Jaroslav Haslinger

A note on a dual finite element method

Commentationes Mathematicae Universitatis Carolinae, Vol. 17 (1976), No. 4, 665--673

Persistent URL: http://dml.cz/dmlcz/105726

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

17,4 (1976)

A NOTE ON A DUAL FINITE ELEMENT METHOD

J. HASLINGER, Praha

Abstract: In [9],[10] the construction of suitable subspaces of linear trial vector-functions, admissible for the dual variational formulation was given as well as the proof of the rate of approximation in C-norm. In the present paper we prove the rate of approximation in L^2-norm. This fact permits us to obtain the same results as in [9],[10] under the weaker assumptions on the regularity of the solution.

Key words: Finite elements, equilibrium model.

AMS: 65N30 Ref. Ž.: 8.33

A number of articles has been written on the dual finite element method (see [1] - [10] etc.). In [9],[10] the authors presented some results, using the simpliest finite element "equilibrium model", applying the piecewise linear polynomials to the solution of a mixed boundary value problem for one second order elliptic equation without the absolute term. The rate of convergence $O(h^2)$ was proved, provided the exact solution is sufficiently smooth. Let us introduce some notations. Let Ω be a bounded domain in R_2. By $H^k(\Omega)(k \geq 0$ integer) we denote the set of real functions, which are square-integrable in Ω together with their generalized derivatives up to the order k.

We write $H^0(\Omega) = L^2(\Omega), \vec{H}^k(\Omega) = H^k(\Omega) \times H^k(\Omega)$

with the norm

$$\| \vec{v} \|_{k,\Omega} = (\| v_1 \|^2_{k,\Omega} + \| v_2 \|^2_{k,\Omega})^{1/2},$$

$(\vec{v} = (v_1, v_2))$,

where

$$\| v_i \|_{k,\Omega} = (\int_\Omega \sum_{|\alpha| \le k} | D^\alpha v_i |^2 \, dx)^{1/2}.$$

By

$$| v |_{j,\Omega} = (\int_\Omega \sum_{|\alpha| = j} | D^\alpha v |^2 \, dx)^{1/2}$$

we denote the j-th seminorm in $H^j(\Omega)$.

$C^k(\bar{\Omega})$ denote the space of continuous functions, the derivatives of which up to the order k are continuous and continuously extensible onto $\bar{\Omega}$ $(C^0(\bar{\Omega}) = C(\bar{\Omega}))$. We write $\vec{C}^k(\bar{\Omega}) = C^k(\bar{\Omega}) \times C^k(\bar{\Omega})$ with the norm

$$\| \vec{v} \|_{C^k(\bar{\Omega})} = \max_{i=1,2} \| v_i \|_{C^k(\bar{\Omega})} \text{ and}$$

$$\| v_i \|_{C^k(\bar{\Omega})} = \max_{\substack{|\alpha| \le k \\ x \in \bar{\Omega}}} | D^\alpha v_i(x) |$$

At first, we recall main results from [9]. Let K be a non-degenerate triangle with vertices a_1, a_2, a_3 and set $a_4 = a_1$. For $\vec{v} \in \vec{H}^1(K)$ we define the outward flux

$$T_i \vec{v} = \vec{v}|_{a_i a_{i+1}} \cdot \vec{n}^{(i)} = \bar{v}_1 n_1^{(i)} + \bar{v}_2 n_2^{(i)},$$

where $\vec{n}^{(i)} = (n_1^{(i)}, n_2^{(i)}) \in R_2$ is the outward unit normal to ∂K on $a_i a_{i+1}$, $\bar{v}_i$ are the traces of v_i on $a_i a_{i+1}$. By $P_k(M)$ $(k \ge 0$ integer) we denote the set of all polynomials of the order at most k, defined on the set $M \subseteq R_2$. Let

$\lambda_1^{(i)}$, $\lambda_2^{(i)}$ be the basic linear functions of the side $a_i a_{i+1}$, i.e.

- $\lambda_k^{(i)} \in P_1(a_i a_{i+1})$, $k = 1,2$;
- $\lambda_1^{(i)}(a_i) = 1$, $\lambda_1^{(i)}(a_{i+1}) = 0$;
- $\lambda_2^{(i)}(a_i) = 0$, $\lambda_2^{(i)}(a_{i+1}) = 1$

and let us denote $\int_{a_i a_{i+1}} uv\, ds = [u,v]_i$, $u,v \in L^2(a_i a_{i+1})$.

