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THE NILPOTENCY OF TORSION-FREE RINGS WITH GIVEN TYPE SET

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Abstract: How does the additive structure of a ring affect its multiplicative structure? In this note we consider torsion-free rings and show that certain simple restrictions on the type set of the additive group of the ring forces the ring to be nilpotent. We obtain bounds on the degree of nilpotency in terms of easily derived invariants of the type set.

Key words: Torsion-free, type set, nilstufe, non-associative.

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Introduction. The nilstufe, n(G), of a torsion-free abelian group G was defined by Szele [4] to be the largest positive integer n such that there is an associative ring on G having a non-zero product of n elements. If no such largest integer exists then n(G) is set equal to ∞. Feigelstock [1] defines the strong nilstufe, N(G), in a similar way but in this case allows non-associative ring structures on G. Other authors have considered related notions, for example Gardner [3] and Wickless [6].

In [5] Webb showed that if G is torsion-free with fini-

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te rank r then either $n(G) = \infty$, or $n(G) \leq r$, and either $N(G) = \infty$, or $N(G) \leq 2^{(r-1)}$. The purpose of this note is to derive improved bounds on n(G) and N(G) in terms of the length $\ell = \ell(T(G))$ of the type set T(G) of G. Here by the type set of G we mean the partially ordered set of types of non-zero elements of G. The length of T(G) is the length of the longest chain in T(G). This notion is well defined since T(G) satisfies both the ascending and descending chain conditions. (Fuchs [2], page 112, Ex. 10.) It is easy to see that $\ell \leq r$.

We introduce the following notions:

(i) The height, $h(\underline{t})$, of a type $\underline{t}$ in T(G) is the length of the longest descending chain in T(G) with $\underline{t}$ as its maximal element. Clearly, $h(\underline{t}) \leq \ell$ for all $\underline{t}$ in T(G).

(ii) If $\underline{s}$, $\underline{t}$ are types we say that $\underline{s}$ absorbs $\underline{t}$ if $\underline{s}\underline{t} = \underline{s}$. In particular idempotent elements are self-absorbing. If $\underline{s}$, $\underline{t}$ are in T(G) and $\underline{s}$ absorbs $\underline{t}$ we call $\underline{s}$ an absorbing element in T(G).

(iii) The order O(T(G)), of T(G) is equal to the greatest positive integer n such that there exists n+1 types $\underline{t}_1, \dots \underline{t}_n$, $\underline{s}$ in T(G) satisfying the condition:

$$\underline{s} \geq \underline{t}_1 \underline{t}_2 \cdots \underline{t}_n.$$

If no such n exists we set O(T(G)) equal to ∞. Clearly a type set of finite order contains no absorbing elements.

Throughout the remainder of this note G denotes a torsion-free abelian group of finite rank r. The length of T(G) is denoted by ℓ. With these conventions the main results are:

Proposition 1: If $N(G) = \infty$ then $O(T(G)) = \infty$.

Proposition 2: If T(G) contains no idempotent elements then,

$$n(G) \leq \min\{2^{\ell} - 1, r\}.$$

Proposition 3: If T(G) contains no absorbing elements and N(G) is finite, in particular if T(G) has finite order, then,

$$N(G) \leq 2^{\ell-1}$$

$$n(G) \leq \ell .$$

We end the note with an example of a group G having $N(G) = \infty$ whose type set contains no idempotent elements.

Proofs of the results. Proposition 1 is a simple consequence of the definition of O(T(G)). Suppose that $N(G) = \infty$. Then there is a ring $(G, *)$ on G and an infinite sequence $g_1, g_2, \ldots,$ of elements of G such that for each positive integer n there is a bracketing such that the product,

$$x_n = g_1 * g_2 * \ldots * g_n$$

is non-zero. Now,

$$\underline{t}(x_n) \geq \underline{t}(g_1)\underline{t}(g_2) \ldots \underline{t}(g_n)$$

so $O(T(G)) = \infty$.

Lemma 4: Suppose that $(G, *)$ is a ring on G and that g_1, g_2 from G are such that $g_1 * g_2 \neq 0$.

(i) If T(G) contains no absorbing elements then,

$$\underline{t}(g_1 * g_2) > \underline{t}(g_i) \text{ for } i = 1,2.$$

(ii) If T(G) contains no idempotent elements then either,

$$\underline{t}(g_1 * g_2) > \underline{t}(g_1)$$

or $$\underline{t}(g_1 * g_2) > \underline{t}(g_2).$$

Proof. Clearly, $\underline{t}(g_1 * g_2) \geq \underline{t}(g_1)\underline{t}(g_2) \geq \underline{t}(g_i) \quad i = 1,2.$

Suppose that $\underline{t}(g_1 * g_2) = \underline{t}(g_1)$, then

(A) $$\underline{t}(g_1) = \underline{t}(g_1)\underline{t}(g_2) \geq \underline{t}(g_2)$$

and $\underline{t}(g_1)$ is an absorbing element of T(G). This proves (i). If T(G) contains no idempotents then (A) implies that $\underline{t}(g_1) \neq \underline{t}(g_2)$ whence $\underline{t}(g_1) > \underline{t}(g_2)$ and (ii) follows.

Lemma 5: Suppose that T(G) contains no idempotent types. Let $(G, \circ)$ be an associative ring on G. Let n, k be positive integers satisfying $k \geq 2^{n-1}$. Suppose that $g_1, g_2, \ldots$ $\ldots, g_k$ from G are such that

$$g = g_1 \circ g_2 \circ \ldots \circ g_k \neq 0.$$

Then the height of $\underline{t}(g)$ in T(G) is at least n.

