varphi_i a_i + \langle N \rangle$. If we show that k + + $\le r_p$ then the equation $p^s z = p(\sum_{i=1}^{k} \varphi_i a_i + \sum_{i=1}^{\ell} \vartheta_i b_i)$ is solvable in G by the hypothesis and Theorem 1 can be applied. It remains to show that $k + \ell \le r_p$. Let the equation $pz = \sum_{i=1}^{k} \lambda_i x_i + \sum_{i=1}^{\ell} \mu_i z_i$ be solvable in G. Then the equation $p^{m_\ell +1}(z + \langle N \rangle) = p^{m_\ell} \sum_{i=1}^{k} \lambda_i x_i + \langle N \rangle$ is solvable in $G/\langle N \rangle$ owing to (4) so that $p \mid \lambda_i$, $i = 1,2,\dots,k$, by Lemma 6. So, again by Lemma 6, $p \mid \mu_i$, $i = 1,2,\dots,\ell$, and the elements $x_1, x_2, \dots, x_k$, $z_1, z_2, \dots, z_\ell$ are p-independent in G. Hence the elements $x_1 + pG$, $x_2 + pG, \dots, x_k + pG$, $z_1 + pG$, $z_2 + pG, \dots, z_\ell + pG$ of G/pG are linearly independent over GF(p) and $k + \ell \le r_p$ by the hypothesis.

Definition 3: A torsionfree group G is said to be almost divisible if the p-height of any non-zero element of G is finite for finitely many primes only.

Theorem 2: An almost divisible torsionfree group G is factor-splitting if and only if G/pG is finite for all primes p.

Proof: The condition is necessary by Theorem 1.

Conversely, let M be an arbitrary basis of G, $N \subsetneqq M$, $M \setminus N = \{a_\lambda \mid \lambda \in \Lambda\}$. If $\{b_\beta \mid \beta < \nu\}$ is an increasing p-height ordering on the basis N of $L = \langle N \rangle_\pi^G$ then in the same manner as in the proof of Theorem 1 one can prove the existence of an integer ℓ and of elements $z_1, z_2, \ldots, z_\ell$, such that the formulas (3),(4) and (5) are valid.

Now we put $m_\lambda = 1$ if $h_p^G(a_\lambda) = \infty$ and $m_\lambda = p^{m_\ell}$ if $h_p^G(a_\lambda) < \infty$. We are going to show that every element $b \in \langle m_\lambda a_\lambda \mid \lambda \in \Lambda \rangle$ satisfies the condition (FSp) with respect to N. Let the equation

(10) $\quad p^s x = pb + u, \; b \in \langle m_\lambda a_\lambda \mid \lambda \in \Lambda \rangle, \; u \in \langle N \rangle,$

be solvable in G. Then $b = p^{m_\ell} \sum_{i=1}^{r} \alpha_i a_i + a$ where $h_p^G(a) = \infty$ and $a_i \in \{a_\lambda \mid \lambda \in \Lambda\}$, $h_G^p(a_i) < \infty$, $i = 1,2,\ldots,r$. From the solvability of the equation (10) follows the solvability of the equation $p^s y = p^{m_\ell+1} \sum_{i=1}^{r} \alpha_i a_i + \sum_{i=1}^{\ell} \mu_i z_i$ owing to (5) and (3). However, the p-independence of $z_1, z_2, \ldots$ $\ldots, z_\ell$ in G yields $\mu_i = p^{m_\ell+1} \mu'_i$, $i = 1,2,\ldots,\ell$, and the equation $p^{s-1} y = p^{m_\ell} \sum_{i=1}^{r} \alpha_i a_i + p^{m_\ell} \sum_{i=1}^{\ell} \mu'_i z_i$ is solvable in G. There is an element $a' \in G$ such that $p^{s-1} a' = a$, a being of infinite p-height in G, and consequently, $p^{s-1}(y + a') = p^{m_\ell} \sum_{i=1}^{r} \alpha_i a_i + a + v = b + v$ where $v \in \langle N \rangle$ by (4).

Since G is almost divisible we can do this procedure for each prime p and finally we obtain the non-zero integers $\overline{m}_\lambda$, $\lambda \in \Lambda$, such that every element from $\langle \overline{m}_\lambda a_\lambda \mid \lambda \in \Lambda \rangle$ satisfies (FSp) with respect to N for each prime p. Now it suffices to apply Theorem 1.