In [9] we proved

<u>Theorem 1.</u> Let $\vec{u} \in \vec{H}^1(K)$. Then the equations

$$(j)\quad [T_i\vec{u}, \lambda_k^{(i)}]_i = \alpha_i\, [\lambda_1^{(i)}, \lambda_k^{(i)}]_i + \beta_i\, [\lambda_2^{(i)}, \lambda_k^{(i)}]_i \qquad (k = 1,2)$$

$$(jj)\quad \Pi\, \vec{u}(a_i).\vec{n}^{(i)} = \alpha_i,\quad \Pi\, \vec{u}(a_{i+1}).\vec{n}^{(i)} = \beta_i$$

define an operator $\Pi \in \mathcal{L}(\vec{H}^1(K), (P_1(K))^2) \cap \mathcal{L}(\vec{C}(K), (P_1(K))^2)$. 1)

In [9] properties of Π were studied. Let us denote

$\mathfrak{M}(K) = \{\vec{v} = (v_1, v_2), v_j \in P_1(K),\ j = 1,2;\ \operatorname{div} \vec{v} = 0\}$

$U(K) = \{\vec{v} \in \vec{H}^1(K),\ \operatorname{div} \vec{v} = 0\}$

We proved:

(1) $\Pi \in \mathcal{L}(U(K), \mathfrak{M}(K))$

(2) $\Pi\, \vec{v} = \vec{v} \quad \forall\, \vec{v} \in (P_1(k))^2$

----

1) $\mathcal{L}(X,Y)$ denotes the space of linear bounded mappings of X into Y.

(3) $\|\vec{v} - \Pi\vec{v}\|_{\vec{C}(K)} \leq 4\left(1 + \frac{6\sqrt{2}}{\sin\alpha}\right)h^2\|\vec{v}\|_{\vec{C}^2(K)}$

$\forall \vec{v} \in \vec{C}^2(K)$,

where h = diam K and α is the minimal interior angle of K (analogously in R_n for $n>2$, see [10]).

Our aim is to prove the following

Theorem 2. Let $\vec{v} \in \vec{H}^j(K)$, $j = 1,2$. Then

(4) $\|\vec{v} - \Pi\vec{v}\|_{o,K} \leq c\cdot\frac{h^j}{\sin\alpha}|\vec{v}|_{j,K}$,

where h = diam K, α is the minimal interior angle of K and c is an absolute constant.

Before the proof we introduce some notations and we recall the well-known facts. Let $\hat{K}$ be the triangle with the following vertices: $Q_1 = (0,0)$, $Q_2 = (1,0)$, $Q_3 = (0,1)$. One can easily show that there exist the unique affine mapping $F: R_2 \to R_2$, $F(\hat{x}) = B\hat{x} + b$, $B \in \mathcal{L}(R_2,R_2)$ regular, $b \in R_2$ such that $F(\hat{K}) = K$. Let h be the diameter of K and ϱ the diameter of a circle inscribed in K ($\hat{h},\hat{\varrho}$ have the same meaning for $\hat{K}$). In [11] was proved that

(5) $\|B\| \leq \frac{h}{\hat{\varrho}}$, $\|B^{-1}\| \leq \frac{\hat{h}}{\varrho}$

and

(5') $\frac{1}{2\,tg\frac{\alpha}{2}} \leq \frac{h}{\varrho} \leq \frac{2}{\sin\alpha}$ (α is the same as in th.2).

Lemma 1. Let Π be defined through (j),(jj). Then

(6) $\|\Pi\vec{v}\|_{o,K} \leq \frac{\hat{c}}{\sin\alpha}|\det B|^{1/2}\|\hat{\vec{v}}\|_{1,\hat{K}}$ $\forall \vec{v} \in \vec{H}^1(K)$,

where $\hat{\vec{v}} = \vec{v} \circ F = (v_1 \circ F, v_2 \circ F)$, $\hat{c}$ is an absolute constant.

