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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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CLOSURE CONDITIONS IN THE NETS

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Abstract: The existence of the sum (product) of permutations of the coordinate algebra of a net with singular points on one line is equivalent to the diagonal (Reidemeister) closure condition of certain type. Some other closure conditions were found equivalent to the group properties of the operations mentioned.

Key words: Net, coordinate algebra, closure conditions.

AMS: 50A20 Ref. Ž.: 3.912

Introduction. In the theory of projective planes a lot is known about the connections between a validity of certain geometric conditions (frequently called closure conditions) and validity of certain algebraic conditions in the domain of coordinates of these projective planes or the existence of some types of automorphisms. In the theory of nets no such similar systematical theory is established hitherto, although there have been published numerous results having the character of one sided implication rather than systematical equivalence. Inspired by V. Havel's papers [1],[2],[3] I was looking for the simplest geometrical conditions such that algebraical conditions equivalent to them would form, in the whole, the known algebraic structures. Some examples will be shown in the author's next article.

I am indebted to V. Havel for calling my attention to this subject and for his permanent help and encouragement during the preparation of the present paper.

Part 1. *Definitions*

Definition: Under a *net* we shall understand a triple $(\mathcal{P}, \mathcal{L}, (V_\iota)_{\iota \in I})$, where $\mathcal{P}$ and I are non-void sets, $\mathcal{L}$ is a set of some at least two-element subsets of $\mathcal{P}$ and $\iota \longmapsto V_\iota$ is an injective mapping of I into $\mathcal{P}$ such that

(1) $v := \{ V_\iota \mid \iota \in I \} \in \mathcal{L}$

(2) $\forall P \in \mathcal{P} \setminus v \qquad \forall \iota \in I \qquad \exists ! \ell \in \mathcal{L} \qquad P, V_\iota \in \ell$

(3) $\#(a \cap b) = 1 \qquad \forall a, b \in \mathcal{L}, \qquad a \neq b.$

The elements of $\mathcal{P}$ are called *points*, the elements of $\mathcal{L}$ are called *lines*, the points V_ι, $\iota \in$ I are said to be *singular* or *improper*, the points of $\mathcal{P} \setminus v$ are said to be *proper*; the line v is said to be *singular* or *improper*; the lines of $\mathcal{L} \setminus \{ v \}$ are said to be *proper*; $\#$ I is called *degree* of the net, $\#(\ell \setminus v)$ (is the same for all $\ell \in \mathcal{L} \setminus \{ v \}$) is called *order* of the given net. The net $(v, \{ v \}, (V_\iota)_{\iota \in I})$ is said to be *trivial*.

In the sequel, we shall restrict ourselves to the study of nets with degree $\geqq 3$. If $A_1, A_2, \dots A_n \in \ell$ hold for $A_1, A_2, \dots A_n \in \mathcal{P}$, $\ell \in \mathcal{L}$, we shall write $\overline{A_1 A_2 \dots A_n}$; for $n = 2$, $A_1 \neq A_2$ we denote $\ell =: A_1 A_2$ and call ℓ the *join line* of A_1, A_2. If $P \in a, b$ holds for a distinct $a, b \in \mathcal{L}$ and for $P \in \mathcal{P}$, then we shall write $P =: a \sqcap b$ and we call P the *intersection point* of a, b. We shall write $X \in \mathcal{N}$ instead of $X \in \mathcal{P} \setminus v$, $p \in \mathcal{N}$ instead of $p \in \mathcal{L} \setminus \{ v \}$, $V_\iota \in \mathcal{N}$ instead of $V_\iota \in \mathcal{P}$, $v \in \mathcal{N}$ instead of $v \in \mathcal{L}$.

<u>Definition:</u> By <u>frame</u> of a net $\mathcal{N}$ we mean a quadruple $(0, \alpha, \beta, \gamma)$ where 0 is a proper point and $\alpha, \beta, \gamma \in I$, $\alpha \neq \beta \neq \gamma \neq \alpha$.

<u>Definition</u>: Under an <u>admissible algebra</u> we shall understand a quadruple $(S, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$, where S is a non-void set, 0 is a distinguished element of S, J is an index set with one prominent index θ , σ_ι is a permutation of S with $0^{\sigma_\iota} = 0$ and $+_\iota$ is a loop operation over S with neutral element 0 such that

(i) $\sigma_\theta = id_S$

(ii) $\forall \xi, \eta \in J,\ \xi \neq \eta \quad \forall b, c \in S \quad \exists !\, a \in S$

$$a^{\sigma_\xi} +_\xi b = a^{\sigma_\eta} +_\eta c.$$

<u>Construction 1.</u> Let $\mathcal{N}$ be a net with a frame $(0, \alpha, \beta, \gamma)$ then define an admissible algebra in such a way that:

