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CONDITIONS FOR LOCAL ASYMPTOTIC NORMALITY OF EXPERIMENT SEQUENCES

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Abstract: Conditions concerning L_2-differenciability of square roots of densities ensuring local asymptotic normality of the sequence of independent experiments are presented to generalize the result of Roussas [2]. Under a little stronger conditions the asymptotic linearity of the mentioned derivations is proved.

Key words and phrases: Experiment sequence, local asymptotic normality, likelihood ratio, differenciability in the second mean.

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In this article only independent experiments will be treated, which means that the following scheme will be used:

$$(\mathcal{X}_m, \mathcal{A}_m, P(\cdot\,,t)) = (\prod_1^m E_q, \prod_1 \mathcal{B}_q\,, \prod_1^m P_k(\cdot\,,t)),\quad t \in E_r,$$

moreover we shall assume that probability measures $P_k(t)$ are absolutely continuous with respect to a σ-finite measure m in a neighbourhood of some parametric point t_o:

$$\frac{dP_k(t)}{dm} = p_k(t), \qquad t \in U(t_o).$$

We call a sequence of experiments locally asymptotically normal (LAN) when the limiting distribution of likelihood ratios $\log \frac{dP_n(t+h/\sqrt{n})}{dP_n(t)}$ is Gaussian. Such a property is utilized when the efficiency of estimates or test criteria are examined (see for instance Hájek [1]). Therefore the main goal of this paper is to find general conditions implying LAN in the case of the described independent experiments.

The regularity conditions for LAN of the sequence under consideration are to be expressed in terms of roots of densities $p_k(t)$, say

$$s_k(t) = (p_k(t))^{1/2}$$

treated as members of the space $L^2(m)$ of square integrable functions. More precisely, our conditions concern Fréchet differenciability in $L^2(m)$ of these roots:

Condition A1: $s_k(t)$ are Fréchet differenciable in parametrical point t_o, i.e.

$$\lim_{\|h\|\to 0} \|h\|^{-2} \int (s_k(t_o+h) - s_k(t_o) - h' d_k(t_o))^2 dm = 0,$$

uniformly in $k = 1,2,..$

Condition A2. Putting

$$G_k = \langle d_k(t_o), d_k(t_o) \rangle_{L^2(m)},$$

then there exists

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} G_k = G.$$

Condition A3.

$$\lim_{n\to\infty} (nG_{[j,j]})^{-1} \sum_{k=1}^{n} \int_{A_{nk}} D^2_{k[j]} dP_k(t_o) = 0, \quad 1 \le j \le r,$$

where $D_{k[j]}$ is the j-th component of

$$D_k = d_k s_k^{-1} \text{ for } s_k > 0$$
$$= o \qquad \text{for } s_k = 0$$

and $A_{nk} = \{x : D^2_{k[j]} \geq n\, G_{[j,j]}\, \varepsilon\}$, $\varepsilon > 0$.

Condition A4. Derivations $d_k(t)$ exist in a neighbourhood of t_o and are $L^2(m)$-continuous at the point t_o uniformly for $k = 1,2,\ldots$.

Let

$$C_{nh} = n^{-1/2} \sum_{k=1}^{n} D_k(t_o + n^{-1/2}h)$$

$$P_{nh} = \prod_{k=1}^{n} P_k(t_o + n^{-1/2}h).$$

Now, we are in position to state our main results.

Proposition 1. Assume that A1,A2,A3 hold. Then for every bounded sequence $\{h_n\}$ is

$$(1) \quad \log\frac{dP_{nh_n}}{dP_{no}} - 2h_n'C_{no} + 2h_n'Gh_n \longrightarrow 0$$

in P_{no} Probability. Moreover, if $h_n \longrightarrow h$, then

$$(2) \quad L\left(\log \frac{dP_{nh_n}}{dP_{no}} \Big/ P_{no}\right) \longrightarrow N(-2h'Gh, 4h'Gh)$$

$$(3) \quad L\left(\log \frac{dP_{nh_n}}{dP_{no}} \Big/ P_{nh_n}\right) \longrightarrow N(2h'Gh, 4h'Gh)$$

$$(4) \quad L\left(C_{no} / P_{no}\right) \longrightarrow N_r(o, G)$$

$$(5) \quad L\left(C_{no} / P_{nh_n}\right) \longrightarrow N_r(2Gh, G),$$

where $L(X/P)$ denotes the distribution of a random vector X with respect to a probability measure P.

