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ON THE POSET OF TENSOR PRODUCTS ON THE UNIT INTERVAL

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Abstract: The paper is concerned with the way in which the poset of all tensor products on the unit interval I of reals is embedded in the complete lattice of all binary operations on I. The main result says that any lower-semicontinuous commutative operation on I that has 0 for zero and 1 for unit can be obtained as the join in $I^{I\times I}$ of a countable family of tensor products on I all of whose members are isomorphic to

$$x \boxplus y = 0 \vee (x + y - 1).$$

Key words: Tensor product, $c\ell$-monoid, residuated lattice, lower-semicontinuity.

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Introduction. In [4] we considered various ways in which I can be endowed with the structure of a symmetric monoidal closed category. Recall that any tensor product on I (that is, an isotone binary operation $\Box : I\times I \longrightarrow I$ with the properties

(0.1) $(I, \Box, 1)$ is a commutative monoid;

(0.2) the distributive law

$$(\bigvee X) \Box a = \bigvee\{ x \Box a \mid x \in X \},$$

where $\bigvee X$ denotes the supremum of X in I, holds for any $X \subseteq I$ and any $a \in I$)

has a right adjoint $h: I\times I \longrightarrow I$, linked with $\Box$ by the formula

(0.3) for all $x,y,z \in I$, $x \sqcap y \leq z$ iff $x \leq h(y,z)$.
The right adjoint h of $\sqcap$ is uniquely determined by the formula

(0.4) $h(x,y) = \max\{t \in I \mid t \sqcap x \leq y\}$; $x, y \in I$.

Also recall that a binary operation on I satisfies (0.2) iff it is isotone, lower-semicontinuous, and has 0 for zero.

If we generalize the above notion to an arbitrary complete lattice L with the least element 0 and the greatest element 1; then a binary operation $\sqcap$ on L is a <u>tensor product</u> iff $(L,\sqcap)$ is an <u>integral</u> $c\ell$<u>-monoid</u> in the sense of Birkhoff [1]. According to Dilworth and Ward [2], a tensor product on L together with its right adjoint h endow L with the structure of a <u>residuated lattice</u>; $\sqcap$ is then called multiplication and h is called <u>residuation</u> in L.

In this paper we shall adhere to the terminology of [4] and use the term "tensor product". Given a complete lattice L we shall denote by $\mathcal{T}(L)$ the set of all tensor products on L partially ordered by the relation

(0.5) $\sqcap \leq \sqcap'$ iff $x \sqcap y \leq x \sqcap' y$ holds for all $x,y \in L$.
Thus, $\mathcal{T}(L)$ is a subposet of the complete lattice $\mathcal{O}(L) = L^{L \times L}$ of all binary operations on L.

1. <u>Some properties of the posets</u> $\mathcal{T}(L)$

1.1. <u>Observation</u>. Given a complete lattice L and $\sqcap$, $\sqcap' \in \mathcal{T}(L)$ let h and h' be the right adjoints of $\sqcap$ and $\sqcap'$, respectively. Then $\sqcap \leq \sqcap'$ iff $h(x,y) \geq h'(x,y)$ holds for any $x,y \in L$.

Proof. It is easy to show that the adjointness condition (0.3) for a couple $(\square, h)$ on L is equivalent to the following couple of inequalities in $(L, \square, h)$

$$(A') \quad x \leq h(y, x\square y) \qquad h(x,y)\square x \leq y \quad (A'')$$

If $\square \leq \square'$ then by (A'') for $(\square', h')$ we have $h'(x,y)\square x \leq \leq h'(x,y)\square' x \leq y$ hence $h'(x,y) \leq h(x,y)$ for all $x,y \in L$. Similarly one proves the converse implication.

1.2. *Observation.* If L is completely distributive then the meet $\wedge$ in L is the greatest element of $\mathcal{T}(L)$.

Proof. By definition, $(x,y) \longmapsto x\wedge y$ is a tensor product on L iff L is completely distributive. If $\square \in \mathcal{T}(L)$ we obtain by the isotony of $\square$ the inequality

$$x\square y \leq (x\square 1)\wedge(1\square y) = x\wedge y$$

for all $x,y \in L$. Thus $\wedge$ is the unit of $\mathcal{T}(L)$ provided L is completely distributive.

1.3. *Remark.* It is easily shown (see [2]) that if L is, moreover, boolean, $\mathcal{T}(L) = \{\wedge\}$.

