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Evelyn Nelson

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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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GALOIS CONNECTIONS AS LEFT ADJOINT MAPS

Evelyn NELSON, Hamilton x)

Abstract: The "tensor product" of two partially ordered sets has been defined in the literature as the set of all Galois connections between them. Investigations of this construct have usually yielded pleasant results only when the p.o. sets under consideration were complete. The approach of the title is used to clarify the reasons for this phenomenon, provide simple proofs of many of the results for complete p.o. sets, and show that in the category of all (bounded) partially ordered sets, most of the usual properties of a "tensor product" are lacking.

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It is well-known fact (see, for example, MacLane [4, p. 93] that for p.o. (partially ordered) sets A and B, the maps from A to B which are the first half of a Galois connection are precisely those order-preserving maps from A to the dual of B which are left adjoint functors between the two p.o. sets, qua categories. The structure of the set of all Galois connections between two p.o. sets, ordered pointwise, and its connection with bimorphisms, especi-

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ally for complete p.o. sets, was investigated by Shmuely [7]. It is the purpose of this note to put Shmuely's results in a somewhat different perspective by using the approach indicated in the title, and to show exactly the extent to which the results for the special case of complete p.o. sets can be extended to arbitrary (bounded) p.o. sets.

The main point seems to be that in the category of all bounded p.o. sets and left adjoint maps, the formation of hom sets, which are ordered quite naturally by the "pointwise" order, is close to being an internal hom functor, but in ways which will be made more explict, is rather badly behaved, while the restriction of this construction to the full subcategory of all complete p.o. sets and complete-join-preserving maps provides a functional (in the sense of Banaschewski-Nelson [2]) internal hom functor for which there is a dualizer (the two-element chain) and hence for which there is a tensor multiplication which also provides universal bimorphisms. In sharp contrast, we will see that the larger category does not have universal bimorphisms.

§ 1. Preliminaries. Recall that for p.o. sets A and B, a functor from A to B, as categories, is just an order preserving map $A \longrightarrow B$, and an order-preserving $f: A \longrightarrow B$ is left adjoint iff there exists a (right) adjoint functor $f^{\#} : B \longrightarrow A$ such that $f(a) \leq b$ iff $a \leq f^{\#}(b)$ for all $a \in A$, $b \in B$. Note that the adjoint, $f^{\#}$, is unique, that

$ff^{\#}f$ and $f^{\#}ff^{\#}$ are the identity maps on A and B respectively, and $a \leq f^{\#}f(a)$, $ff^{\#}(b) \leq b$ for all $a \in A$, $b \in B$.

The dual A^d of a p.o. set has the same underlying set as A, and $a \leq b$ in A^d iff $b \leq a$ in A.

We will consider the category PA of all bounded p.o. sets (i.e. thos p.o. sets with a largest element, 1, and a smallest element, 0) and all left adjoint maps between them. Note that every morphism in PA preserves all existing joins (since these are coproducts in A, as a category) and so in particular maps 0 (which is the join of the empty set) to 0. Moreover if A is complete then it is an immediate consequence of the special adjoint functor theorem that every order-preserving map $f: A \longrightarrow B$ which preserves all joins is a left adjoint map; in fact, for $b \in B$, $f^{\#}(b) = \bigvee \{ a \in A \mid f(a) \leq b \}$. Consequently the category CJSL of all complete p.o. sets and complete-join-preserving maps (i.e. complete join semilattices and their homomorphisms) is a full subcategory of PA.

For a left adjoint $f: A \longrightarrow B$ in PA, let $\hat{f}: \hat{A} \longrightarrow \hat{B}$ (where $\hat{A}$, $\hat{B}$ are the MacNeille completions of A and B) be defined by

$$\hat{f}(x) = \bigvee \{ f(a) \mid a \in A,\ a \leq x \}$$

and define $g: \hat{B} \longrightarrow \hat{A}$ by

$$g(y) = \bigwedge \{ f^{\#}(b) \mid b \in B,\ b \geq y \}.$$

Since the MacNeille completion of a p.o. set is a join- and meet-dense extension (see Banaschewski-Bruns [1]) it follows that for $x \in \hat{A}$, $y \in \hat{B}$,

$$\begin{aligned} \hat{f}(x) \leq y \text{ iff } & f(a) \leq b \text{ for all } a \in A,\ b \in B \text{ with } a \leq x,\ y \leq b \\ \text{iff } & a \leq f^{\#}(b) \text{ for all } a \in A,\ b \in B \text{ with } a \leq x, \\ & \quad y \leq b \\ \text{iff } & x \leq g(y). \end{aligned}$$

Consequently $\hat{f}$ is a left-adjoint map, with adjoint g. Again because $\hat{A}$ is a join-dense extension of A, it follows that $\hat{f}$ is the only left-adjoint extension of f; consequently <u>every morphism</u> f: A $\longrightarrow$ B <u>has a unique extension to a morphism</u> $\hat{f}$: $\hat{A} \longrightarrow \hat{B}$. Since the dual of $\hat{B}$ is the completion of the dual of B, this yields Theorem 1.2 of Shmuely [7].

