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ON SYSTEMS OF GRAPHS INTERSECTING IN PATHS

V. RÖDL, Praha

Abstract: Let $\mathcal{K}$ be a class of graphs. Let $f(n,\mathcal{K})$ denote the largest number so that there exist graphs $G_1, G_2, \dots, G_{f(n,\mathcal{K})}$ with vertex set $V(|V| = n)$ every two of them intersect in a graph which belongs to $\mathcal{K}$. In this note we prove that $c_1 n^5 \le f(n,\mathcal{K}) \le c_2 n^5$ for $\mathcal{K} = \mathcal{P}$ = class of all paths. Our result gives an answer to a question of Prof. V.T. Sós.x)

Key words: Graph, path, cycle.

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Notation: A graph G on a set of vertices V is a subset of $[V]^2$. The elements of a G are called edges. If $x \in V$, degree of x is $d(x) = |\{y; y \in V, \{x,y\} \in G\}|$. The degree of a graph G with the vertex set V is defined as the $D(G) = \max_{x \in V} d(x)$. Let $X = \{x_1, x_2, \dots, x_k\}$ be a k-element set. By a path of length $k - 1 \ge 2$ we understand a set $\{\{x_1,x_2\}, \{x_2,x_3\}, \dots, \{x_{k-1},x_k\}\}$. The path of length 1 is $\emptyset$. The cycle of length $k \ge 2$ is a set $\{\{x_1,x_2\}, \{x_2,x_3\}, \dots, \{x_{k-1},x_k\}, \{x_k,x_1\}\}$.
The cycle of length k = 1 is $\emptyset$. For $k \ge 2$ a k - 1 star is

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x) The same result was obtained independently by V.T. Sós and M. Simonovits and it is going to be published in Proceedings of the conference in Orsay.

defined as a set $\{\{x_1,x_2\},\{x_1,x_3\},\dots,\{x_1,x_k\}\}$

Results:

I. - Upper bound.

I.1 There are at most $k\binom{n}{k}$ graphs $G_1,G_2,\dots G_p$ with the following properties:

1) $D(G_i) \geq k - 1$ for every $i \in \{1,2,\dots,p\}$

2) $G_i \subset [V]^2$ for every $i \in \{1,2,\dots,p\}$ where $|V| = n$

3) $D(G_i \cap G_j) < k - 1$ for every $i,j \in \{1,2,\dots,p\}$ $i \neq j$

Proof: There exist $k\binom{n}{k}$ $(k - 1)$-stars on the n-point set. From this follows that for every system $\{G_1,G_2,\dots G_p\}$ $(p > \binom{n}{k}k)$ with the properties 1) and 2) there exists $(k - 1)$-star S and $i,j \in \{1,2,\dots p\}$ $i \neq j$ so that $S \subset G_i$ and $S \subset G_j$ which contradicts 3).

I.2 Let $\mathcal{G}$ be a system of graphs with the following properties:

1) $D(G) \leq 2$ for every $G \in \mathcal{G}$

2) $G \cap G'$ is a path for every $G, G' \in \mathcal{G}$, $G \neq G'$

Then for every $\mathcal{G}' \subset \mathcal{G}$ with $|\mathcal{G}'| > 1$ the graph $\bigcap_{G \in \mathcal{G}'} G$ is a path.

Proof: Let $e, e' \in \bigcap_{G \in \mathcal{G}'} G$ and $e \cap e' = \emptyset$.
As $G \cap G'$ is a path for every $G, G' \in \mathcal{G}$ there must exist for every $G \in \mathcal{G}'$ a path $P_G = \{e_1,e_2,\dots e_{k(G)}\}$ such that $e_1 = e$, $e_{k(G)} = e'$, $e_i \cap e_{i+1} \neq \emptyset$ for $i = 1,2,\dots K(G) - 1$ and $P_G \subset G$. As $D(G) \leq 2$ for every $G \in \mathcal{G}' \subset \mathcal{G}$ we get $P_G = P'_G$ for all $G,G' \in \mathcal{G}'$. As every two disjoint edges in $\bigcap_{G \in \mathcal{G}'} G$ are joined by a path which is a subset of $\bigcap_{G \in \mathcal{G}'} G$ and $D(G) \leq 2$ for every $G \in \mathcal{G}'$ it follows that $\bigcap_{G \in \mathcal{G}'} G$ is a path.