1.4. *Proposition.* Let L be a complete chain. Then $\mathcal{T}(L)$ has the least element iff 1 is isolated in L.

Proof. Given a complete chain L consider the operation

$$(1.1) \qquad x\Delta y = \begin{cases} 0 & \text{if } x\vee y<1 \\ x\wedge y & \text{otherwise.} \end{cases}$$

Clearly, $\Delta \in \mathcal{T}(L)$ iff $1 > \bigvee\{x\in L \mid x \neq 1\}$ in L. Since $\Delta \leq \leq \square$ holds for any $\square \in \mathcal{T}(L)$ it suffices to show that for any $A \subseteq L\setminus\{1\}$ such that $\bigvee A = 1$ there exists a system $\{\square_a;\ a\in A\}$ of tensor products on L such that $\Delta = = \bigwedge\{\square_a \mid a\in A\}$ in the complete lattice $\mathcal{T}(L)$. To this end, put

(1.2) $$x \,\square_a\, y = \begin{cases} 0 & \text{if } x \vee y \leq a \\ x \wedge y & \text{otherwise} \end{cases}$$

for any $a \in A$ and $x, y \in L$. Then it is easily verified that the family $\{\square_a;\ a \in A\}$ has the desired properties.

1.5. Proposition. If L is a complete lattice and $\mathcal{C}$ is a nonempty chain in $\mathcal{T}(L)$ then the join of $\mathcal{C}$ in $\mathcal{O}(L)$ is again a tensor product on L.

Proof. Assume that $\emptyset \neq \mathcal{C}$ is a chain of tensor products on L. We have to verify that

(1.3) $$x \triangle y = \bigvee \{x \square y \mid \square \in \mathcal{C}\}$$

is a tensor product on L. Obviously, $\triangle$ is commutative, distributive with respect to all joins in L, and it has 0 for zero and 1 for unit. As to the associativity, take any $x, y, z \in L$. We have $(x \triangle y) \triangle z =$

$$= \bigvee \{ (\bigvee \{ x \square y \mid \square \in \mathcal{C}\}) \square' z \mid \square' \in \mathcal{C}\} =$$
$$= \bigvee \{ \bigvee \{ (x \square y) \square' z \mid \square \in \mathcal{C}\} \mid \square' \in \mathcal{C}\} =$$
$$= \bigvee \{ (x \square'' y) \square'' z \mid \square'' = \max(\square, \square');\ \square, \square' \in \mathcal{C}\} =$$
$$= \bigvee \{ x \square'' (y \square'' z) \mid \square'' = \max(\square, \square');\ \square, \square' \in \mathcal{C}\} =$$
$$= \bigvee \{ \bigvee \{ x \square (y \square' z) \mid \square' \in \mathcal{C}\} \mid \square \in \mathcal{C}\} =$$
$$= \bigvee \{ x \square \bigvee \{ y \square' z \mid \square' \in \mathcal{C}\} \mid \square \in \mathcal{C}\} = x \triangle (y \triangle z).$$

2. A result concerning $\mathcal{T}(I)$. Let us now consider the case when L = I is the unit interval of real numbers. Let $\mathcal{C} \subseteq \mathcal{T}(I)$, $\mathcal{C} \neq \emptyset$, and let $\triangle = \bigvee \mathcal{C}$ in $\mathcal{O}(I)$. If we omit the requirement that $\mathcal{C}$ be a chain, $\triangle$ is again isotone, commutative, lower-semicontinuous, and has 0 for zero and 1 for unit. On the other hand, it need not by far be associative; in fact, we shall show that any binary operation $\triangle$

on I that fulfils the above mentioned conditions can be obtained as a join in $\mathcal{O}(I)$ of a countable family $\{\square_i; i \in \omega\}$ of tensor products on I. Moreover, we can ensure that each $\square_i$ is continuous, the semigroup $(I, \square_i)$ has no idempotents other than 0 and 1 and all elements of $I \setminus \{1\}$ are nilpotent in $(I, \square_i)$; in other words ([5]), that each semigroup $(I, \square_i)$ is isomorphic to $(I, \boxplus)$ where

(2.1) $\qquad x \boxplus y = 0 \vee (x + y - 1)$ for all $x, y \in I$.