Corollary 4: If an infinite direct sum $G = \sum_{i \in I} G_i$ of torsionfree groups G_i, $i \in I$, is factor-splitting then each group G_i, $i \in I$, is factor-splitting and, for each prime p, the group G_i is p-divisible for all but a finite number of indices $i \in I$.

Proof: It follows immediately from Lemma 2 and Theorem 1.

At the end of this paper we shall present two examples. The first one proves the existence of a non-factor-splitting group every rank finite pure subgroup of which is factor-splitting.

Example 1: Let $\{p_1, p_2, \ldots\}$ be an infinite set of primes, $H = \sum_{i=0}^{\infty} \langle x_i \rangle + \sum_{i=1}^{\infty} \langle y_i \rangle$ be a free group and $K = \langle p_i x_0 + x_i + p_i^2 y_i \mid i = 1,2,\ldots \rangle$ be its subgroup. If $m(\sum_{i=0}^{n} \alpha_i x_i + \sum_{i=1}^{n} \beta_i y_i) = \sum_{i=1}^{n} \lambda_i (p_i x_0 + x_i + p_i^2 y_i)$ then $m\alpha_0 = \sum_{i=1}^{n} \lambda_i p_i$, $m\alpha_i = \lambda_i$, $m\beta_i = \lambda_i p_i^2$, $i = 1,2,\ldots,n$. Hence $\beta_i = \alpha_i p_i^2$, $i = 1,2,\ldots,n$, and $\alpha_0 = \sum_{i=1}^{n} \alpha_i p_i$ from which it easily follows the purity of K in H.
Thus the group $G = H/K$ is torsionfree and $\{x_0 + K, x_1 + K, \ldots\}$ is a basis of G.

Let us consider the subgroups $U = \langle \{x_1, x_2, \ldots\} \cup K \rangle$ and $V = \langle \{x_1, x_2, \ldots\} \cup \{x_0 + p_i y_i \mid i = 1,2,\ldots\} \rangle$ of H. It is an easy exercise to show that $V = \langle U \rangle_{\pi}^{H}$ and that G/U is a mixed group of rank one. Moreover, $h_{p_i}^{H/V}(x_0 + V) \geq 1$ for all $i = 1, 2, \ldots$. Let the equation $p_j(x + U) = mx_0 + U$ be solvable in

H/U. Then $p_j(\sum_{i=0}^{n} \alpha_i x_i + \sum_{i=1}^{n} \beta_i z_i) = mx_o + \sum_{i=1}^{n} \lambda_i x_i +$ $+ \sum_{i=1}^{n} \mu_i (p_i x_o + x_i + p_i^2 y_i)$ and so

(11) $$p_j \alpha_o = m + \sum_{i=1}^{n} \mu_i p_i,$$

(12) $$p_j \beta_i = \mu_i p_i^2, \; i = 1,2,\dots,n.$$

By (12), $p_j \mid \mu_i$ for all $i = 1,2,\dots,n$, $i \neq j$, and hence $p_j \mid m$ by (11). Consequently, no non-zero multiple of $x_o + U$ has the same type in H/U as $x_o + V$ in H/V. By [1, Theorem 2] the factor-group $H/U \cong {}^{H/K}/_{U/K}$ does not split and hence G is not factor-splitting.

Let us show that the subgroup $X = \langle \{x_o, x_1, \dots, x_n, y_1, y_2, \dots, y_n\} \cup K \rangle$ is pure in H. Indeed, from the equality $m(\sum_{i=0}^{n+k} \alpha_i x_i + \sum_{i=1}^{n+k} \beta_i y_i) = \sum_{i=0}^{n} \lambda_i x_i + \sum_{i=1}^{n} \mu_i y_i +$ $+ \sum_{i=1}^{n+k} \nu_i (p_i x_o + x_i + p_i^2 y_i)$ it easily follows that $\beta_i = \alpha_i p_i^2$, $i = n + 1,\dots,n + k$, so that $\sum_{i=0}^{n+k} \alpha_i x_i + \sum_{i=1}^{n+k} \beta_i y_i =$ $+ (\alpha_o - \sum_{i=n+1}^{n+k} \alpha_i p_i) x_o + \sum_{i=1}^{n} \alpha_i x_i + \sum_{i=1}^{n} \beta_i y_i +$ $+ \sum_{i=n+1}^{n+k} \alpha_i (p_i x_o + x_i + p_i^2 y_i)$.

