Nicole De Grande-De Kimpe

$L_f(a,r)$-spaces between which all the operators are compact. II.

Commentationes Mathematicae Universitatis Carolinae, Vol. 19 (1978), No. 1, 1--12

Persistent URL: http://dml.cz/dmlcz/105827

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

19,1 (1978)

$L_\varphi(a,r)$-SPACES BETWEEN WHICH ALL THE OPERATORS ARE COMPACT, II

Nicole De GRANDE-De KIMPE, Brussel

Abstract: Certain couples of $L_\varphi(a,r)$-spaces, between which all the operators are compact, are characterized. This result is related to the existence of a common step-space in both spaces.

Key words: Nuclear Fréchet space, compact operator.

AMS: 46A45, 47B05 Ref. Ž.: 7.972.23, 7.972.56

---

§ 1. Introduction. This paper is a continuation of the investigation begun in [1]. The problem considered here is to characterize certain couples of $L_\varphi(a,r)$-spaces between which all the operators are compact. The relation "All the operators from $L_\varphi(a,r)$ to $L_g(b,s)$ are compact" is denoted by $(L_\varphi(a,r),L_g(b,s))) \in \mathcal{R}$.
Our main result shows that, in all the cases considered, the relation $(L_\varphi(a,r),L_g(b,s)) \in \mathcal{R}$ is symmetric and is equivalent to the statement

"the spaces $L_\varphi(a,r)$ and $L_g(b,s)$ have no common step-space".

The definitions and terminology not explained here, as well as the situation of the problem in the theory of nuclear Fréchet spaces (and of $L_\varphi(a,r)$-spaces in particular), can be found in [1]. In that paper we restrict ourselves to the diagonal operators. It turned out that the existence of a non-compact diagonal operator was related to the existence of a non-compact generalized diagonal operator.

The question "What happens in the case when all the generalized diagonal operators between the spaces are compact?" will be solved in this paper.

§ 2. Necessary and sufficient conditions for $(L_f(a,\infty),\ L_g(b,\infty)) \in \mathcal{R}$.

Lemma. Let T be an operator from $L_f(a,r)$ to $L_g(b,s)$, $0 < r,s \leq \infty$, defined by $T(e_j) = (e^{r_{ij}})_i$. Then there exists a k_o such that

$$\overline{\lim_j}\left(\frac{\sup_i r_{ij}}{f(r_{k_o} a_j)}\right) \leq 1.$$

Proof: Consider the topological dual spaces $(L_f(a,r))' = L_f(a,r)^\times$ and $(L_g(b,s))' = L_g(b,s)^\times$.
Then the transpose

$$\tau_T:\ L_g(b,s)^\times \longrightarrow L_f(a,r)^\times$$

of T, is continuous when both spaces are equipped with their strong topology.
Since the set $\{e_i \mid i = 1,2,\dots\}$ is strongly bounded in $L_g(b,s)^\times$ and since $L_f(a,r)^\times$ is nuclear under its strong topology, the set $B = \{{}^\tau T(e_i) \mid i = 1,2,\dots\}$ will be a simple subset of $L_f(a,r)^\times$ (see [4]).
I.e.

$$(\sup_i e^{r_{ij}})_j = (e^{\sup_i r_{ij}})_j \in L_f(a,r)^\times .$$

Hence

$$\exists M,\ \exists k_o \text{ such that } \forall j:\ e^{\sup_i r_{ij}} \le e^{M+f(r_{k_o} a_j)}$$

or

$$\forall j:\ \frac{\sup_i r_{ij}}{f(r_{k_o} a_j)} \le \frac{M}{f(r_{k_o} a_j)} + 1,$$

from which the conclusion follows.