<u>Proof</u>: using Fubini's theorem:

$$\| \Pi \vec{v} \|_{o,K} = |\det B|^{1/2} \| \widehat{\Pi \vec{v}} \|_{o,\hat{K}} \leq 2 \det|B|^{1/2}$$

$$(\text{mes } \hat{K})^{1/2} \| \widehat{\Pi \vec{v}} \|_{\vec{C}(\hat{K})} = \sqrt{2} |\det B|^{1/2} \| \Pi \vec{v} \|_{\vec{C}(K)}.$$

Let $a_i a_{i+1} = F(I)$, where I is a side of $\hat{K}$, which is determined by (0,0),(1,0) and let $F|_I$ be the restriction of F on I. Then it holds:

$$\widehat{T_i \vec{v}} = \hat{\bar{v}}_1 n_1^{(i)} + \hat{\bar{v}}_2 n_2^{(i)} \qquad (\hat{\bar{v}}_i = \bar{v}_i \circ F \,|\, _I).$$

Hence

$$| [T_i \vec{v}, \lambda_k^{(i)}]_i | = | \int_{a_i a_{i+1}} T_i \vec{v} \, \lambda_k^{(i)} \, ds | = q_i | \int_0^1 \widehat{T_i \vec{v}} \, \hat{\lambda}_k^{(i)} d\hat{s} | \leq$$

$$\leq q_i (\int_0^1 | \widehat{T_i \vec{v}} |^2 d\hat{s})^{1/2} \leq \hat{\beta} q_i \| \hat{\vec{v}} \|_{1,\hat{K}},$$

where q_i is the length of $a_i a_{i+1}$, $\hat{\lambda}_k^{(i)} = \lambda_k^{(i)} \circ F|_I$ and $\hat{\beta}$ is the norm of the mapping $\gamma : \vec{H}^1(K) \longrightarrow \vec{L}^2(\partial K)$ such that $\gamma \vec{v} = (\bar{v}_1, \bar{v}_2)$ ($\bar{v}_i$ are the traces of v_i on ∂K). A direct calculation yields that

$$\det A^{(i)} = \frac{1}{12} q_i^2,$$

where $A^{(i)}$ is the matrix of the system (j). Using Cramer's rule we obtain

$$|\alpha_i| \leq \hat{c} \| \hat{\vec{v}} \|_{1,\hat{K}}, \quad |\beta_i| \leq \hat{c} \| \hat{\vec{v}} \|_{1,\hat{K}}.$$

From (jj) and Cramer's rule it follows e.g. for $\Pi \vec{v}(a_2) =$ $= (w_1(a_2), w_2(a_2))$:

$$|w_1(a_2)| = \left|\det \begin{Vmatrix} \beta_1, & n_2^{(1)} \\ \alpha_2, & n_2^{(2)} \end{Vmatrix} \cdot \det (\vec{n}^{(1)}, \vec{n}^{(2)}\right| \leq$$

$$\leq \hat{c} \, \frac{1}{\sin \alpha} \, \|\hat{\vec{v}}\|_{1,\hat{K}}$$

because the det $(\vec{n}^{(1)}, \vec{n}^{(2)})$ is equal to the sinus of the angle between $\vec{n}^{(1)}, \vec{n}^{(2)}$. Similar estimates hold for the remaining values of $\vec{w}$ at the vertices. The assertion of our lemma now follows from the fact that $\Pi \vec{v} \in (P_1(K))^2$.

Proof of Theorem 2: for $j = 2$ (analogously for $j = 1$). It holds:

$$(7) \quad \|\vec{v} - \Pi \vec{v}\|_{0,K} = \sup_{\vec{g} \neq 0} \frac{(\vec{v} - \Pi \vec{v}, \vec{g})}{} \quad .$$

Let us denote

$$(8) \quad f(\vec{v}) = (\vec{v} - \Pi \vec{v}, \vec{g})_{0,K} = |\det B| \, (\hat{\vec{v}} - \widehat{\Pi \vec{v}}, \hat{\vec{g}})_{0,\hat{K}} =$$

$$= |\det B| \hat{f}(\hat{\vec{v}}),$$

where $\hat{\vec{v}} = (\hat{v}_1, \hat{v}_2)$, $\hat{v}_i = v_i \circ F$. Let us examine the functional $\hat{f}$. From (2) and (8):

$$(9) \quad \hat{f}(\hat{\vec{v}}) = 0 \qquad \forall \hat{\vec{v}} \in (P_1(K))^2$$