Obviously, every pure subgroup of G of finite rank is contained in some subgroup X/K and with respect to Lemma 2 it suffices to prove that X/K is factor-splitting. However, $\{x_o + K, x_1 + K, \dots, x_n + K\}$ is a basis of X/K and for $\sum_{i=0}^{n} \lambda_i x_i + \sum_{i=1}^{n} \mu_i y_i + K$ and $m = p_1^2 p_2^2 \dots p_n^2 = m_i p_i^2$ we have $m(\sum_{i=0}^{n} \lambda_i x_i + \sum_{i=1}^{n} \mu_i y_i + K) = \sum_{i=0}^{n} m \lambda_i x_i + \sum_{i=1}^{n} m_i \mu_i (p_i x_o +$

$+ x_i + p_i^2 y_i) - (\sum_{i=1}^{n} m_i (\mu_i p_i) x_0 - \sum_{i=1}^{n} m_i \mu_i x_i + K = (m\lambda_0 - \sum_{i=1}^{n} m_i (\mu_i p_i) x_0 + \sum_{i=1}^{n} (m\lambda_i - m_i \mu_i \; x_i + K$. Thus X/K is a primitive group of finite rank and so it is factor-splitting by [10, Theorem 2].

The second example will show that the class of factor-splitting groups is not closed under direct sums.

<u>Example 2</u>: 1. Let $\{p_1 p_2, \dots\}$ be an infinite set of primes such that $p_i > i^2$ for each $i = 1,2,\dots$. Further, let $H_1 = \langle a \rangle \oplus \langle b \rangle \oplus \sum_{i=1}^{\infty} \langle x_i \rangle$ be a free group and $K_1 = \langle a + ib + p_i^2 x_i \mid i = 1,2,\dots \rangle$ be its subgroup. Let us show that K_1 is pure in H_1. For $p(\lambda a + \mu b + \sum_{i=1}^{n} \nu_i x_i) = \sum_{i=1}^{n} \alpha_i (a + ib + p_i^2 x_i)$ we have

$$p\lambda = \sum_{i=1}^{n} \alpha_i, \tag{13}$$

$$p\nu_i = p_i^2 \alpha_i, \quad i = 1,2,\dots,n. \tag{14}$$

If $p \notin \{p_1, p_2, \dots\}$ then by (14) $p \mid \alpha_i$, $i = 1,2,\dots,n$, and $\lambda a + \mu b + \sum_{i=1}^{n} \nu_i x_i \in K_1$. If $p_j = p$ for some $j = 1,2,\dots$ then $p \mid \alpha_i$ for each $i = 1,2,\dots,n$, $i \neq j$, by (14), hence $p \mid \alpha_j$ by (13) and again $\lambda a + \mu b + \sum_{i=1}^{n} \nu_i x_i \in K_1$.

2. We proceed to show that the torsionfree group $G_1 = H_1/K_1$ is homogeneous of the type Z (i.e. every non-zero element of G_1 has the same type as the infinite cyclic group) and consequently it is factor-splitting by [2, Theorem 2].

Let the equation $p(x + K_1) = \lambda a + \mu b + K_1$ be solvable in G_1. Then $p(\varrho a + \sigma b + \sum_{i=1}^{n} \varrho_i x_i) = \lambda a + \mu b + \sum_{i=1}^{n} \lambda_i (a + ib + p_i^2 x_i)$ in H and