Remark 1. Given the increasing sequences $(f(a_m))_n$ and $(g(b_n))_n$, we can find increasing sequences of indices (p_i) and (q_i) such that:

$$\ldots \le g(b_{q_i-1}) \le f(a_{p_{i-1}}) \le f(a_{p_{i-1}+1}) \le \ldots \le f(a_{p_i-1})$$

$$< g(b_{q_i}) \le \ldots \le g(b_{q_{i+1}-1}) \le f(a_{p_i}) \le \ldots, \text{ for all } i.$$

For each j we denote by $p_{i(j)}$ (resp. $q_{i(j)}$) the smallest integer between the p_i (resp. q_i), such that

$$f(a_j) \le f(a_{p_{i(j)}-1}) < g(b_{q_{i(j)}}).$$

Proposition 1. Let (p_i) and (q_i) be the increasing sequences of indices defined in the preceding remark.
If

(a) $\exists k_1$ such that $\forall m: \lim_i \dfrac{g(m\ b_{q_i-1})}{f(k_1 a_{p_{i-1}})} = 0$

and

(b) $\exists m_1$ such that $\forall k: \lim_i \dfrac{f(k\ a_{p_i-1})}{g(m_1\ b_{q_i}}) = 0,$

then all the operators from $L_f(a,\infty)$ to $L_g(b,\infty)$ are compact. I.e. $(L_f(a,\infty), L_g(b,\infty)) \in \mathcal{R}$.

Proof: Let the operator $T: L_f(a,\infty) \longrightarrow L_g(b,\infty)$ be represented by the matrix (t_{ij}).
It is easy to see that, for our purpose, it is sufficient to consider those operators T for which $t_{ij} > 0$, $\forall i$, $\forall j$.

So put $T(e_j) = (e^{r_{ij}})_i$. Then

$$\| T(e_j) \|_m = \sup_i e^{r_{ij}+g(m\, b_i)}$$

and this sup is attained somewhere; at the index $i(m,j)$ say.
Thus

$$\| T(e_j) \|_m = e^{r_{i(m,j),j}+g(m\, b_{i(m,j)})}.$$

Put

$$c_{mj} = \log \| T(e_j) \|_m .$$

I.e.

(1) $$c_{mj} = r_{i(m,j),j} + g(m\, b_{i(m,j)}).$$

The continuity of T is then expressed by

$$\forall_m, \exists k_m \text{ such that } \sup_j e^{c_{mj}-f(k_m a_j)} < \infty .$$

So:

(2) $\forall m$, $\exists k_m$, $\exists j_m$ such that $c_{mj} \leq f(k_m a_j)$ for $j \geq j_m$.

The compactness of T will be proved if

(3) $\exists k$ such that $\forall m$, $\exists j_m : c_{mj} \leq f(k\, a_j)$ for $j \geq j_m$.

We put $J_m = \{ j \mid c_{mj} > 0 \}$.
(Remark that, if J_m is finite or empty for all m, the operator T is compact.)

Since c_{mj} increases with m we have $J_m \supset J_m$, whenever

$m > m'$. Denote by m_0 the smallest value of m for which J_m is infinite. Suppose we had:

(4) $\exists\, m_2 \geq m_0,\ \exists\, J'_{m_2}$ infinite subsequences of J_{m_2} such that

for $j \in J'_{m_2} : g(b_{i(m_2,j)}) > f(a_j)$.

Then take m_1 from assumption (b), $t > \max\,(m_2, m_1)$, k_{m_2} and k_t from (2) and finally $k > \max\,(k_{m_2}, k_t)$.

Then for $j \in J_{m_2}$ we have:

$$0 < \frac{c_{tj}}{f(k\,a_j)} < \frac{c_{tj}}{f(k_t a_j)} < 1 \text{ for } j \geq j_t.$$

So

$$(*) \qquad \sup_{j \in J_{m_2}} \frac{c_{tj} - c_{m_2 j}}{f(k\,a_j)} < \infty\,.$$

On the other hand, by the definition of c_{mj}, we obtain

$$c_{tj} \geq r_{i(m_2,j),j} + g(t\, b_{i(m_2,j)}).$$

So

$$c_{tj} - c_{m_2 j} \geq g(t\, b_{i(m_2,j)}) - g(m_2 b_{i(m_2,j)}),$$

whence

$$\frac{c_{tj} - c_{m_2 j}}{f(k\,a_j)} \geq \frac{g(t\, b_{i(m_2,j)})}{f(k\,a_j)} \left[1 - \frac{g(m_2 b_{i(m_2,j)})}{g(t\, b_{i(m_2,j)})} \right]$$

It follows from (4) that $\lim\limits_{j \in J'_{m_2}} i(m_2, j) = \infty$.