Proof. If n = 1 the result is clear. Suppose that the result holds for all $m \leq n$, and that $k \geq 2^n$. Then,

$$g = (g_1 \circ g_2 \circ \ldots \circ g_{2^{n-1}}) \circ (g_{2^{n-1}+1} \circ \ldots \circ g_k)$$

$$= x \circ y,$$

say. It follows from Lemma 1(ii) that one of the inequalities

$$\underline{t}(g) > \underline{t}(x) \text{ or } \underline{t}(g) > \underline{t}(y)$$

holds. Thus,

$$h(\underline{t}(g)) > h(\underline{t}(x)) \text{ or } h(\underline{t}(g)) > h(\underline{t}(y)).$$

But both x and y are products of at least 2^{n-1} elements so that,

$$h(\underline{t}(g)) > n$$

as desired.

Proof of Proposition 2. Suppose that for some $k \geq 2^{\ell}$ there is an associative ring $(G, \circ)$ on G with a non-zero product $g = g_1 \circ g_2 \circ \dots \circ g_k$. Lemma 5 implies that $h(\underline{t}(g)) \geq$ $\geq \ell + 1$, contradicting the definition of ℓ. Thus $n(G) <$ $< 2^{\ell}$. Webb [4] shows that if $n(G)$ is finite then $n(G) \leq r$. Taking these results together we obtain,

$$n(G) \leq \min\{2^{\ell} - 1, r\}.$$

Let R be a non-associative ring. For each positive integer k we define $R^{(k)}$ to be the subring of R generated by all products of k elements in R. Also we let F(R) denote the subring of the ring of endomorphisms E(R) of the additive group of R generated by the endomorphisms L_a, R_a for a in R where,

$$xL_a = ax, \quad xR_a = xa, \text{ for } x \text{ in } R.$$

Lemma 6: Let R be a torsion-free ring. Let n and k be positive integers satisfying $k > 2^{n-1}$. Then,

$$R^{(k)} \subseteq R(F(R))^n.$$

Proof. We proceed by induction on n, the result being clear when $n = 1$. Suppose that the result holds for $n = m > 1$, and that $k > 2^m$. Let x be the product of k elements in R, then $x = uv$ where at least one of u and v, u say, is a product of at least 2^{m-1} elements. So, u is in $R(F(R))^m$ and uv is in $R(F(R))^{m+1}$.

Proof of Proposition 3. Suppose that N(G) equals the positive integer k. Let $(G,*)$ be a ring on G such that,

$$(G,*)^{(k)} \neq 0.$$

It follows from Lemma 6 that given any integer n satisfying $2^{n-1} < k$ then,

$$G[F(G,*)]^{(n)} \neq 0.$$

In particular there must be non-zero monomials in $G[F(G,*)]^{(n)}$. Recalling that $F(G,*)$ is associative we see that such a monomial $\mathcal{G}$ may be written in the form,

$$\mathcal{G} = (\ldots ((gX_1)X_2) \ldots X_n) \neq 0,$$

where g is in G and, for each i, X_i denotes $*$ multiplication on the left or right by an element of G. It now follows from Lemma 1(i) that,

$$\underline{t}(g) < \underline{t}(gX_1) < \underline{t}((gX_1)X_2) < \ldots < \underline{t}(\mathcal{G}),$$

is a strictly ascending chain in T(G) of length n + 1. By hypothesis $n + 1 \leq \ell$ and so,

$$N(G) = k \leq 2^{\ell-1}$$

so completing the proof of the first inequality.

Next since $n(G) \leq N(G)$, we may assume that $n(G) = n$, a positive integer. Then there is an associative ring $(G, \circ)$ on G and a collection $g_1, g_2 \ldots, g_n$ of n elements of G such that,

$$g_1 \circ g_2 \circ \ldots \circ g_n \neq 0.$$

For each i satisfying $1 \leq i \leq n$ set,

$$\underline{t}_i = \underline{t}(g_1 \circ g_2 \circ \ldots \circ g_i).$$

Then using Lemma 1(i), we have

$$\begin{aligned} \underline{t}_{i+1} &= \underline{t}((g_1 \circ g_2 \circ \ldots \circ g_i) \circ g_{i+1}) \\ &> \underline{t}(g_1 \circ g_2 \circ \ldots \circ g_i) = \underline{t}_i. \end{aligned}$$

Thus there is a strictly ascending chain, $\underline{t}_1 < \underline{t}_2 < \ldots < \underline{t}_n$ in T(G). Whence we deduce that $n \leq \ell$.

Finally we give an example of a group G which has:

(i) Finite type set,

(ii) no idempotent types,

(iii) $N(G) = \infty$.

Let $R_1 \subseteq R_2 \subseteq Q$ be rational groups such that neither $\underline{t}(R_1)$ nor $\underline{t}(R_2)$ is idempotent, but $\underline{t}(R_1)\underline{t}(R_2) = \underline{t}(R_2)$. Let $G = R_1 x \oplus R_2 y$ and define a multiplication $*$ on G by putting

$$x * x = 0 = y * y, \quad x * y = y = y * x$$

and using linearity. Note that the fact that $\underline{t}(R_2)$ absorbs $\underline{t}(R_1)$ implies that given r in R_1, s in R_2 then rs is in R_2 where the product is taken in the rationals. Note also that Conditions (i) and (ii) above hold. Moreover, for any n the product,

$$x * (\ldots * (x * (x * y)) \ldots) = y \neq 0,$$

where x appears n times. Thus $N(G) = \infty$.

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