(15) $$p\rho = \lambda + \sum_{i=1}^{n} \lambda_i,$$

(16) $$p\sigma = \mu + \sum_{i=1}^{n} i\lambda_i,$$

(17) $$p\rho_i = p_i^2 \lambda_i, \ i = 1,2,\dots,n.$$

If $p \notin \{p_1, p_2, \dots\}$ then $p \mid \lambda_i$, $i = 1,2,\dots,n$, by (17) and so $p \mid \lambda$, $p \mid \mu$ by (15) and (16). If $p = p_j$ for some $j = 1, 2,\dots$, then by (17) we have $\lambda_i \equiv 0 \pmod{p_j}$ for each $i = 1, 2,\dots,n$, $i \neq j$. Then (15) and (16) give $\lambda + \lambda_j \equiv 0 \pmod{p_j}$ and $\mu + j\lambda_j \equiv 0 \pmod{p_j}$ so that $\lambda j - \mu \equiv 0 \pmod{p_j}$. However, if $\lambda \neq 0$ and $j > \max(|\mu/\lambda|, 2|\lambda|)$ then $0 < |\lambda j - \mu| \leq |\lambda| j + |\mu| < 2|\lambda| j < j^2 < p_j$. If $\lambda = 0$ then $\mu \equiv 0 \pmod{p_j}$. In both cases the element $\lambda a + \mu b + K$ can be divisible either by primes p for which $p \mid \lambda$, $p \mid \mu$ or by primes p_j with $0 < j \leq \max(|\mu/\lambda|, 2|\lambda|)$ for $\lambda \neq 0$ and with $p_j \mid \mu$ for $\lambda = 0$.

3. Let $H_2 = \langle c \rangle \oplus \langle d \rangle \oplus \sum_{i=1}^{\infty} \langle y_i \rangle$ be a free group and $K_2 = \langle c + (p_i - i)d + p_i^2 y_i \mid i = 1,2,\dots \rangle$ be its subgroup. Similarly as above one can prove that $G_2 = H_2/K_2$ is a homogeneous torsionfree group of the type Z and, consequently, it is factor-splitting.

4. Consider the group $G = H/K \cong G_1 \oplus G_2$ where $H = H_1 \oplus H_2$, $K = K_1 \oplus K_2$ and let $S = \langle u + K, v + K, w + K \rangle_{\pi}^{G}$ where $u = a + c$, $v = b - d$, $w = d$. Obviously, $\{u + K, v + K, w + K\}$ is a basis of S and $u + iv + p_i w + K = a + ib + p_i^2 x_i + c + (p_i - i)d + p_i^2 y_i - p_i^2(x_i + y_i) + K = -p_i^2(x_i + y_i) + K$ for all $i = 1,2,\dots$. Now let the equation $p_i(x + K) = \lambda u +$

$+ \mu v + w + K$ be solvable in G/K. Then $p_i(\alpha a + \beta b + \gamma c + \delta d + \sum_{j=1}^{n} (\rho_j x_j) + \sigma_j y_j) = \lambda(a + c) + \mu(b - d) + d + \sum_{j=1}^{n} (\vartheta_j(a + jb + p_j^2 x_j) + \omega_j(c + (p_j - j)d + p_j^2 y_j))$ from which

$$(18) \qquad p_i\alpha = \lambda + \sum_{j=1}^{n} \vartheta_j,$$

$$(19) \qquad p_i\beta = \mu + \sum_{j=1}^{n} j\,\vartheta_j,$$

$$(20) \qquad p_i\gamma = \lambda + \sum_{j=1}^{n} \omega_j,$$

$$(21) \qquad p_i\delta = -\mu + 1 + \sum_{j=1}^{n} (p_j - j)\,\omega_j,$$

$$(22) \qquad p_i\,\rho_j = p_j^2\,\vartheta_j, \ j = 1,2,\dots,n,$$

$$(23) \qquad p_i\,\sigma_j = p_j^2\,\omega_j, \ j = 1,2,\dots,n.$$

By (22) and (23) we have $\vartheta_j \equiv 0 \pmod{p_i}$ and $\omega_j \equiv 0 \pmod{p_i}$ for each $j = 1,2,\dots,n$, $j \neq i$. Then (18),(20) and (19),(21) yield

$$(24) \qquad \lambda + \vartheta_i \equiv 0 \pmod{p_i}, \ \lambda + \omega_i \equiv 0 \pmod{p_i},$$

$$(25) \qquad \mu + i\vartheta_i \equiv 0 \pmod{p_i}, \ -\mu + 1 - i\omega_i \equiv 0 \pmod{p_i}.$$

Thus $\mu - \lambda i \equiv 0 \pmod{p_i}$, $-\mu + 1 + \lambda i \equiv 0 \pmod{p_i}$ and hence $1 \equiv 0 \pmod{p_i}$ - a contradiction. Thus the basis $\{u + K, v + K, w + K\}$ of S does not satisfy the condition (FSp) for infinitely many primes and S is not factor-splitting by [4, Theorem 3]. Consequently, d is not factor-splitting by Lemma 2.

We conclude these investigations by presenting a sufficient condition under which a direct aum of two factor-splitting groups is factor-splitting.

For $\pi' \subseteq \pi$ let $R_{\pi'}$ denote the group of rationals with denominators prime to every $p \in \pi'$.