$S := 0\,V_\alpha \setminus \{V\}$; $J := I \setminus \{\alpha, \beta\}$; $\theta := \gamma$; $\sigma_\iota : S \longrightarrow S$,

$x \longmapsto ((x\,V_\beta \sqcap 0\,V_\gamma)V_\alpha \sqcap 0\,V_\iota)V_\beta \sqcap 0\,V_\alpha \quad \forall \iota \in J$;

$a^{\sigma_\iota} +_\iota b = ((a\,V_\beta \sqcap 0\,V_\gamma)V_\alpha \sqcap b\,V_\iota)V_\beta \sqcap 0\,V_\alpha \quad \forall a, b \in S.$

This admissible algebra is called the <u>coordinate algebra of $\mathcal{N}$ with respect to the frame</u> $(0, \alpha, \beta, \gamma)$. A map $S \times S \longrightarrow \mathcal{P} \setminus v$ defined by the rule $(a,b) \longmapsto (a\,V_\beta \sqcap 0\,V_\gamma)V_\alpha \sqcap (b\,V_\beta)$ is said to be the <u>coordinate map</u>.

<u>Construction 2.</u> Let $(S, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$ be an admissible algebra $\# S > 1$, with a prominent index θ . Define $I := J \cup \{\omega_1, \omega_2\}$, where $\{\omega_1, \omega_2\} \cap J = \emptyset$, $\#\{\omega_1, \omega_2\} = 2$;

$\mathcal{P} := (S\times S)\cup I;\ v := I;$

$\mathcal{L} := \{\{(x,y) \mid x = a\} \cup \{\omega_2\} \mid a\in S\} \cup \{\{(x,y) \mid y = b\}\cup$
$\cup\{\omega_1\} \mid b\in S\}\cup\{\{(x,y) \mid y = x^{\sigma_\iota} +_\iota c\} \cup \{\iota\} \mid c \in$
$\in S,\ \iota \in J\}\cup\{v\}.$

Then $(\mathcal{P},\mathcal{L},(V_\iota)_{\iota\in J})$ is a net called the <u>net over admissible algebra</u> $(S,0,(\sigma_\iota)_{\iota\in J},(+_\iota)_{\iota\in J})$.

The proof are omitted here.

Part 2. <u>Closure conditions</u>

<u>Definition:</u> Let $\mathcal{N}$ be a net of degree $\geqq 4$ and $\alpha,\beta,\gamma,\delta$ be pairwise distinct elements of I. By the <u>Minor Desargues Condition</u> (denoted by MDC) <u>of type</u> $(\alpha,\beta,\gamma,\delta)$ in $\mathcal{N}$ we mean the following implication: $(\forall A,B,C,A',B',C' \in \mathcal{P}\setminus v)$ $(\overline{AA'V_\delta} \wedge \overline{BB'V_\delta} \wedge \overline{CC'V_\delta} \wedge \overline{ABV_\gamma} \wedge \overline{A'B'V_\gamma} \wedge \overline{ACV_\beta} \wedge \overline{A'C'V_\beta} \wedge$ $\wedge \overline{BCV_\alpha} \Longrightarrow \overline{B'C'V_\alpha})$.

If MDC of type $(\alpha,\beta,\gamma,\delta)$ in $\mathcal{N}$ holds for fixed $\delta \in$ $\in$ I and all distinct $\alpha,\beta,\gamma \neq \delta$ we shall say that <u>MDC of type</u> (δ) holds in $\mathcal{N}$. If MDC of type (δ) holds for all $\delta \in$ I in $\mathcal{N}$ we shall say that <u>MDC holds universally</u> in $\mathcal{N}$.

<u>Theorem 2.1.</u> Let $\mathcal{N}$ be a net of degree $\geqq 4$ with the frame $(0,\alpha,\beta,\gamma)$ and let MDC of type (α) hold in $\mathcal{N}$. Then in a coordinate algebra of $\mathcal{N}$ with respect to the frame $(0,\alpha,\beta,\gamma)$, $+_\gamma$ is a group operation and $+_\gamma = +_\iota$ for all $\iota \in I\setminus\{\alpha,\beta\}$. Under the previous assumptions, MDC of type (β) holds in the net $\mathcal{N}$ if and only if σ_ι is an automorphism of the group $(OV_\alpha\setminus\{V_\alpha\};+_\gamma) = (S,+)$ for all

$\iota \in I \setminus \{\alpha, \beta\}$.

Proof: see [1] page 21.