Proposition 2. Assume that A1-A4 hold. Then for every bounded sequences $\{h_n\}$, $\{g_n\}$ is

(6) $\quad \| C_{nh_n} - C_{no} + 2Gh_n \| \longrightarrow 0$ in P_{no}-probability

and

(7) $$L\left(\begin{pmatrix} C_{no} \\ C_{nh_n} \end{pmatrix} \Big| P_{no}\right) \longrightarrow N_{2r}\left(\begin{pmatrix} o \\ -2Gh \end{pmatrix}, \begin{pmatrix} G & G \\ G & G \end{pmatrix}\right)$$

(8) $$L\left(\begin{pmatrix} C_{no} \\ C_{nh_n} \end{pmatrix} \Big| P_{ng_n}\right) \longrightarrow N_{2r}\left(\begin{pmatrix} 2Gg \\ 2G(g-h) \end{pmatrix}, \begin{pmatrix} G & G \\ G & G \end{pmatrix}\right)$$

whenever $\{h_n\}$, $\{g_n\}$ are convergent sequences in E_r with limits h, g, respectively.

Assertions of Proposition 1 are proved by Roussas-Philippou (1973) under a little bit stronger and not so compact assumptions. A close examination of their proofs leads immediately to the verification of our Proposition 1, the proof of which will be therefore omitted here. Our main goal is Proposition 2. In what follows we present its proof.

Relation (2) implies that sequences $\{P_{nh_n}\}$, $\{P_{no}\}$ are contiguous (see Roussas 1972, Theorem 3.1). It will be convenient to construct a sequence of probability measures, say $\{R_n\}$, also contiguous with $\{P_{no}\}$. Let R_n be defined by

$$\frac{dR_n}{dP_{no}} = a_n \prod_k s_{kh_n} s_{kg_n}$$

where

$$a_n^{-1} = \iint \prod_k s_{kh_n} s_{kg_n} dm = \iint \prod_k \frac{1}{2} [s_{kh_n}^2 + s_{kg_n}^2 - \\ - (s_{kh_n} - s_{kg_n})^2] \, dm.$$

It is possible to assume, without loss of generality, that

$$h_n \longrightarrow h \quad \text{and} \quad g_n \longrightarrow g.$$

Then

$$a_n \longrightarrow \exp\,(h-g)'G(h-g)$$

and

$$\log \frac{dR_n}{dP_{no}} - \log a_n - \frac{1}{2}\log \frac{dP_{nh_n}}{dP_{no}} - \frac{1}{2}\log \frac{dP_{ng_n}}{dP_{no}} \longrightarrow 0$$

in P_{no}-probability. These relations, together with (1) and (4), give

$$(9) \qquad L\,(\log \frac{dR_n}{dP_{no}} \mid P_{no}) \longrightarrow N(-\tfrac{1}{2}(h+g)'G(h+g), (h+g)'g(h+g))$$

and, consequently, $\{R_n\}$, $\{P_{no}\}$ are contiguous sequences again according to Roussas (1972).

The contiguity of sequences $\{R_n\}$,$\{P_{nh_n}\}$,$\{P_{no}\}$ provides a substantial simplification because we need not distinguish among them when proving convergence in probability. This convenient tool will be now used for checking following asymptotic relations.

$$(10) \quad \sum \frac{s_{kh_n} - s_{ko}}{s_{kh_n}} - \sum \frac{s_{kh_n} - s_{ko}}{s_{kg_n}} - h_n'G(g_n - h_n) \longrightarrow 0$$

$$(11) \quad \sum \frac{s_{kh_n} - s_{ko}}{s_{kh_n}} - h_n'C_{ng_n} - \frac{1}{2}h_n'G(4g_n - 3h_n) \longrightarrow 0$$

in probabilities R_n, P_{nh_n}, P_{no}, where $\{h_n\}$, $\{g_n\}$ are bounded.

The convergence (10) will be verified with respect to underlying probability measure R_n. The statistics

$$U_n = \sum_{k=1}^{n} U_{nk} = \sum \left(\frac{s_{kh_n} - s_{ko}}{s_{kh_n}} - \frac{s_{kh_n} - s_{ko}}{s_{kg_n}} \right) =$$

$$= \sum_{k=1}^{n} (s_{kh_n} - s_{ko})(s_{kg_n} - s_{kh_n})(s_{kh_n}\, s_{kg_n})^{-1}$$

have finite means and it is easy to prove that

(12) $$E_{R_n} U_n - h_n' G(g_n - h_n) \longrightarrow 0.$$

Put

$$U_{nk}^{c} = U_{nk} \quad |U_{nk}| < c$$
$$= 0 \quad |U_{nk}| \geq c$$

for some $c > 0$.