Thus

$$\lim_{j \in J'_{m_2}} \frac{g(m_2 b_{i(m_2,j)})}{g(t\, b_{i(m_2,j)})} = 0.$$

Moreover, with the notation of the remark we obtain from (4) that for $j \in J'_{m_2}$:

$$\frac{g(t\ b_{i(m_2,j)})}{f(k\ a_j)} \geq \frac{g(t\ b_{q_{i(j)}})}{f(k\ a_{p_{i(j)}-1})} \geq \frac{g(m_1 b_{q_{i(j)}})}{f(k\ a_{p_{i(j)}-1})} ,$$

which by assumption b) implies

$$\lim_{j \in J'_{m_2}} \frac{g(t\ b_{i(m_2,j)})}{f(k\ a_j)} = \infty .$$

This is in contradiction with ($*$).

Therefore (4) cannot be true.

I.e.

$\forall m \geq m_o$, $\exists j_m$ such that $g(b_{i(m_2,j)}) \leq f(a_j)$, for $j \geq j_m, j \in J_m$.

Taking k_1 from assumption a) and making use of the notations in the remark, we then obtain

$$\forall m \geq m_o,\ \exists j_m \text{ such that } 0 \leq \frac{g(m\ b_{i(m,j)})}{f(k_1 a_j)} \leq \frac{g(m\ b_{q_{i(j)}-1})}{f(k_1 a_{p_{i(j)}-1})} ,$$

for $j \in J_m$, $j \geq j_m$.

From assumption (a) it then follows that

$$(5) \qquad \forall m \geq m_o \lim_{j \in J_m} \frac{g(m\ b_{i(m,j)})}{f(k_1 a_j)} = 0.$$

We are now in a position to prove that T is compact. Take k_o from the lemma, k_1 as above and $k_2 > \max(k_1, k_o)$. We'll prove that (3) is satisfied for $k = k_2$. Choose any m.

If $m < m_o$, (3) is satisfied since then $c_{mj} \leq 0$.

If $m \geq m_o$, (3) is satisfied for $j \in \mathbb{N} \setminus J_m$ and it is left to check on (3) for $j \in J_m$ (j sufficiently large).

Now

$$\frac{c_{mj}}{f(k_2 a_j)} = \frac{r_{i(m,j),j}}{f(k_2 a_j)} + \frac{g(m\, b_{i(m,j)})}{f(k_2 a_j)}$$

So, by the lemma and (5) the desired conclusion follows.

Theorem. If $f^{-1} \circ g$ and $g^{-1} \circ f$ are slowly increasing then the following are equivalent:

(i) $((L_f(a,\infty), L_g(b,\infty)) \in \mathcal{R}$.

(ii) All the generalized diagonal operators from $L_f(a,\infty)$ to $L_g(b,\infty)$ are compact.

(iii) $L_f(a,\infty)$ and $L_g(b,\infty)$ have no common step-space.

(iv) $((L_g(b,\infty), L_f(a,\infty)) \in \mathcal{R}$.

(v) All the generalized diagonal operators from $L_g(b,\infty)$ to $L_f(a,\infty)$ are compact.

(vi) The conditions in the preceding proposition.

Proof: (ii) $\Longrightarrow$ (vi): Consider, with obvious notations, the couples of step-spaces:

(A) $(L_f(a_{p_i-1},\infty),\ L_g(b_{q_i-1},\infty))$ and

(B) $(L_f(a_{p_i-1},\infty),\ L_g(b_{q_i},\infty))$.

Then all the diagonal operators between the spaces under (A) (resp. (B)) are compact.