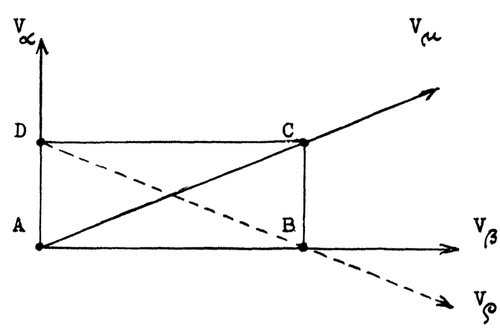
Definition: We denote $\Sigma = \{(\sigma_\iota)_{\iota \in J}\} \cup \{\sigma_\beta\}$, where the endomorphism σ_β is defined by $a^{\sigma_\beta} = 0 \quad \forall a \in S$. The sum $\sigma_\varphi + \sigma_\mu$ is defined for $\sigma_\varphi, \sigma_\mu \in \Sigma$ by the rule

$$a^{\sigma_\varphi + \sigma_\mu} := a^{\sigma_\varphi} + a^{\sigma_\mu} \qquad \forall a \in S.$$

Lemma 2.2: For every $\sigma_\mu \in \Sigma \quad \sigma_\beta + \sigma_\mu = \sigma_\mu + \sigma_\beta = \sigma_\mu$.

The proof is trivial.

Definition: Let $\mathcal{N}$ be a net of degree ≥ 4, and α, β, γ, φ pairwise distinct indexes of I. By diagonal condition (shortly DC) of type $(\alpha, \beta, \mu, \varphi)$ in $\mathcal{N}$ we shall mean the following implication: $(\forall A,B,C,D \in \mathcal{P} \setminus v)(\overline{ABV_\beta} \wedge \overline{CDV_\beta} \wedge \overline{ADV_\alpha} \wedge \overline{BCV_\alpha} \wedge \overline{ACV_\mu} \Longrightarrow \overline{BDV_\varphi})$.



Proposition 2.3: Let $\mathcal{N}$ be a net of degree ≥ 4, with a frame $(0, \alpha, \beta, \gamma)$ and let MDC of type (α) hold in $\mathcal{N}$. Then for $\mu \in I \setminus \{\alpha, \beta\}$ there is exactly one $\varphi \in I \setminus \{\alpha, \beta\}$ such that $x^{\sigma_\varphi} = -x^{\sigma_\mu} \quad \forall x \in S$ iff DC of type $(\alpha, \beta, \mu, \varphi)$,

restricted to $A = 0$, $A \neq B$, holds in $\mathcal{N}$.

Proof: Choose an element $0 \neq a \in S$ and let $A = 0$, $B = (a,0)$, $C = (a, a^{\sigma_\mu})$, $D = (0, a^{\sigma_\mu})$. These points are satisfying all assumptions of DC of type $(\alpha, \beta, \mu, \varphi)$ in $\mathcal{N}$ where $A = 0$ and $A \neq B$ hold. Then the conclusion of this DC BDV_φ holds iff the point $B = (a,0)$ is an element of the line $n =: DV_\varphi$ of the equation $y = x^{\sigma_\varphi} + a^{\sigma_\mu}$, i.e. iff $a^{\sigma_\varphi} = - a^{\sigma_\mu}$ $\forall a \in S$. If there is an index $\omega \neq \varphi$ such that $a^{\sigma_\omega} = - a^{\sigma_\mu}$, then the point $(a,0)$ is an element of the line DV_ω of the equation $y = x^{\sigma_\omega} + a^{\sigma_\mu}$. Hence there is a line m, $B,D,V_\omega \in m$, $m \neq n$ which contradicts the definition of the net. (If $B = D$ then $a = 0$ i.e. $A = B$.) Thus there is exactly one index φ of the previous property.

Corollary. Let $\mathcal{N}$ be a net of degree ≥ 4 with a frame $(0, \alpha, \beta, \gamma)$ and with a coordinate algebra $(S, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$ with respect to $(0, \alpha, \beta, \gamma)$, and let MDC of type (α) hold in $\mathcal{N}$. Then for each $\sigma_\mu \in \Sigma$ exactly one $\sigma_\varphi \in \Sigma$ exists such that

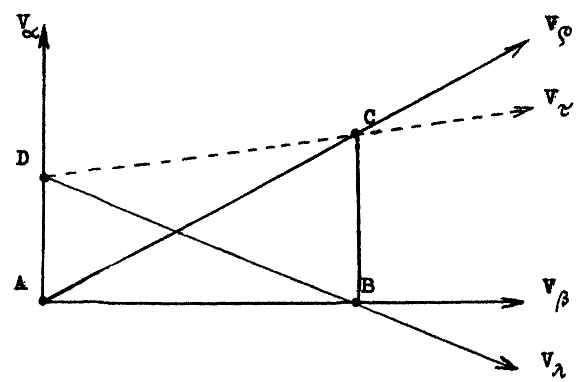
$$\sigma_\varphi + \sigma_\mu = \sigma_\beta = \sigma_\mu + \sigma_\varphi \quad ,$$

iff DC of type $(\alpha, \beta, \mu, \varphi)$ together with $A = 0$, $A \neq B$ holds in $\mathcal{N}$.

The proof follows from Proposition 2.3 and from the properties of the group $(S,+)$ (see [4] pp. 14 - 15).