2.1. Theorem. Let $\triangle$ be an isotone, commutative and lower-semicontinuous binary operation on I such that $x \triangle 0 = 0$ and $x \triangle 1 = x$ holds for any $x \in I$. Then there exists a countable set $\mathcal{O}\!\mathcal{L}$ or tensor products on I isomorphic to the product $\boxplus$ given by (2.1) so that

(2.2) $\qquad x \triangle y = \bigvee \{ x \square y \mid \square \in \mathcal{O}\!\mathcal{L} \}$

holds for all $x, y \in I$.

Proof. We shall need the following lemma which follows immediately from the lower semicontinuity of $\triangle$.

2.1.1. Lemma. With $\triangle$ as in the assumptions of 2.1 let D be a dense subset of I and let $x, y_1, \ldots, y_n, z_1, \ldots, z_n, w \in I$ so that $x \triangle x > w$ and $x \triangle y_i > z_i$ for each $i = 1, \ldots, n$. Then for every $u < x$ there exists $d \in D$ with the properties $u < d < x$, $d \triangle d > w$, and $d \triangle y_i > z_i$.

2.1.2. Assume given $\triangle$ that satisfies the assumptions of 2.1 and some a, b, ε with

(2.3) $\qquad 0 < b \leq a < 1, \quad 0 < \varepsilon < a \triangle b.$

We are going to prove that there exists an order-isomorphism $f: I \approx I$ such that the tensor product $\boxplus^f$ on I defined by the formula

(2.4) $\qquad x \boxplus^f y = f^{-1}(fx \boxplus fy)$, all $x,y \in I$

satisfies the inequalities

(2.5) $\qquad a \boxplus^f b > a \triangle b - \varepsilon \quad , \quad x \boxplus^f y \leq x \triangle y$ for all $x,y \in I$.

Choose a countable dense subset $D \subseteq I$ so that $0, 1 \notin D$. Now assume we have constructed a family

(2.6) $\qquad \{ d_{n,k};\ n \geq 5,\ 3 \leq k \leq 2^n \}$

with the properties

(a) $D = \{ d_{n,k} \mid n \geq 5,\ 3 \leq k < 2^n \}$;

(b) $1 > d_{n,3} > d_{n,4} > \dots > d_{n,2^n-1} > d_{n,2^n} = 0$ for any $n \geq 5$;

(c) $d_{n,k} = d_{n+1,2k}$ for any $n \geq 5$, $3 \leq k \leq 2^n$;

(d) $d_{n,k} \triangle d_{n,p} > d_{n,k+p-2}$ whenever $n \geq 5$, $3 \leq k$, p, and $k + p \leq 2^n + 2$;

(e) $a > d_{5,13}$, $b > d_{5,18}$, and $a \triangle b - \varepsilon < d_{5,31}$.

Then the map $d_{n,k} \longmapsto 1 - k/2^n$ is an order-preserving bijection between $D \cup \{0\}$ and the set of all (notice that $d_{n+1,4} \longmapsto 1 - 2/2^n$ and $d_{n+2,4} \longmapsto 1 - 1/2^n$) dyadic rationals in the interval $[0,1[$, which is dense in I, too. Its unique extension f to the whole of I is an order-isomorphism $I \approx I$ with the property

(2.7) for any $n \geq 5$ and any $k,p = 3,\dots,2^n$,

$$d_{nk} \boxplus^f d_{n,p} = d_{n,\min(2^n,k+p)}.$$

We have $x \triangle 1 = x \boxplus^f 1 = x$, $x \triangle 0 = x \boxplus^f 0 = 0$ for any $x \in I$. Next, if $0 < x$, $y < 1$ we can take the first $n \geq 5$ with $d_{n,3} > x$, $y > d_{n,2^n-1}$ (this n certainly exists because D is dense in I) and consider the last k and p in $\{3,\dots,2^n\}$ with $d_{n,k} \geq x$ and $d_{n,p} \geq y$, respectively. Then $x > d_{n,k+1}$, $y > d_{n,p+1}$, and

$$\begin{cases} \text{either } k + p > 2^n \text{ whence } x \boxplus^f y \leq d_{n,k} \boxplus^f d_{n,p} = 0 \leq x \Delta y, \\ \text{or } k + p \leq 2^n \text{ whence } x \boxplus^f y \leq d_{n,k} \boxplus^f d_{n,p} = \end{cases}$$

$$= d_{n,k+p} < d_{n,k+1} \Delta\, d_{n,p+1} \leq x \Delta y.$$

Finally we obtain from (e) that $a \boxplus^f b \geq d_{5,13} \boxplus^f d_{5,18} = d_{5,31} > a \Delta b - \varepsilon$.