In particular, each morphism A $\longrightarrow$ B in PA where B $\in$ CJSL has a unique extension to a morphism $\hat{A} \longrightarrow$ B. However, this does <u>not</u> provide a reflection from PA into CJSL, since the restriction to A of a morphism $\hat{A} \longrightarrow$ B in CJSL, although it will preserve all joins existing in A, need not be a left-adjoint map. An easy example of such a situation is the following. Let A, B and C be the p.o. sets pictured in Figure 1. B is clearly an essential extension of A in the category of all p.o. sets and order-preserving maps, which is join- and meet-dense, and hence (see [1]), B is the MacNeille completion of A. The map f: B $\longrightarrow$ C which maps e and everything below it to 0, and everything else identically, is a left adjoint map. However if g: C $\longrightarrow$ A were the right adjoint of fi(i: A $\longrightarrow$ B the inclusion map) then fi(a)= = 0 = fi(b) would imply $a \leq g(0)$ and $b \leq g(0)$ and so without loss of generality c g(0), which would imply $fi(c) \leq 0$, a contradiction.

§ 2. <u>The</u> hom <u>functors</u>. For A, $B \in PA$, let H(A,B) be the set of all morphisms $A \longrightarrow B$ with the pointwise partial order. H(A,B) has as smallest element the constant map with value 0, and as largest element the map which takes 0 to 0 and everything else to 1, and so $H(A,B) \in PA$. In fact H(,) provides a functor $PA^{op} \times PA \longrightarrow PJ$, the category of all bounded p.o. sets and maps preserving all existing joins; for $f: A' \longrightarrow A$, $g: B \longrightarrow B'$ in PA, H(f,g): : $H(A,B) \longrightarrow H(A',B')$ is given by H(f,h)(h) = ghf.

Note that if A and B are complete then H(A,B), being closed under pointwise joins, is also complete, and so for any $f: A' \longrightarrow A$, $g: B \longrightarrow B'$ in PA, H(f,g) is a left adjoint map. Thus H(,), restricted to CJSL, provides an internal hom functor on the smaller category.

For each $A \in PA$ and $a \in A$, the map $\mu_a: A \longrightarrow 2$ (2 the two-element chain) such that $\mu_a(x) = 0$ iff $x \leq a$, has a right adjoint ($\mu_a^{\#}(0) = a$, $\mu_a^{\#}(1) = 1$) and moreover an easy calculation shows that the correspondence $a \rightsquigarrow \mu_a$ provides an isomorphism of A^d with $A^* = H(A,2)$.

Now for each left adjoint map $f: A \longrightarrow B$, $f^{\#}$ is a left adjoint map $B^d \longrightarrow A^d$; in view of the fact that $A^{dd} =$ $= A$ this gives an isomorphism $H(A,B) \longrightarrow H(B^d,A^d)$. Furthermore for $b \in B$, $\mu_{f^{\#}(b)}(x) = 0$ iff $x \leq f^{\#}(b)$ iff $f(x) \leq b$ iff $\mu_b(f(x)) = 0$ and so $\mu_{f^{\#}(b)} = \mu_b f$. In view of the aboe remarks it follows that the map $H(A,B) \longrightarrow H(B^*,A^*)$ given by $f \rightsquigarrow f^*$ where $f^*(h) = hf$ is an isomorphism. Consequently the functor H(,2) = ($)^*$ provides a selfduality of PA.

One comment in passing: a simple calculation shows that for all $A \in PA$, $H(A, n+1)$ (where $n+1$ is the $n+1$-element chain $0 < 1/n < 2/n < \ldots < 1$) is isomorphic to the dual of the set of all n-tuples $(a_1, \ldots, a_n) \in A^n$ with $a_1 \leq \leq a_2 \leq \ldots \leq a_n$ and with the pointwise partial ordering. For such an n-tuple $(a_1, \ldots, a_n) \in A^n$, the map $f: A \longrightarrow n+1$ corresponding to it is given by $f(a) = j/n$ if j is the largest number with $a \leq a_j$ if such exists, $f(a) = 1$ otherwise.