Remark. The element σ_φ is said to be opposite to the element σ_μ ; let us write $\sigma_\varphi =: -\sigma_\mu$; $a^{\sigma_\beta} = a^{-\sigma_\mu + \sigma_\mu} = - a^{\sigma_\mu} + a^{\sigma_\mu} = 0$ holds provided that the element $-\sigma_\mu \in \Sigma$ exists.

Definition: Let $\mathcal{N}$ be a net of degree ≥ 5; let α, $\beta \in I$, $\alpha \neq \beta$; $\varrho, \lambda, \tau \in I$ all distinct from α, β. Then the Generalized diagonal condition (shortly GDC) of type $(\alpha, \beta, \varrho, \lambda, \tau)$ in $\mathcal{N}$ is defined as the following implication: $(\forall A,B,C,D \in \mathcal{P} \setminus v)\ (\overline{ABV_\beta} \wedge \overline{ADV_\alpha} \wedge \overline{BCV_\alpha} \wedge \overline{ACV_\varrho} \wedge \overline{BDV_\lambda} \Longrightarrow \overline{CDV_\tau})$.



Proposition 2.4: Let $\mathcal{N}$ be a net of degree ≥ 5 with a frame $(0, \alpha, \beta, \gamma)$ and with a coordinate algebra $(S, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$ with respect to this frame. Let MDC of type (α) hold in $\mathcal{N}$. Then for all $\varrho, \lambda \in I \setminus \{\alpha, \beta\}$ there exists exactly one $\tau \in I \setminus \{\alpha, \beta\}$ such that

$$x^{\sigma_\tau} = x^{\sigma_\varrho} + x^{\sigma_\lambda} \qquad \forall x \in S$$

iff GDC of type $(\alpha, \beta, \varrho, \lambda, \tau)$ with $A = 0$, $A \neq B$ holds in $\mathcal{N}$.

Proof: Choose an element $0 \neq a \in S$ and let $A = 0$, $B = (a, 0)$. Then $C = (a, a^{\sigma_\varrho})$, $D = (0, -a^{\sigma_\lambda})$ (the points D, B, V_λ are elements of a line of the equation $y = x^{\sigma_\lambda} - a^{\sigma_\lambda}$)

and the line CV_τ has the equation $y = x^{\sigma_\tau} - a^{\sigma_\tau} + a^{\sigma_\varphi}$. These points satisfy all assumptions of GDC where A = 0, $A \neq B$. The point D is an element of the line CV_τ iff

$$- a^{\sigma_\lambda} = - a^{\sigma_\tau} + a^{\sigma_\varphi} \quad \text{i.e.} \quad a^{\sigma_\tau} = a^{\sigma_\varphi} + a^{\sigma_\lambda} \qquad \forall a \in S.$$

The assumption that an index $\omega \neq \tau$ exists such that $a^{\sigma_\omega} = a^{\sigma_\varphi} + a^{\sigma_\lambda}$ leads to the contradiction with the definition of the net.

Corollary: Let $\mathcal{N}$ be a net of degree ≥ 5 with a frame $(0, \alpha, \beta, \gamma)$ and with a coordinate algebra $(S, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$ with respect to this frame. Let MDC of type (α) hold in $\mathcal{N}$. Then for all $\sigma_\varphi, \sigma_\lambda \in \Sigma$ there exists exactly one $\sigma_\tau \in \Sigma$ such that

$$\sigma_\tau = \sigma_\varphi + \sigma_\lambda$$

iff GDC of type $(\alpha, \beta, \varphi, \lambda, \tau)$ with A = 0, $A \neq B$ holds in $\mathcal{N}$. The proof follows from Proposition 2.4 and from the definition of the operation + in Σ .

Definition: Let $\mathcal{N}$ be a net of degree ≥ 4, let α, $\beta \in I$, $\alpha \neq \beta$. If for each $\mu \in I$, $\mu \neq \alpha$, $\mu \neq \beta$, a $\lambda \in I$ exists such that DC of type $(\alpha, \beta, \mu, \lambda)$ holds in $\mathcal{N}$ then we say that DC of type (α, β) holds in $\mathcal{N}$.

Let $\mathcal{N}$ be a net of degree ≥ 5, let $\alpha, \beta \in I$; $\alpha \neq \beta$. If for all φ , $\lambda \in I$, distinct from α, β, a $\tau \in I$ exists such that GDC of type $(\alpha, \beta, \varphi, \lambda, \tau)$ holds, then we say that GDC of type (α, β) holds in $\mathcal{N}$.