Thus we only have to construct the family (2.6). Choose a sequence $e_5 < e_6 < \dots < e_n \dots$ with $e_n \nearrow 1$ and fix a well-ordering of the countable dense set D (when we mention the first element of some nonempty subset of D in the sequel we shall be referring to just this ordering). We shall proceed by induction on n.

I. For $n = 5$ first choose $d_{29} \in D$ with $a \Delta b - \varepsilon < d_{29} < a \Delta b$.

Since $a \Delta b > d_{29}$ it follows from 2.1.1 that there exists $d_{18} \in D$ such that $d_{29} < d_{18} < b$, $a \Delta d_{18} > d_{29}$.

Similarly we can use 2.1.1 and the last inequality to ensure the existence of some $d_{13} \in D$ with $d_{18} < d_{13} < a$, $d_{13} \Delta d_{18} > d_{29}$.

Next there exists $d_{17} \in D$ so that $d_{18} < d_{17} < d_{13}$ and $d_{17} \Delta d_{18} > d_{29}$.

Now pick d_{14} through d_{16}, and d_{19} through d_{23} so that $d_{17} < d_{16} < d_{15} < d_{14} < d_{13}$ and $d_{29} < d_{23} < d_{22} < d_{21} < d_{20} < d_{19} < d_{18}$. Because Δ is isotone we have

$$d_k \Delta d_p \geq d_{17} \Delta d_{18} > d_{29}$$

whenever $13 \leq k \leq 17$, $13 \leq p \leq 18$ so that we can successively pick elements d_{24} through d_{28} with the properties

$$d_{29} < d_{24} < d_{23} \wedge (d_{13} \triangle d_{13}),$$
$$d_{29} < d_{25} < d_{24} \wedge (d_{13} \triangle d_{14}),$$
$$d_{29} < d_{26} < d_{25} \wedge (d_{13} \triangle d_{15}) \wedge (d_{14} \triangle d_{14}),$$
$$d_{29} < d_{27} < d_{26} \wedge (d_{13} \triangle d_{16}) \wedge (d_{14} \triangle d_{15}),$$
$$d_{29} < d_{28} < d_{27} \wedge (d_{13} \triangle d_{17}) \wedge (d_{14} \triangle d_{16}) \wedge (d_{15} \triangle d_{15}).$$

Finally we choose d_{30} and d_{31} so that $a \triangle b - \varepsilon < d_{31} < d_{30} < d_{29}$ and put $d_{32} = 0$.

Since $1 \triangle 1 > d_{22}$ and $1 \triangle d_k = d_k > d_{10+k}$ for each $k = 13,\dots,22$, Lemma 2.1.1 guarantees the existence of some $d_{12} \in D$ such that $d_{12} \triangle d_{12} > d_{22}$ and $d_{12} \triangle d_k > d_{10+k}$ for all $k = 13,\dots,22$. We pick one and proceed similarly in all the remaining steps. Thus we obtain in turn:

$d_{11} \in D$ with $d_{11} \triangle d_{11} > d_{20}$ and $d_{11} \triangle d_k > d_{9+k}$; $k = 12,\dots,23$;

$d_{10} \in D$ with $d_{10} \triangle d_{10} > d_{18}$ and $d_{10} \triangle d_k > d_{8+k}$; $k = 11,\dots,24$;

$\vdots$

$d_4 \in D$ with $d_4 \triangle d_4 > d_6$ and $d_4 \triangle d_k > d_{2+k}$; $k = 5,\dots,30$;

and finally $d_3 \in D$ with $d_3 > e_5$, $d_3 \triangle d_3 > d_4$, and $d_3 \triangle d_k > d_{1+k}$; $k = 4,\dots,31$.

Since $\triangle$ is commutative, putting $d_{5,k} = d_k$ for $k = 3,\dots,32$ yields a finite sequence that fulfils, for the fixed $n = 5$, the conditions (b), (d), and (e).

II. Induction step. Assume given a family $\{d_{m,k};\ 5 \le m \le n,\ 3 \le k \le 2^m\}$ such that every $d_{m,k}$ belongs to D, the conditions (b) and (d) are satisfied for all $m \le n$, the condition (c) is satisfied for all $m \le n - 1$, the condition

(e) is satisfied, and $d_{m,3} > e_m$ holds for each $m = 5,\dots,n$.