Now

$$(10) \quad |\hat{f}(\hat{\vec{v}})| \leq \|\hat{\vec{g}}\|_{0,\hat{K}} \|\hat{\vec{v}} - \widehat{\Pi \vec{v}}\|_{0,\hat{K}} \leq \|\hat{\vec{g}}\|_{0,\hat{K}} (\|\hat{\vec{v}}\|_{1,\hat{K}} +$$

$$+ \|\widehat{\Pi \vec{v}}\|_{0,\hat{K}}).$$

Using (6) we estimate $\|\widehat{\Pi \vec{v}}\|_{0,\hat{K}}$:

$$\|\widehat{\Pi \vec{v}}\|_{0,\hat{K}} = |\det B|^{-1/2} \|\Pi \vec{v}\|_{0,K} \leq \frac{\hat{c}}{\sin\alpha} \|\hat{\vec{v}}\|_{1,\hat{K}}.$$

From this and (10):

$$(11) \quad |\hat{f}(\hat{\vec{v}})| \leq \|\hat{\vec{g}}\|_{0,\hat{K}} \left(1 + \frac{\hat{c}}{\sin\alpha}\right) \|\hat{\vec{v}}\|_{1,\hat{K}} \leq$$

$$\leq |\det B|^{-1/2} \|\vec{g}\|_{0,K} \left(1 + \frac{\hat{c}}{\sin\alpha}\right) \|\hat{\vec{v}}\|_{2,\hat{K}}.$$

Using (11) and Bramble-Hilbert lemma (see [11],[12]) we obtain:

$$(12) \quad |\hat{f}(\hat{\vec{v}})| \leq c|\det B|^{-1/2} \|\vec{g}\|_{0,K} \left(1 + \frac{\hat{c}}{\sin\alpha}\right) |\hat{\vec{v}}|_{2,\hat{K}}$$

where c is an absolute constant. Using the well-known fact that (see [11])

$$|\hat{\vec{v}}|_{2,\hat{K}} \leq \|B\|^2 |\det B|^{-1/2} |\vec{v}|_{2,K}$$

and (8),(12):

$$|f(\vec{v})| \leq c\left(1 + \frac{\hat{c}}{\sin\alpha}\right) \|B\|^2 \|\vec{g}\|_{0,K} |\vec{v}|_{2,K}.$$

From this, (5),(5') and (7) we obtain the assertion of our theorem.

For details how to use Theorem 2, see [9],[10].

References

[1] B. Fraeijs de VEUBEKE: Displacement and equilibrium models in the finite element method, Stress Analysis, ed. by O.C. Zienkiewicz and G. Holister, J. Wiley, 1965, 145-197.

[2] B. Fraeijs de VEUBEKE, O.C. ZIENKIEWICZ: Strain energy bounds in finite-element analysis by slab analogies, J. Strain Analysis 2(1967), 265-271.

[3] V.B.Jr. WATWOOD, B.J. HARTZ: An equilibrium stress field model for finite element solution of two-dimensional elastostatic problems, Int.J. Solids Structures 4(1968), 857-873.

[4] B. Fraeijs de VEUBEKE, M. HOGGE: Dual analysis for heat conduction problems by finite elements, Int. J. Numer. Meth. Eng.(1972), 65-82.

[5] J.P. AUBIN, H.G. BURCHARD: Some aspects of the method of the hypercicle applied to elliptic variational problems, Numer. Sol. Part. Dif. Eqs. II, Synspade (1970), 1- 67.

[6] J. VACEK: Dual variational principles for an elliptic partial differential equation, Apl. mat. 18 (1976), 5-27.

[7] G. GRENACHER: A posteriori error estimates for elliptic partial differential equations. Inst. Fluid Dynamics and Appl. Math.,Univ. Maryland, TN-BN-T 43, July 1972.

[8] J.M. THOMAS: Méthode des éléments finis équilibre pour les problèmes elliptiques du 2-ème ordre. To appear.

[9] J. HASLINGER; I. HLAVÁČEK: Convergence of a finite element method based on the dual variational formulation, Apl. mat. 21(1976), 43-65.

[10] J. HASLINGER; I. HLAVÁČEK: Convergence of a dual finite element method in R_n, Comment. Math. Univ. Carolinae 16(1975), 369-486.

[11] P.G. CIARLET, P.A. RAVIART: General Lagrange and Hermite interpolation in R^n with applications to finite element method, Arch. Rat. Mech. Anal. 46(1972),177-199.

[12] J.H. BRAMBLE, M. ZLÁMAL: Triangular elements in the finite element method, Math. Comp. 24(1970), 809-820.

Matematicko-fyzikální fakulta
Karlova universita
Malostranské nám.25,11000 Praha 1
Československo

(Oblatum 19.5.1976)