Proposition 2.5: Let $\mathcal{N}$ be a net of degree ≥ 5 with a frame $(0, \alpha, \beta, \gamma)$ and with a coordinate algebra $(S, 0, (\sigma_\iota)_{\iota \in J} (+_\iota)_{\iota \in J})$ with respect to this frame. Let MDC of type (α) hold in $\mathcal{N}$. If we denote $\Sigma = \{(\sigma_\iota)_{\iota \in J}\} \cup \cup \{\sigma_\beta\}$ (where $a^{\sigma_\beta} := 0 \quad \forall a \in S$) and define

$$+ : a^{\sigma_\rho + \sigma_\mu} := a^{\sigma_\rho} + a^{\sigma_\mu} \qquad \forall a \in S;\ \sigma_\rho, \sigma_\mu \in \Sigma ,$$

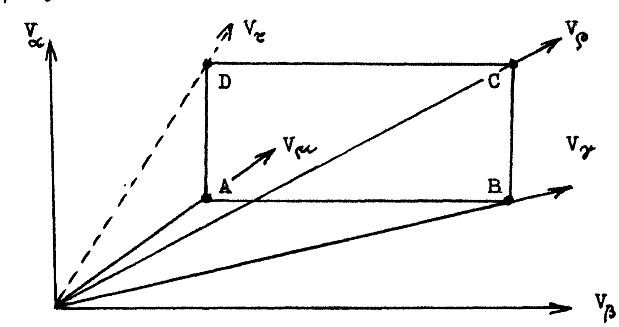
then DC of type (α, β) and GDC of type (α, β) together with $A = 0$, $A \neq B$ hold in $\mathcal{N}$ iff $(\Sigma, +)$ is a group.

The proof follows from Propositions 2.3 and 2.4. The validity of the associative law follows immediately.

<u>Definition:</u> Let $\mathcal{N}$ be a net of degree ≥ 3, $\alpha, \beta, \gamma, \mu, \rho, \tau \in I$, $\alpha \neq \beta \neq \gamma \neq \alpha$. By the <u>Reidemeister condition (denoted by RC) of type</u> $(\alpha, \beta, \gamma, \mu, \rho, \tau)$ in $\mathcal{N}$ we mean the implication:

$$(\forall A,B,C,D,H \in \mathcal{P} \setminus v)\ (\overline{HAV_\mu} \wedge \overline{HBV_\gamma} \wedge \overline{HCV_\rho} \wedge \overline{ABV_\beta} \wedge \overline{DCV_\beta} \wedge \wedge \overline{ADV_\alpha} \wedge \overline{BCV_\beta} \Longrightarrow \overline{HDV_\tau}).$$

If RC of type $(\alpha, \beta, \gamma, \mu, \rho, \tau)$ holds for fixed $\alpha, \beta, \gamma \in I$ and all $\mu, \rho, \tau \in I$ we shall say that RC <u>of type</u> (α, β, γ) holds in $\mathcal{N}$.



Proposition 2.6: Let $\mathcal{N}$ be a net of degree ≥ 3 with a frame $(0, \alpha, \beta, \gamma)$ and a coordinate algebra $(S, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$ with respect to this frame. Then for $\mu, \varphi \in I \setminus \{\alpha, \beta\}$; $\mu \neq \varphi$, exactly one $\tau \in I \setminus \{\alpha, \beta\}$ exists such that

$$(x^{\sigma_\mu})^{\sigma_\varphi} = x^{\sigma_\tau} \qquad \forall x \in S$$

iff RC of type (α, β, γ) with $H = 0$ holds in $\mathcal{N}$.

Remark: We denote $x^{\sigma_\mu \sigma_\varphi} := (x^{\sigma_\mu})^{\sigma_\varphi}$.

Proof: Choose an element $x \in OV_\alpha \setminus \{V_\alpha\}$ and denote $R := x V_\beta \sqcap OV_\gamma$, $A := RV_\alpha \sqcap OV_\mu$, $x^{\sigma_\mu} = AV_\beta \sqcap OV_\alpha$, $B := AC_\beta \sqcap OV_\gamma$, $C := BV_\alpha \sqcap OV_\varphi$, $(x^{\sigma_\mu})^{\sigma_\varphi} = CV_\beta \sqcap OV_\alpha$ (such an element exists in accordance with the definition). The mapping $\sigma_\tau : x \longmapsto x^{\sigma_\mu \sigma_\varphi}$ is a permutation provided that there is a point V_τ such that $\overline{ODV_\tau}$ for $D := RV_\alpha \sqcap CV_\beta$. But such a point V_τ exists iff RC of type (α, β, γ) with $H = 0$ holds.

Proposition 2.7: Let $\mathcal{N}$ be a net of degree ≥ 3 with a frame $(0, \alpha, \beta, \gamma)$ and with a coordinate algebra $(S, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$ with respect to this frame. Then for $\mu \in I \setminus \{\alpha, \beta\}$, exactly one index $\varphi \in I \setminus \{\alpha, \beta\}$ exists such that

$$(x^{\sigma_\mu})^{\sigma_\varphi} = (x^{\sigma_\varphi})^{\sigma_\mu} = x \qquad \forall x \in S$$

iff RC of type $(\alpha, \beta, \gamma, \mu, \varphi, \gamma)$ with $H = 0$ holds in $\mathcal{N}$.