Lemma 8: Let $\pi = \bigcup_{i=1}^{m} \pi_i$ and let G be a torsionfree group. If $G \otimes R_{\pi_i}$, $i = 1,2,\dots,m$, is factor-splitting, then G is factor-splitting.

Proof: Let M be an arbitrary basis of G, $N \subsetneq M$, $M \setminus N = \{a_\lambda \mid \lambda \in \Lambda\}$. Since the group $G \otimes R_{\pi_i}$ is factor-splitting, it follows from Theorem 1 that there are non-zero integers $m_\lambda^{(1)}$, $\lambda \in \Lambda$, such that every element from $\langle m_\lambda^{(1)} a_\lambda \mid \lambda \in \Lambda \rangle$ satisfies (FSp) with respect to N for all primes $p \in \pi_1$. If $m_\lambda^{(i)}$, $\lambda \in \Lambda$, are non-zero integers such that every element from $\langle m_\lambda^{(i)} a_\lambda \mid \lambda \in \Lambda \rangle$ satisfies (FSp) with respect to N for all primes $p \in \bigcup_{j=1}^{i} \pi_j$ then the above argument yields the existence of non-zero integers $m_\lambda^{(i+1)}$, $\lambda \in \Lambda$ such that every element from $\langle m_\lambda^{(i+1)} a_\lambda \mid \lambda \in \Lambda \rangle$ satisfies (FSp) with respect to N for all primes $p \in \bigcup_{j=1}^{i+1} \pi_j$. From this the assertion easily follows.

Theorem 3: Let G_1, G_2 be factor-splitting torsionfree groups such that both G_1 and G_2 are not p-divisible for a finite number of primes p only. Then the direct sum $G = G_1 \oplus G_2$ is factor-splitting.

Proof: Let π_1 be the set of all primes p for which the group G_1 is p-divisible and π_2 be the set of all primes p for which G_2 is p-divisible. By hypothesis, the set $\pi_3 = \pi \setminus (\pi_1 \cup \pi_2)$ is finite and $\pi = \pi_1 \cup \pi_2 \cup \pi_3$. Now the group $G_1 \otimes R_{\pi_1}$ is divisible and so $G \otimes R_{\pi_1} = (G_1 \otimes R_{\pi_1}) \oplus (G_2 \otimes R_{\pi_1})$ is factor-splitting by [10, Lemma 2]. The same argument shows that $G \otimes R_{\pi_2}$ is factor-splitting. Finally, $G \otimes R_{\pi_3}$ is almost divisible and so it is factor-splitting by Theorem 2 since $G/pG \cong G_1/pG_1 \oplus G_2/pG_2$ is finite for each prime p. Lemma 8 now finishes the proof.

R e f e r e n c e s

[1] L. BICAN: Mixed abelian groups of torsionfree rank one, Czech. Math. J. 20(1970), 232-242.

[2] L. BICAN: Factor-splitting abelian groups of rank two, Comment. Math. Univ. Carolinae 11(1970),1-8.

[3] L. BICAN: Splitting in abelian groups (to appear in Czech. Math. J.).

[4] L. BICAN: Factor-splitting abelian groups of finite rank, Comment. Math. Univ. Carolinae 17(1976), 473-480.

[5] L. BICAN: A splitting criterion for abelian groups, Comment. Math. Univ. Carolinae 19(1978),763-774.

[6] L. FUCHS: Abelian groups, Budapest, 1958.

[7] L. FUCHS: Infinite abelian groups I, II, Academic Press, 1970.

[8] L. PROCHÁZKA: Zametka o rasčepljajemosti smešannych abelevych grupp, Czech. Math. J. 10(1960), 479-492.

[9] L. PROCHÁZKA: O rasčepljajemosti faktor-grupp abelevych grupp bez kručenija konečnogo ranga, Czech. Math. J. 11(1961), 521-557.

[10] L. PROCHÁZKA: Zametka o faktorno rasčepljajemych abelevych gruppach, Čas. pěst. mat. 87(1962), 404-414.

[11] A.E. STRATTON: A splitting theorem for mixed abelian groups, Symposia Mathematica, Vol.XIII, Academic Press, London, 1974, 109-125.

Matematicko-fyzikální fakulta
Universita Karlova
Sokolovská 83, 18600 Praha 8
Československo

(Oblatum 30.6. 1978)