I.e. [1] (§ 4 Theorem, (i) $\Longrightarrow$ (v) and § 3 Lemma 1, (ii)):

(*) $\forall k,\ \exists i_k$ such that $f(k\, a_{p_{i-1}}) < g(b_{q_i-1})$ for $i \geq i_k$

or

(**) $\forall m,\ \exists i_m$ such that $f(m\, b_{q_i-1}) < f(a_{p_{i-1}})$ for $i \geq i_m$

or

(***) there exists a partition $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2$ such that

(*) holds on $\mathbb{N}_1$ and (**) holds on $\mathbb{N}_2$.

Now (*) and (***) lead to a contradiction with the sequence of inequalities stated in the preceding proposition, while, taking $k_1 > 1$ in (**), we obtain assumption (a) in this proposition. In the same way it is proved that case (B) leads to assumption (b) in the proposition.

Since (i) $\Longrightarrow$ (ii) is trivial we have already, by the proposition, (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (vi).

Proof of (iii) $\Longleftrightarrow$ (ii): Suppose that T is a non-compact generalized diagonal operator from $L_f(a,\infty)$ to $L_g(b,\infty)$. Then T can be considered as a (non-compact) diagonal operator from $L_f(a,\infty)_I$ to $L_g(b,\infty)_J$ for suitable subsequences I and J of $\mathbb{N}$.
Then by the Theorem in [1] § 4 (proof of (i) $\Longrightarrow$ (v)) the spaces $L_f(a,\infty)_I$ and $L_g(b,\infty)_J$ and hence also the spaces $L_f(a,\infty)$ and $L_g(b,\infty)$ have a common step-space. This contradicts (iii). On the other hand, if $L_f(a,\infty)$ and $L_g(b,\infty)$ have a common step-space there is obviously a non-compact generalized diagonal operator between them.

The remaining equivalences follow, by symmetry, also from the Theorem in [1] § 4.

Remark 2. As can be seen in [1] (§ 2, d) and § 4, Prop. 2 + Remark 1), the case $r = s = 0$ behaves in exactly the same way as the case $r = s = \infty$.
Therefore the preceding results are also true (with the same proof) in the case $r = s = 0$.

§ 3. Necessary and sufficient conditions for $L_f(a,r),\ L_g(b,s) \in \mathcal{R}$, $(0 < r,s < \infty)$

Remark 1. Given the increasing sequences $(f(r\ a_n))_n$ and $(g(s\ b_n))_n$, we can find increasing sequences of indices (p_i) and (q_i) such that

$$\dots \leq g(s\ b_{q_i-1}) \leq f(r\ a_{p_{i-1}}) \leq \dots \leq f(r\ a_{p_i-1}) < g(s\ b_{q_i}) \leq \dots$$
$$\dots \leq g(s\ b_{q_{i+1}-1}) \leq f(r\ a_{p_i}) \leq \dots$$

As before, for each j we denote by $p_{i(j)}$ (resp. $q_{i(j)}$) the smallest integer between the p_i (resp. q_i) such that

$$f(r\ a_j) \leq f(r\ a_{p_{i(j)}-1}) < g(s\ b_{q_{i(j)}}).$$

Proposition 1. Let (p_i) and (q_i) be the increasing sequences of indices defined in the preceding remark. If

(a) $\exists\, k_1$ such that $\lim\limits_i \dfrac{g(s\ b_{q_i-1})}{f(r_{k_1} a_{p_{i-1}})} = 0$

and

(b) $\exists\, m_1$ such that $\lim\limits_i \dfrac{f(r\ a_{p_i-1})}{g(s_{m_1} b_{q_1})} = 0,$

then $(L_f(a,r),\ L_g(b,s)) \in \mathcal{R}$.

We omit the proof, which is similar to the one in the case $r = s = \infty$.

As an example we treat a special case in which the technical conditions in the preceding proposition are easy to check. The result $(*)$ obtained is similar to those obtained for power series spaces (see [3] Theorem 6 and [4] Theorem 3).