Proof: Choose an element $x \in OV_\alpha \setminus \{V_\alpha\}$ and let $A := x V_\beta \sqcap OV_\gamma$, $D := AV_\alpha \sqcap OV_\mu$ then $x^{\sigma_\mu} = DV_\beta \sqcap OV_\alpha$. Let

$C := x^{\sigma_\mu} V_\beta \sqcap OV_\gamma$, $B := CV_\alpha \sqcap OV_\varphi$; the point $(x^{\sigma_\mu})^{\sigma_\varphi} =$ $= BV_\beta \sqcap OV_\alpha$ is equal to the point x iff for the points O,A,B,C,D of $\mathcal{N}$, RC of type $(\alpha, \beta, \gamma, \mu, \varphi, \gamma)$ holds. The second part of the proposition follows similarly.

Remark: The permutation σ_φ from Proposition 2.7 is said to be inverse to σ_μ and will be denoted by σ_μ^{-1}. Clearly if $\sigma_\varphi = \sigma_\mu^{-1}$ then $\sigma_\mu = \sigma_\varphi^{-1}$.

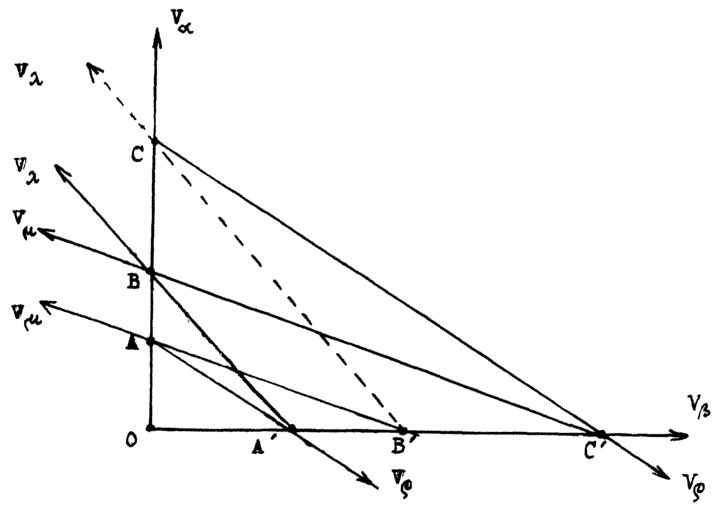
Corollary: If for $\mathcal{N}$ from Proposition 2.7 RC of type $(\alpha, \beta, \gamma, \mu, \varphi, \gamma)$ with H = O holds for fixed $\alpha, \beta, \gamma \in I$ and each μ (resp. φ), then for each permutation σ_μ (resp. σ_φ), exactly one inverse permutation $\sigma_\mu^{-1} = \sigma_\varphi$ (resp. $\sigma_\varphi^{-1} = \sigma_\mu$) exists.

Proof: The existence of σ_μ^{-1} follows from Proposition 2.7; the existence of σ_φ^{-1} follows analogously.

Definition: Let $\mathcal{N}$ be a net of degree $\geqq 5$ and let $\alpha, \beta, \mu, \varphi, \lambda$ be pairwise distinct indexes from I. By the Pappus condition (shortly PC) of type $(\alpha, \beta, \varphi, \mu, \lambda)$ in $\mathcal{N}$ we mean the following implication:

$$(\forall O,A,B,C,A',B',C' \in \mathcal{P} \setminus v)\ (\overline{OABCV_\alpha} \wedge \overline{OA'B'C'V_\beta} \wedge \overline{AA'V_\varphi} \wedge \wedge \overline{CC'V_\varphi} \wedge \overline{AB'V_\mu} \wedge \overline{BC'V_\mu} \quad \overline{A'BV_\lambda} \Longrightarrow \overline{B'CV_\lambda}).$$

If PC of type $(\alpha, \beta, \varphi, \mu, \lambda)$ holds in $\mathcal{N}$ for fixed $\alpha, \beta, \mu \in I$ (resp. $\alpha, \beta \in I$) and for any $\varphi, \lambda \in I$ (resp. $\varphi, \mu, \lambda \in I$), then we shall say that PC of type (α, β, μ) (resp. PC of type (α, β)) holds in $\mathcal{N}$.



<u>Proposition 2.8:</u> Let $\mathcal{N}$ be a net of degree $\geqq 8$, with a frame $(0', \alpha, \beta, \gamma)$ and with a coordinate algebra $(S, 0, (\sigma_\iota)_{\iota \in I}, (+_\iota)_{\iota \in I})$ with respect to this frame. Let MDC of type (α) and DC of type (α, β) with $A = 0'$, $A \neq B$ hold in $\mathcal{N}$.

