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A SOMEWHAT SURPRISING SUBSPACE OF $\beta N - N$

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Abstract: The purpose of this short note is to show that under some assumption on set theory there exists a linearly ordered topological space which can be densely embedded into $\beta N - N$.

Key words and phrases: Čech-Stone compactification, linearly ordered topological space, base matrix, Novák number.

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Denote, as usual, by N^* the space of all uniform ultrafilters on the countable discrete set, i.e. $N^* = \beta N - N$, the remainder of integers to their Čech-Stone compactification.

If P is a dense-in-itself topological space, call n(P), the Novák number of P, the least cardinality of a family of nowhere dense sets which covers P.

Writing c for the cardinality 2^ω, we can state the main result of the present paper as follows:

Theorem. Suppose $n(N^*) > c$. Then there exists a linearly ordered topological space which can be embedded as a dense subspace into N^*.

The proof of this theorem turns out to be an easy exercise on a machinery developed in [BPS]. Let us summarize several notions and facts from that paper, which will be needed further on.

<u>Definition</u>. A family $\mathcal{G} \subset \mathrm{Open}(N^*)$ is an almost-partition of N^*, if $\mathcal{G}$ is pairwise disjoint and $\bigcup\mathcal{G}$ is dense in N^*. If $\mathcal{G}$ and $\mathcal{H}$ are almost-partitions of N^*, then $\mathcal{G}$ refines $\mathcal{H}$ ($\mathcal{G} \prec \mathcal{H}$) if for each $G \in \mathcal{G}$ there is an $H \in \mathcal{H}$ with $G \subset H$.

The family $\Theta \subset \mathcal{P}(\mathrm{Open}(N^*))$ is called a matrix, if each member of Θ is an almost-partition of N^*.

A matrix Θ is shattering, if for each non-void open set $U \subset N^*$ there is some $\mathcal{G} \in \Theta$ such that U meets at least two members of $\mathcal{G}$.

A matrix Θ is a base-matrix, if the ordering $\succ$ well-orders the whole Θ and if $\bigcup\Theta$ is a π-base for N^*.

Given two matrices Θ and Θ', we shall say that Θ' strongly refines Θ ($\Theta' \prec\prec \Theta$) if there is a bijection $b: \Theta \longrightarrow \Theta'$ such that $b(\mathcal{G}) \prec \mathcal{G}$ for each $\mathcal{G} \in \Theta$.

If Θ is a matrix, call a family $\mathcal{C}$ to be a chain in Θ, if $\mathcal{C}$ is centered, contained in $\bigcup\Theta$ and maximal with respect to those two properties. If $|\mathcal{C}| = |\Theta|$, then the chain $\mathcal{C}$ is called long.

The cardinal number $\varkappa(N^*)$ is defined as $\min\{|\Theta| : \Theta$ is a shattering matrix in $N^*\}$.

<u>Fact 1</u>. ([BPS], 2.11(c)) For each shattering matrix Θ with $|\Theta| = \varkappa(N^*)$ there exists a base-matrix Θ' such that $|\Theta'| = \varkappa(N^*)$, $\Theta' \prec\prec \Theta$ and $\bigcup\Theta' \subset \mathrm{Clopen}(N^*)$.

<u>Fact 2</u>. ([BPS], 3.5(iii)) $n(N^*) > c$ if and only if $\varkappa(N^*) = c$ and each shattering matrix Θ, $|\Theta| = c$, con-

tains a long chain.

Proof of the Theorem. Well-order the family of all clopen subsets of N^* : $\mathrm{Clopen}(N^*) = \{H_\xi : \xi < c\}$. Let $\mathcal{G}_\xi = \{H_\xi, N^* - H_\xi\}$. The matrix $\theta = \{\mathcal{G}_\xi : \xi < c\}$ is clearly shattering. Assuming $n(N^*) > c$, we have $|\theta| = c = \varkappa(N^*)$ by Fact 2, hence according to Fact 1, there is a base-matrix θ' with $|\theta'| = c$, $\theta' \prec\prec \theta$, $\bigcup \theta' \subset \mathrm{Clopen}(N^*)$.

Let us write $\theta' = \{\mathcal{V}_\xi : \xi < c\}$; we may assume without any loss of generality that $\mathcal{V}_0$ is infinite, $\mathcal{V}_\xi \prec \mathcal{V}_\eta$ whenever $\eta < \xi < c$ and that $|\{W \in \mathcal{V}_\xi : W \subset V\}| \geq \omega$ whenever $\eta < \xi < c$, $V \in \mathcal{V}_\eta$.

Using Fact 2, we know that there are long chains in θ'. If $\mathcal{C}$ is such a chain and if H is a clopen subset of N^*, then by the choice of θ there is some $C \in \mathcal{C}$ such that either $C \subset H$ or $C \subset N^* - H$ holds. Thus $|\bigcap \mathcal{C}| = 1$. We shall show that the set

$$D = \{x \in \bigcap \mathcal{C} : \mathcal{C} \text{ is a long chain in } \theta'\}$$

is the desired subspace.