I.3 Let $\mathcal{G}$ be a system of graphs with the following properties:

1) There exists V with $|V| = n$ so that $0 \neq G \subset [V]^2$ for every $G \in \mathcal{G}$

2) G is a path for every $G \in \mathcal{G}$

3) $G \cap G'$ is a path for every $G, G' \in \mathcal{G}$

Then $|\mathcal{G}| \leq \binom{\binom{n}{2}}{2} + \binom{n}{2}$

Proof: Let $P = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{k-1}, x_k\}\} \in \mathcal{G}$

$P = \{\{y_1, y_2\}, \{y_2, y_3\}, \ldots, \{y_{\ell-1}, y\}\} \in \mathcal{G}$

Let $\{x_1, x_2\} = \{y_1, y_2\}$ and $\{x_{k-1}, x_k\} = \{y_{\ell-1}, y\}$

From 3) follows that $P = P'$. So we have $|\mathcal{G}| \leq \binom{\binom{n}{2}}{2} + \binom{n}{2}$

I.4 Let $\mathcal{G}$ be a system of graphs with the following properties:

1) There exists V with $|V| = n$ so that $G \subset [V]^2$ for every $G \in \mathcal{G}$

2) G is connected for every $G \in \mathcal{G}$

3) $G \cap G'$ is a path for every $G, G' \in \mathcal{G}$

4) $D(G) \leq 2$ for every $G \in \mathcal{G}$

Then $|\mathcal{G}| \leq 2(n-2)\left[\binom{\binom{n}{2}}{2} + \binom{n}{2}\right] + \binom{n}{2}$

Proof: From I.2 I.3 and 4) it follows that

$$|\{G \cap G', G, G' \in \mathcal{G}\}| \leq \binom{\binom{n}{2}}{2} + \binom{n}{2} + 1$$

Define the sets $\mathcal{F}_i$ as follows:

$$\mathcal{F}_1 = \{G \cap G'; G, G' \in \mathcal{G} \wedge \forall H, H' \in \mathcal{G} \neg (G \cap G' \subsetneq H \cap H')\} - \{0\}$$

$$\mathcal{F}_{i+1} = \{G \cap G'; G, G' \in \mathcal{G} \wedge \forall H, H' \in \mathcal{G} \ ((H \cap H' \in \mathcal{G} - \bigcup_{j=1}^{i} \mathcal{F}_j) \rightarrow$$
$$\longrightarrow \neg (G \cap G' \subsetneq H \cap H'))\} - \{0\}$$

Denote by $\mathcal{E}'_i = \{G;\ G \in \mathcal{G}$ and $P \subset G$ for some $P \in \mathcal{F}_i\}$ and put $\mathcal{E}_1 = \mathcal{E}'_1$

$$\mathcal{E}_{i+1} = \mathcal{E}'_{i+1} - \mathcal{E}'_i$$

Let $P \in \mathcal{F}_i$. There are at most $2(n-2)$ graphs in $\mathcal{E}_i$ containing P (the graphs in $\mathcal{E}_i$ are connected)

As $\bigcup_i \mathcal{F}_i = \{G \cap G';\ G,G' \in \mathcal{G}\} - \{0\}$ and $\mathcal{G} - \bigcup \mathcal{E}_i$ consists of pairwise disjoint graphs we have

$$|\mathcal{G}| \le 2(n-2)\left[\binom{\binom{n}{2}}{2} + \binom{n}{2}\right] + \binom{n}{2}$$