Then for $\omega, \tau \in I \setminus \{\alpha, \beta\}$, $\omega \neq \tau$,

$$(x^{\sigma_\omega})^{\sigma_\tau} = (x^{\sigma_\tau})^{\sigma_\omega} \qquad \forall\, x \in S \text{ holds,}$$

iff PC of type (α, β, μ) with $x^{\sigma_\mu} = -x \qquad \forall\, x \in S$, $0 = 0'$, holds in $\mathcal{N}$.

<u>Proof:</u> Choose an element $a \in 0'V_\alpha \setminus \{V_\alpha\}$ and denote

$0 := 0'$, $A := (0, a^{\sigma_\tau})$, $B := (0, a^{\sigma_\omega})$, $C := (0, (a^{\sigma_\omega})^{\sigma_\tau})$, $\bar{C} := (0, (a^{\sigma_\tau})^{\sigma_\omega})$, $A' := (a, 0)$, $B' := (a^{\sigma_\tau}, 0)$, $C' := (a^{\sigma_\omega}, 0)$. Then the line AB' is expressed by the equation $y = x^{\sigma_\iota} + a^{\sigma_\tau}$, where $0 = (a^{\sigma_\tau})^{\sigma_\iota} + a^{\sigma_\tau}$ holds for σ_ι. Similarly, the

line BC′ can be expressed by the equation $y = x^{\bar{\sigma}_\iota} + a^{\sigma_\omega}$ with $0 = (a^{\sigma_\omega})^{\bar{\sigma}_\iota} + a^{\sigma_\omega}$ for $\bar{\sigma}_\iota$. Both of these relations imply the condition $x^{\sigma_\iota} = -x = x^{\bar{\sigma}_\iota}$ $\forall x \in S$, i.e., $\sigma_\iota = \bar{\sigma}_\iota = \sigma_\mu$ and also $\overline{AB'V_\mu} \wedge \overline{BC'V_\mu}$ hold in $\mathcal{N}$. Analogously, for σ_φ the line AA′ yields the condition $0 = a^{\sigma_\varphi} + a^{\sigma_\tau}$ as well as CC′ for $\bar{\sigma}_\varphi$ the condition $0 = (a^{\sigma_\omega})^{\bar{\sigma}_\varphi} + (a^{\sigma_\omega})^{\sigma_\tau}$ yields. These conditions imply $x^{\sigma_\varphi} = -x^{\sigma_\tau} = x^{\bar{\sigma}_\varphi}$ $\forall x \in S$ i.e. $\sigma_\varphi = \bar{\sigma}_\varphi$ and $\overline{AA'V_\varphi} \wedge \overline{CC'V_\varphi}$.
For σ_λ (resp. $\bar{\sigma}_\lambda$), the line A′B (resp. B′$\bar{C}$) provides the condition $0 = a^{\sigma_\lambda} + a^{\sigma_\omega}$ (resp. $0 = (a^{\sigma_\tau})^{\bar{\sigma}_\lambda} + (a^{\sigma_\tau})^{\sigma_\omega}$).
Hence $x^{\sigma_\lambda} = -x^{\sigma_\omega} = x^{\bar{\sigma}_\lambda}$ $\forall x \in S$ i.e. $\sigma_\lambda = \bar{\sigma}_\lambda$ and $\overline{A'BV_\lambda} \wedge \overline{B'\bar{C}V_\lambda}$.

If $(a^{\sigma_\tau})^{\sigma_\omega} = (a^{\sigma_\omega})^{\sigma_\tau}$ holds then $\bar{C} = C$ and PC of type (α, β, μ) with $x^{\sigma_\mu} = -x$ $\forall x \in S$ holds in $\mathcal{N}$. Conversely, if PC of type (α, β, μ) with $x^{\sigma_\mu} = -x$ $\forall x \in S$ holds in $\mathcal{N}$, then $C = \bar{C}$ i.e. $(a^{\sigma_\omega})^{\sigma_\tau} = (a^{\sigma_\tau})^{\sigma_\omega}$ for any element $a \in S$.

<u>Proposition 2.9</u>: Let $\mathcal{N}$ be a net of degree $\geqq 8$ with a frame $(0', \alpha, \beta, \gamma)$ and a coordinate algebra $(S, 0, (\sigma_\iota)_{\iota \in I}, (+_\iota)_{\iota \in I})$ with respect to this frame. Let MDC of type (α), DC of type (α, β) with $A = 0'$, $A \neq B$ and RC of type (α, β, γ) with $H = 0'$ hold in $\mathcal{N}$.

Then for all distinct $\tau, \lambda, \omega \in I \setminus \{\alpha, \beta\}$,

$$(x^{\sigma_\tau})^{(\sigma_\lambda \sigma_\omega)} = (x^{(\sigma_\tau \sigma_\lambda)})^{\sigma_\omega} \qquad \forall x \in S,$$

$$x^{\sigma_\omega \sigma_\tau} = x^{\sigma_\tau \sigma_\omega} \qquad \forall x \in S$$

hold iff PC of type (α, β) with $0 = 0'$ holds in $\mathcal{N}$.