For any $k = 3,\dots,2^n$ put $d_{n+1,2k} = d_{n,k}$. Then take the first element d of the nonempty subset

$$\{ t \in D \mid t < d_{n,3} \} \setminus \{ d_{n,k} \mid k = 3,\dots,2^n \}$$

in D. There exists the unique k_0 such that $3 \le k_0 \le 2^n - 1$ and $d_{n,k_0+1} < d < d_{n,k_0}$. Put $d_{n+1,2k_0+1} = d$ (this, together with $d_{n,3} > e_n \nearrow 1$, ensures that all elements of D will eventually get included in our family). For $k \neq k_0$, $3 \le k \le 2^n - 1$ pick an arbitrary element $d_{n+1,2k+1} \in D$ so that $d_{n,k+1} < d_{n+1,2k+1} < d_{n,k}$. We have defined all the members $d_{n+1,k}$; $6 \le k \le 2^n$. Obviously $1 > d_{n+1,6} > d_{n+1,7} > \dots > d_{n+1,2^{n+1}} = 0$.

Now we shall verify that

$$d_{n+1,k} \triangle d_{n+1,p} > d_{n+1,k+p-2}$$

holds whenever $6 \le k$, p and $k + p \le 2^{n+1} + 2$. We shall distinguish the following three cases.

1. If $k = 2r$ and $p = 2s$ then $r + s \le 2^n + 1$ and by the induction hypothesis we have $d_{n+1,k} \triangle d_{n+1,p} = d_{n,r} \triangle d_{n,s} > d_{n,r+s-2} = d_{n+1,k+p-4} > d_{n+1,k+p-2}$.

2. If exactly one of the numbers k, p is odd, e.g. $k = 2r$, $p = 2s + 1$ then $r + s \le 2^n + 1$ and we have $d_{n+1,k} \triangle d_{n+1,p} \ge d_{n,r} \triangle d_{n,s+1} > d_{n,r+s-1} = d_{n+1,k+p-3} > d_{n+1,k+p-2}$.

3. If $k = 2r + 1$ and $p = 2s + 1$ then $r + s \le 2^n$ and we have $d_{n+1,k} \triangle d_{n+1,p} \ge d_{n,r+1} \triangle d_{n,s+1} > d_{n,r+s} = d_{n+1,k+p-2}$.

It remains to define $d_{n+1,k}$ for $k = 3,4$, and 5. Again we recall 2.1.1 and choose successively

$d_{n+1,5} \in D$ so that $d_{n+1,5} \Delta d_{n+1,5} > d_{n+1,8}$ and $d_{n+1,5} \Delta d_{n+1,k} >$

$> d_{n+1,3+k}$ for each $k = 6, \ldots, 2^{n+1} - 3$;

$d_{n+1,4} \in D$ so that $d_{n+1,4} \Delta d_{n+1,4} > d_{n+1,6}$ and $d_{n+1,4} \Delta d_{n+1,k} >$

$> d_{n+1,2+k}$ for each $k = 5, \ldots, 2^{n+1} - 2$;

and finally

$d_{n+1,3} \in D$ so that $d_{n+1,3} > e_{n+1}$, $d_{n+1,3} \Delta d_{n+1,3} > d_{n+1,4}$, and

$d_{n+1,3} \Delta d_{n+1,k} > d_{n+1,1+k}$ for each $k = 4, \ldots, 2^{n+1} - 1$.

2.1.3. Let Δ satisfy the assumptions of 2.1. Take a countable dense subset D of I which misses 0 and 1. Since 1 is the unit in (I, Δ) and Δ is lower-semicontinuous the set

(2.8) $\quad A = \{(a,b,m) \mid a,b \in D, a \geq b,\ a \Delta b > 1/m\}$

is infinite countable. Owing to 2.1.2 we can select for each $(a,b,m) \in A$ a tensor product $\square_{a,b,m}$ on I so that the ordered semigroups $(I, \square_{a,b,m})$ and $(I, \boxplus)$ are isomorphic, $x \square_{a,b,m} y \leq$ $\leq x \Delta y$ holds for all $x,y \in I$, and $a \square_{a,b,m} b > a \Delta b - 1/m$.

We set

(2.9) $\quad x \circ y = \bigvee \{x \square_{a,b,m} y \mid (a,b,m) \in A\}$, all $x,y \in I$.