Now, to prove (10), it is necessary (and sufficient) to show that

(13) $$P_{no}(U_n \neq \sum U_{nk}^{c}) \longrightarrow 0$$

(14) $$E\, U_n - E \sum U_{nk}^{c} \longrightarrow 0$$

and

(15) $$P_{no}(|\sum U_{nk}^{c} - E \sum U_{nk}^{c})| > \varepsilon) \longrightarrow 0, \quad \varepsilon > 0.$$

The first task is somewhat formally complicated. We may write

$$P_{no}(U_n \neq \sum U_{nk}^{c}) \leq P_{no}(\max_k |U_{nk}| \geq c \wedge \min_k s_{kh_n} s_{kg_n} s_{ko}^{-2} \geq \tfrac{1}{2}) +$$

$$+ P_{no}(\min_k s_{kh_n} s_{kg_n} s_{ko}^{-2} \leq \tfrac{1}{2}) \leq P_{no}(|(s_{kh_n} - s_{ko})(s_{kh_n} - s_{kg_n})|$$

$$|s_{ko}^{-2} \geq c) + P_{no}(\max_k |s_{kh_n} s_{kg_n} s_{ko}^{-2} - 1| \geq \tfrac{1}{2}) \leq$$

$$\leq \sum_{k=1}^{n} P_{ko}(|\, s_{ko}^{-2}(s_{kh_n} - s_{ko})[(s_{kg_n} - s_{ko} - n^{-1/2} g_\nu' d_\nu) -$$

$$-(s_{kh_n} - s_k - n^{-1/2}h_n'd_k)]| \geq \tfrac{c}{6}) +$$

$$+\sum_{k=1}^{n} P_{ko}(|(s_{kh_n} - s_{ko} - n^{-1/2}h_n'd_k)n^{-1/2}d_k'(g_n - h_n)s_{ko}^{-2}| \geq \tfrac{c}{6})$$

$$+\sum_{k=1}^{n} P_{ko}(n^{-1}|h_n'd_kd_k'(g_n - h_n)| \geq \tfrac{c}{6}) +$$

$$P_{no}(\max_k |s_{kh_n}\, s_{kg_n}\, s_{ko}^{-2} - 1| \geq \tfrac{1}{2}) \leq \tfrac{6}{c}\,[\Sigma \int (s_{kh_n} - s_{ko})^2 dm]^{1/2}.$$

$$\cdot \{[\Sigma \int (s_{kh_n} - s_{ko} - n^{-1/2}h_n'd_k)^2 dm]^{1/2} +$$

$$+ [\Sigma \int (s_{kg_n} - s_{ko} - n^{-1/2}g_n'd_k)^2 dm]^{1/2}\} +$$

$$+ \tfrac{6}{c}[\Sigma \int (s_{kh_n} - s_{ko} - n^{-1/2}h_n'd_k)^2 dm]^{1/2}\,.$$

$$\cdot [\Sigma \int (d_k'(g_n - h_n))^2 dm\; n^{-1}]^{1/2} +$$

$$+ \frac{6}{cn}\Sigma \int_{\{|h_n'd_kd_k'(g_n - h_n)| \geq \frac{cn}{6}s_{ko}^2\}} |h_n'd_kd_k'(g_n - h_n)|\, dm +$$

$$+ P_{no}(\max_k |s_{kh_n}s_{kg_n}s_{ko}^{-2} - 1| \geq \tfrac{1}{2}).$$

But $\Sigma \int (s_{kh_n} - s_{ko})^2 dm$ and $\Sigma \int (d_k'(g_n - h_n))^2 n^{-1} dm$ are bounded while sums of the type $\Sigma \int (s_{kh_n} - s_{ko} - n^{-1/2}h_n'd_k)^2 dm$ tend to zero and the last sume of integrals does the same according to condition A3. Therefore it is sufficient to prove that

$$P_{no}(\max_{k=1,..,n} |s_{kh_n}\, s_{kg_n}\, s_{ko}^{-2} - 1| \geq \tfrac{1}{2}) \longrightarrow 0.$$