Proof: The commutativity law for the permutations $\{(\sigma_\iota)_{\iota\in J}\}$ follows from PC, the existence of the product of permutations follows from RC (see Proposition 2.6). It is necessary to prove the associativity law for those permutations. Choose an element $a \in O'V_\alpha \setminus \{V_\alpha\}$ and put $O := O'$, $A := (O, a^{\sigma_\tau \sigma_\lambda})$, $B := (O, a^{\sigma_\omega \sigma_\lambda})$, $C := (O, a^{\sigma_\tau (\sigma_\omega \sigma_\lambda)})$, $\bar{C} := (O, a^{\sigma_\omega (\sigma_\tau \sigma_\lambda)})$, $A' := (a^{\sigma_\lambda}, O)$, $B' := (a^{\sigma_\tau \sigma_\lambda}, O)$, $C' := (a^{\sigma_\omega \sigma_\lambda}, O)$.

From the equations of the lines AA′ and CC′ it follows $O = (a^{\sigma_\lambda})^{\sigma_\varphi} + a^{\sigma_\tau \sigma_\lambda}) = (a^{\sigma_\lambda})^{\sigma_\varphi} + a^{\sigma_\lambda \sigma_\tau}$ and $O = (a^{\sigma_\omega \sigma_\lambda})^{\bar{\sigma}_\varphi} + a^{\sigma_\tau (\sigma_\omega \sigma_\lambda)} = (a^{\sigma_\omega \sigma_\lambda})^{\bar{\sigma}_\varphi} + a^{(\sigma_\omega \sigma_\lambda)\sigma_\tau}$ i.e. $\sigma_\varphi = \bar{\sigma}_\varphi$ and $\overline{AA'V_\varphi} \wedge \overline{CC'V_\varphi}$. From the equations of the lines AB′ and BC′ it follows $O = (a^{\sigma_\tau \sigma_\lambda})^{\sigma_\mu} + a^{\sigma_\tau \sigma_\lambda}$ and $O = (a^{\sigma_\omega \sigma_\lambda})^{\sigma_\mu} + a^{\sigma_\omega \sigma_\lambda}$, and therefore $\overline{AB'V_\mu} \wedge \overline{BC'V_\mu}$. The equations of the lines A′B and B$\bar{C}$′ yield $O = (a^{\sigma_\lambda})^{\sigma_\iota} + a^{\sigma_\omega \sigma_\lambda} = (a^{\sigma_\lambda})^{\sigma_\iota} + a^{\sigma_\lambda \sigma_\omega}$ and $O = (a^{\sigma_\tau \sigma_\lambda})^{\bar{\sigma}_\iota} + a^{\sigma_\omega (\sigma_\tau \sigma_\lambda)} = (a^{\sigma_\tau \sigma_\lambda})^{\bar{\sigma}_\iota} + (a^{\sigma_\tau \sigma_\lambda})^{\sigma_\omega}$ i.e. $\sigma_\iota = \bar{\sigma}_\iota$. Thus $\overline{A'BV_\iota} \wedge \overline{A'\bar{C}V_\iota}$ holds.

The condition

$$(*) \qquad a^{\sigma_\tau (\sigma_\omega \sigma_\lambda)} = a^{\sigma_\omega (\sigma_\tau \sigma_\lambda)} \qquad \forall a \in S$$

holds iff $C = \bar{C}$ (see Proposition 2.8) which is equivalent to PC of type (α, β) with $O = O'$. The condition $(*)$ holds iff $a^{\sigma_\tau (\sigma_\omega \sigma_\lambda)} = a^{(\sigma_\tau \sigma_\lambda)\sigma_\omega}$ $\forall a \in S$ holds i.e., $a^{\sigma_\tau (\sigma_\lambda \sigma_\omega)} = a^{(\sigma_\tau \sigma_\lambda)\sigma_\omega}$ $\forall a \in S$, which completes the proof.

Proposition 2.10: Let $\mathcal{N}$ be a net of degree ≥ 8 together with a frame $(O', \alpha, \beta, \gamma)$ and a coordinate algebra $(S, O, (\sigma_\iota)_{\iota\in J}, (+_\iota)_{\iota\in J})$ associated with this frame. Let MDC of type (α) and DC of type (α, β) with $A = O'$, $A \neq B$ hold in $\mathcal{N}$.

Then $\Sigma^* = \{(\sigma_\iota)_{\iota \in J}\}$ with the product operation is a commutative group iff RC of type (α, β, γ) with H = 0′ and PC of type (α, β) with 0 = 0′ hold in $\mathcal{N}$.
The proof follows from Propositions 2.6, 2.7 and 2.8.

R e f e r e n c e s

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