Proposition 2. If $\lim_n \frac{a_{n+1}}{a_n} = \infty$ and $s > r$, then $(L_f(a,r),\ L_f(a,s)) \in \mathcal{R}$. Hence in this case the space $L_f(a,s)$ contains no subspace which is isomorphic to a space $L_f(a,r)$, for $r < s$ ($*$).

Proof: We first determine the nature of the sequences of indices (p_i) and (q_i) in this particular case.

We have $\frac{a_{p_{i-1}}}{a_{q_i-1}} \geq \frac{s}{r} > 1$ and thus is $p_{i-1} \geq q_i$.

If $p_{i-1} > q_i$ then $\lim_i \frac{a_{p_{i-1}}}{a_{q_i}} = \infty$, whence $\lim_i \frac{a_{p_i-1}}{a_{q_i}} = \infty$,

which contradicts $\frac{a_{p_i-1}}{a_{q_i}} < \frac{s}{r}$.

So $p_{i-1} = q_i$ for i sufficiently large.

So, for i sufficiently large, putting $q_i = i + 1$, we obtain for the sequence of inequalities:

$$\ldots < s\ a_i \leq r\ a_{i+1} < s\ a_{i+1} \leq r\ a_{i+2} < \ldots$$

It is left to check on the conditions (a) and (b) in proposition 1.

Fix $k = k_o$. Then $\lim_i \frac{s\ a_i}{r_{k_o} a_{i+1}} = 0$, or $s\ a_i < r_{k_o} a_{i+1}$, for i sufficiently large.

Take now $k_1 > k_o$, then

$$0 \leq \frac{f(s\ a_i)}{f(r_{k_1} a_{i+1})} < \frac{f(r_{k_o} a_{i+1})}{f(r_{k_1} a_{i+1})}$$

Since $\lim_i \frac{f(r_{k_o} a_{i+1})}{f(r_{k_1} a_{i+1})} = 0$, condition (a) is satisfied.

On the other hand, since $s > r$ we can take m_o such that $s_{m_o} > r$. Then $f(ra_{i+1}) < f(a_{m_o} a_{i+1})$.

Taking $m_1 > m_o$ we obtain

$$0 \leq \frac{f(r\ a_{i+1})}{f(s_{m_1} a_{i+1})} < \frac{f(s_{m_o} a_{i+1})}{f(s_{m_1} a_{i+1})},$$

from which contidion (b) follows.

The proof of the next theorem is the same as in the case $r = s = \infty$ and is therefore omitted.

<u>Theorem</u>. If $f^{-1} \circ g$ and $g^{-1} \circ f$ are slowly increasing, then the following are equivalent:

(i) $(L_f(a,r),\ L_g(b,s)) \in \mathcal{R}$.

(ii) All the generalized diagonal operators from $L_f(a,r)$ to $L_g(b,s)$ are compact.

(iii) $L_f(a,r)$ and $L_g(b,s)$ have no common step-space.

(iv) $(L_g(b,s),\ L_f(a,r)) \in \mathcal{R}$.

(v) All the generalized diagonal operators from $L_g(b,s)$ to $L_f(a,r)$ are compact.

(vi) The conditions in proposition 1.

<u>Remark 2</u>. In the case $-\infty < r,\ s < 0$ analogous results can be obtained with still the same proof (cfr. Remark 2, § 2).

References

[1] DE GRANDE-DE KIMPE N.: $L_f(a,r)$-spaces between wh the operators are compact, I, Comment. Mat. Univ. Carolinae 18(1977,659-674.

[2] DUBINSKY Ed.: Infinite type power series subspaces of finite type power series subspaces, Israel J. Math. 15(3)(1973), 257-281.

[3] DUBINSKY Ed.: Infinite type power series subspaces of infinite type power series spaces, Israel J. Math. 20(3-4)(1975), 359-368.

[4] KOTHE G.: Nukleare (F)- und (DF)-Folgenräume, Theory of sets and topology (1972), 327-332.

Departement Wiskunde

Vrije Universiteit Brussel

Pleinlaan 2(F 7)

1050 Brussel

Belgium

(Oblatum 20.4. 1977)