Clearly $\circ \leq \Delta$ holds in $\mathcal{T}(I)$. Now suppose there exist x, $y \in I$ with $x \circ y < x \Delta y$. Then $x,y \neq 0,1$. Since Δ is lower-semicontinuous there exist $x_1 < x$ and $y_1 < y$ such that $x \quad y < x_1 \Delta y_1$. Because D is dense in I we can take some $a,b \in D$ with $x_1 < a <$ $< x$, $y_1 < b < y$, and, say, $a \geq b$. For every natural number $m >$ $> 1/(x_1 \Delta y_1 - x \circ y)$ we then have $a \circ b \geq a \square_{a,b,m} b > a \Delta b -$ $- 1/m \geq x_1 \Delta y_1 - 1/m > x \quad y \geq a \quad b$, which is absurd. Thus $\circ = \Delta$ and the proof of 2.1 is complete.

2.2. <u>Corollary</u>. For any $\square$, $\square' \in \mathcal{T}(I)$ the operation

$\triangle$ defined on I by the formula

$$x \triangle y = (x \square y) \wedge (x \square' y) \tag{2.10}$$

fulfils the assumptions of 2.1 hence $\triangle = \bigvee \mathfrak{A}$ in $\mathcal{O}(I)$ for some subset $\emptyset \neq \mathfrak{A} \subseteq \mathcal{T}(I)$. Thus, if the couple $\{\square, \square'\}$ has a meet in $\mathcal{T}(I)$ then the meet necessarily coincides with (2.10). Conclusion: $\{\square, \square'\}$ has a meet in $\mathcal{T}(I)$ iff the operation (2.10) is associative.

2.3. <u>Corollary</u>. Owing to 2.2 it now suffices to find an example of two tensor products on I whose meet in $\mathcal{O}(I)$ is not associative in order to prove that $\mathcal{T}(I)$ is not a lower semilattice.

<u>Example</u>. Let $\square = \boxplus$ and let $\square' = \boxplus^f$ where the order isomorphism $f\colon I \approx I$ is defined by the formula

$$fx = \begin{cases} x & \text{if } 0 \leq x \leq 1/8 \text{ or } 1/2 \leq x \leq 1 \\ 2x - 1/8 & \text{if } 1/8 \leq x \leq 1/4 \\ x/2 + 1/4 & \text{if } 1/4 \leq x \leq 1/2. \end{cases} \tag{2.11}$$

Then

$$\begin{aligned} 3/4 \boxplus^f 7/8 &= 3/4 \boxplus 7/8 = 5/8, \\ 5/8 \boxplus^f 1/2 &= 5/8 \boxplus 1/2 = 1/8, \\ 7/8 \boxplus^f 1/2 &= f^{-1}(3/8) = 1/4 < 3/8 = 7/8 \boxplus 1/2, \\ 3/4 \boxplus^f 1/4 &= f^{-1}(3/4 \boxplus 3/8) = f^{-1}(1/8) = 1/8 > 0 = \\ &= 3/4 \boxplus 1/4 \end{aligned}$$

hence

$$\begin{aligned} (3/4 \triangle 7/8) \triangle 1/2 &= 5/8 \triangle 1/2 = 1/8 > 0 = 3/4 \triangle 1/4 = \\ &= 3/4 \triangle (7/8 \triangle 1/2) \end{aligned}$$

and the meet $\triangle$ of $\square$ and $\square'$ in $\mathcal{O}(I)$ is not associative. Conclusion: $\mathcal{T}(I)$ is not a lower semilattice.

2.4. <u>Corollary</u>. If $\mathcal{T}(I)$ were an upper semilattice then by Proposition 1.5 all nonempty joins would exist in $\mathcal{T}(I)$. In particular, for any $\square$, $\square' \in \mathcal{T}(I)$ the nonempty set of all lower bounds of $\{\square, \square'\}$ in $\mathcal{T}(I)$ would have a join in $\mathcal{T}(I)$, which contradicts 2.3. Conclusion: $\mathcal{T}(I)$ is not an upper semilattice either.

2.5. <u>Remark</u>. On the other hand, it follows trivially from 2.1 that any $\square \in \mathcal{T}(I)$ is a join in $\mathcal{T}(I)$ of a countable set of elements isomorphic to $\boxplus$. In view of 1.5 it is natural to conjecture that there always exists even a non-decreasing sequence $\{\square_n; n \in \omega\}$ of isomorphs of $\boxplus$ so that $\square_n \nearrow \square$. This, however, remains an open question.

R e f e r e n c e s

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