D is dense in N^*. Let U be a non-void open subset of N^*. Since $\bigcup\theta'$ is a π-base for N^*, there is some $\xi < c$ and some non-void $V \in \mathcal{V}_\xi$ with $V \subset U$. Consider the family θ_V consisting of all $\mathcal{W}_\eta$ $(\xi < \eta < c)$, where $\mathcal{W}_\eta = \{W \in \mathcal{V}_\eta : W \subset V\}$. Obviously θ_V is a shattering matrix for V, but V being a clopen subset of N^* is homeomorphic to N^*, hence θ_V contains a long chain $\mathcal{C}_V$. Let $\mathcal{C}$ be a maximal chain in θ' such that $\mathcal{C} \supset \mathcal{C}_V$. Then $\mathcal{C}$ is long and $\bigcap \mathcal{C} \subset V \subset U$. We have proved that U meets D, but U was chosen arbitrarily, thus D is dense in N^*.

D <u>can be linearly ordered</u>. (The basic idea of the following teehnique was first used in [M].) For $x \in D$, denote by $\mathcal{C}_x$ the long chain in θ' with $\bigcap \mathcal{C}_x = \{x\}$. Before defining an order for D, we shall order each $\mathcal{V}_\xi$ as follows:

Let $<_0$ be a linear ordering of $\mathcal{V}_0$ without the first or the last element. Proceeding by the transfinite induction, let $\xi < c$ and suppose that every $\mathcal{V}_\eta$ ($\eta < \xi$) is ordered by $<_\eta$ in such a manner that if $\xi < \eta < \xi$, $V, W \in \mathcal{V}_\xi$, V', $W' \in \mathcal{V}_\eta$, $V' \subset V$, $W' \subset W$ and if $V <_\xi W$, then $V' <_\eta W'$.

Call two members V, W of $\mathcal{V}_\xi$ to be equivalent if for each $\eta < \xi$ there is some $\mathcal{U} \in \mathcal{V}_\eta$ such that $V \cup W \subset U$. Order every equivalence class E by $<_E$ such that $(E, <_E)$ has neither the first nor the last element. Having done this, we may define an order $<_\xi$ by the rule

$V <_\xi W$ iff either V is equivalent to W and $V <_E W$ or there is some $\eta < \xi$ and $V', W' \in \mathcal{V}$ such that $V \subset V'$, $W \subset W'$ and $V' <_\eta W'$.

Finally, for $x, y \in D$ define $x < y$ iff for some $\xi < c$, $V_x \in \mathcal{C}_x \cap \mathcal{V}_\xi$, $V_y \in \mathcal{C}_y \cap \mathcal{V}_\xi$, $V_x <_\xi V_y$ holds.

It is easy to check that $<$ is a linear ordering of D.

If $]x,y[$ is an interval-neighborhood of a point $z \in D$, then there is some $\xi < c$ such that the sets $\mathcal{C}_x \cap \mathcal{V}_\xi$, $\mathcal{C}_y \cap \mathcal{V}_\xi$, $\mathcal{C}_z \cap \mathcal{V}_\xi$ are distinct. Let $V \in \mathcal{C}_z \cap \mathcal{V}_\xi$. Then $z \in V$ and $V \cap D \subset\ ]x,y[$.

If U is a neighborhood of a point $z \in D$, then there is some $\xi < c$ such that for $V \in \mathcal{C}_z \cap \mathcal{V}_\xi$, V is contained in U. Obviously the family $\{W \in \mathcal{V}_{\xi+1} : W \subset V\}$ is an equivalence class in $\mathcal{V}_{\xi+1}$ and the order $<_E$ has not the first and the last element. It follows that one can choose three clopen sets $W_1, W_2, W_3 \in \{W \in \mathcal{V}_{\xi+1} : W \subset V\}$ such that $W_1 <_{\xi+1} W_2 <_{\xi+1}$ $<_{\xi+1} W_3$ and $W_2 \in \mathcal{C}_2 \cap \mathcal{V}_{\xi+1}$. Pick two long chains $\mathcal{C}_1$, $\mathcal{C}_3$ with $W_1 \in \mathcal{C}_1 \cap \mathcal{V}_{\xi+1}$ and $W_3 \in \mathcal{C}_3 \cap \mathcal{V}_{\xi+1}$, let $\{x\} =$ $= \bigcap \mathcal{C}_1$, $\{y\} = \bigcap \mathcal{C}_3$. Then $z \in]x,y[\subset V \cap D \subset U \cap D$.

We have proved that the order-topology of D coincides with its subspace topology, which completes the proof.

Remarks. (a) The assumption $n(N^*) > c$ holds e.g. if V = L, if CH holds or if MA is true. The situation under the assumption of MA is somewhat simpler, since then all chains in Θ' have to be long. By a simple modification of the given proof (use well-ordering in the induction on each stage where the linear ordering without the first and last element was needed) one can show that under MA, N^* contains a densely embedded copy of ${}^c c$ with the lexicographical order.

(b) Each point of the linearly ordered subset constructed in the proof was a P(c)-point in N^*. One can moreover require it to be selective. This is possible, but it is necessary to start the proof with a more careful choice of the matrix Θ .

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