This probability, however, is not larger than

$$P_{no}(\max_k |s_{ko}^{-2}(s_{kh_n}s_{kg_n} - s_{kh_n} s_{ko})| \geq \tfrac{1}{4}) +$$

$$+ P_{no}(\max_k | s_{kh_n}s_{ko}^{-1} - 1| \geq \tfrac{1}{4}) \leq P_{no}(\max_k | s_{kg_n}s_{ko}^{-1} - 1| \geq \tfrac{1}{8}) +$$

$$+ P_{no}(\max_k | s_{ko}s_{kh_n}^{-1} - 1| \geq \tfrac{1}{2}) + P_{no}(\max_k | s_{kh_n} s_{ko}^{-1} - 1| \geq \tfrac{1}{4}) =$$

$$= A_n + B_n + C_n.$$

For A_n we have

$$A_n = P_{no}(\max_k | s_{kg_n}s_{ko}^{-1} - 1| \geq \tfrac{1}{8}) \leq 64 \ [\Sigma \int (s_{kg_n} - s_{ko} -$$

$$- n^{-1/2}g_n' d_k)^2 dm]^{1/2} + n^{-1}\Sigma \int_{\{d_k' g_n \geq n s_{ko}^2/64\}} (g_n' d_k)^2 dm \rightarrow$$
$$\rightarrow 0$$

and the behaviour of B_n and C_n is evidently the same when P_{nh_n}, P_{no} are used, respectively.

Now, we show that the difference between means $E\,U_n$ and $E\Sigma\, U_{nk}^c$ with respect to R_n-probability is asymptotically negligible.

$$| EU_n - E\Sigma\, U_{nk}^c | \leq E\,| U_n - \Sigma\, U_{nk}^c | \leq \int_{\{\max|U_{nk}| \geq c\}} \cdots \int a_n \Sigma |$$
$$|(s_{kh_n} - s_{ko})(s_{kh_n} - s_{kg_n})| \, dm \leq$$
$$\leq a_n \Sigma \int_{\{|U_{nk}| \geq c\}} |(s_{kh_n} - s_{ko})(s_{kh_n} - s_{kg_n})| \, dm \leq$$
$$\leq a_n [\Sigma \int (s_{kh_n} - s_{kg_n})^2 dm]^{1/2}$$

$$\cdot \left\{ \left[\sum \int (s_{kh_n} - s_{ko} - n^{-1/2} h_n' d_k)^2 dm \right]^{1/2} + \right.$$

$$\left. + \left[n^{-1} \sum \int_{\{|U_{nk}| \geq c\}} (h_n' d_k)^2 dm \right]^{1/2} \right\} = A_n(B_n + C_n).$$

It is easy to see that the A_n are bounded and the sequence $\{B_n\}$ tends to zero. Moreover, we may write

$$C_n^2 = n^{-1} \sum \int_{\{|U_{nk}| \geq c\}} (h_n' d_k)^2 dm \leq n^{-1} \sum \int_{\{|U_{nk}| \geq c\}} n s_{ko}^2 dm + n^{-1} \sum \int_{\{(h_n' d_k)^2 \geq n s_{ko}^2\}} (h_n' d_k)^2 dm .$$

But the first term is equal to

$$\sum P_{no} \left\{ \left| (s_{kh_n} - s_{ko})(s_{kh_n} - s_{kg_n})(s_{kh_n} s_{kg_n})^{-1} \right| \geq c \right\}$$

and therefore tends to zero as it has been proved previously. The second one also tends to zero according to A3. Hence $C_n \longrightarrow 0$ and (14) holds. The simple fact that Var $\sum U_{nk}^c \longrightarrow$ $\longrightarrow 0$ leads us to the verification of (15).

To show that (11) is implied by (10) it is sufficient to prove that

$$\sum \frac{s_{kh_n} - s_{ko}}{s_{kg_n}} - h_n' C_{ng_n} - \frac{1}{2} h_n' G(2g_n - h_n) \longrightarrow 0$$

which is obvious when considering P_{ng_n} as the underlying probability measure. Now, using (10) and (11) with $g_n = 0$ we have

$$h'(C_{ng_n} - C_{no} + 2Gg_n) \longrightarrow 0$$

for each bounded sequence $\{h_n\}$, from which (6) follows readily. Finally, the asymptotic normality in (7) and (8) is implies by (6) and Proposition 1.

R e f e r e n c e s

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