I.5: Let $\mathcal{G}$ be a system of graphs satisfying the properties 1) 3) 4) from I.4

Then

$$|\mathcal{G}| \le 2(n-2)\left[\binom{\binom{n}{2}}{2} + \binom{n}{2}\right] + \binom{n}{2}\left(\binom{n-2}{2} + 1\right)$$

Proof: Denote by $\bar{\mathcal{G}}$ the subset of $\mathcal{G}$ consisting of all graphs which are not connected. Choose from every $G \in \bar{\mathcal{G}}$ one component $c(G)$. For every $e \in [V]^2$ the cardinality of the set $\mathcal{G}_e = \{G; G \in \bar{\mathcal{G}},\ e \in G - c(G)\}$ is at most $\binom{n-2}{2}$ which necessitates $|\bar{\mathcal{G}}| \le \binom{n}{2}\binom{n-2}{2}$. Thus we have

$$|\mathcal{G}| \le 2(n-2)\left[\binom{\binom{n}{2}}{2} + \binom{n}{2}\right] + \binom{n}{2}\left(\binom{n-2}{2} + 1\right)$$

II. Lower bound

II.1 Let V be a set with the n elements. There exists

(*) a system of graphs $\mathcal{G}$, every two of them intersect in a path such that $|\mathcal{G}| \ge \left(\frac{n}{5}\right)^5$

Proof: Take a partition $V = V_1 \cup V_2 \cup \ldots \cup V_5$ with

$$|V_i| = \left[\frac{n+i-1}{5}\right] \text{ for } i = 1,2,\ldots 5$$

Put $G \in \mathcal{G}$ iff α) G is isomorphic to C_5 (cycle of length 5) and β) $|V(G) \cap V_i| = 1$ for every $i = 1,2,\dots 5$ where V(G) is a vertex set of G.

It is easy to see that $\mathcal{G}$ satisfies ($*$).

(Iterating this procedure one can get slightly better result.)

Summarizing I.1 I.5 and II.1 we get the following

<u>Theorem</u> $\left(\frac{n}{5}\right)^5 \leq f(n,\mathcal{P}) \leq \binom{n}{2}\left[(n-2)\binom{n}{2} + \frac{5}{3}\binom{n-2}{2} + n - 1\right]$

III. - Concluding remarks:

III.1 It would be interesting to prove that $\lim_{n\to\infty} \frac{f(n,\mathcal{P})}{n^5}$ exists and determine it. From I.1 I.3 and I.5 it follows that we can restrict ourselves to cycles. The following can be shown easily:

($**$) The $\lim_{n\to\infty} \frac{f(n,\mathcal{P},5)}{n^5}$ exists where $f(n,\mathcal{P},5)$ denotes the maximal number of graphs isomorphic to C_5 intersecting in a path.

Proof: Obviously $f(n-1,\mathcal{P},5) \geq f(n,\mathcal{P},5)(1-\frac{5}{n})$

Elementary calculation gives that for $n > 5$ the sequence $\frac{f(n,\mathcal{P},5)}{n^5} \prod_{j=6}^{n} (1-\frac{25}{j^2})$ is decreasing. As $\lim_{n\to\infty} \prod_{j=6}^{n} (1-\frac{25}{j^2})$ exists, ($**$) holds.

III.2 Using similar methods as in I. and II. one can prove the following:

There exist positive constants d_1, d_2 such that $d_1 n^4 \leq f(n,C) \leq d_2 n^4$ where C is a class of all cycles and for every positive integer $\ell \geq 3$ there exist positive constants

$e_1(\ell)$, $e_2(\ell)$ such that

$$e_1(\ell)n^4 \le f(n, C(\ell) \cup \{0\}) \le e_2(\ell)n^4$$

where $C(\ell)$ denotes the class of all cycles of length ℓ .

R e f e r e n c e s :

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