Since every order-preserving map from a chain into a p.o. set B which maps 0 to 0 is left adjoint, the above discussion implies that for p.o. sets A and B, and $a_1 \leq a_2 \leq \leq \ldots \leq a_n$ in A, $b_1 \leq b_2 \leq \ldots \leq b_n$ in B, the map $f: A \longrightarrow B$ such that $f(a) = b_j$ if j is the largest number with $a \leq a_j$ if such exists, $f(a) = 1$ otherwise, is a left adjoint map.

<u>Lemma 1</u>: <u>For all</u> $A, B \in PA$ <u>and</u> $a \in A$, <u>the map</u> $e_{AB}(a)$: $: H(A,B) \longrightarrow B$ <u>given by</u> $e_{AB}(a)(f) = f(a)$ <u>is a left adjoint map</u>.

<u>Proof</u>: For $b \in B$, let $\psi(b): A \longrightarrow B$ be given by $\psi(b)(0) = 0$, $\psi(b)(x) = b$ if $0 < x \leq a$, $\psi(b)(x) = 1$ if $x \nleq a$. The discussion in the preceding paragraph shows that $\psi(b) \in H(A,B)$. Moreover $\psi: B \longrightarrow H(A,B)$ is clearly order-preserving, and for $f \in H(A,B)$, $e_{AB}(a)(f) \leq b$ iff $f(a) \leq b$ iff $f \leq \psi(b)$; consequently ψ is right adjoint to $e_{AB}(a)$ and this establishes the proof.

In the special case that A is complete, e_{AB}, since it clearly preserves all joins, is a left adjoint map. Also for all a, $e_{A2}: A \longrightarrow A^{**}$ is an isomorphism, and hence a left adjoint map.

Proposition 1: For A the six-element p.o. set in Figure 1, e_{A3} is not a left adjoint map.

Proof: Let f, g: $A \longrightarrow 3$ be given by $f(0) = 0$, $f(a) = f(b) = f(c) = 1/2$, $f(d) = f(1) = 1$, and $g(0) = g(a) = g(b) = g(d) = 0$, $g(c) = g(1) = 1/2$. Then f and g are left adjoint maps, and $g \leq f$. Now let $\phi : H(A,3) \longrightarrow 3$ be the map defined by

$$\phi(h) = \begin{cases} 0 & \text{if } h \leq g \\ 1/2 & \text{if } h \not\leq g \text{ and } h \leq f \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi \in H(H(A,3),3)$.

Also, for all $h \in H(A,3)$, if $\phi(h) = 0$ then $h(a) \leq g(a) = 0$, if $\phi(h) = 1/2$ then $h(a) \leq f(a) = 1/2$, and consequently $e_{A3}(a) \leq \phi$. Similarly $e_{A3}(b) \leq \phi$. However, if e_{A3} had an adjoint then we would have $a \leq e_{A3}{}^{\#}(\phi)$ and $b \leq e_{A3}{}^{\#}(\phi)$ and consequently either $e_{A3}(c) \leq \phi$ or $e_{A3}(d) \leq \phi$. But $e_{A3}(c)(g) = g(c) = 1/2 > 0 = \phi(g)$, and $e_{A3}(d)(f) = 1 > 1/2 = \phi(f)$ and so this would give a contradiction. Consequently e_{A3} does not have a right adjoint.

For all A, B, $C \in PA$ and $\phi \in H(A,H(B,C))$ there is a map $\tilde{\phi} : B \longrightarrow H(A,C)$ given by $\tilde{\phi}(b)(a) = \phi(a)(b)$; the fact that for each $b \in B$, $\tilde{\phi}(b) \in H(A,C)$ follows from Lemma 1 and the fact that $\tilde{\phi}(b) = e_{BC}(b)\phi$. If B is complete then $\tilde{\phi}$, since it clearly preserves all joins, is a left adjoint map, and thus there is an order-embedding S_{ABC}: $H(A,H(B,C)) \longrightarrow H(B,H(A,C))$. In general $\tilde{\phi}$ is not a left adjoint map; for example if $\phi \in H(H(A,3), H(A,3))$ is

the identity map, A as in Figure 1, then $\tilde{\phi}: A \longrightarrow$
$\longrightarrow H(H(A,3),3)$ is e_{A3} which by Proposition 1 is not a left adjoint map.

If A and B are both complete this evidently gives a natural isomorphism between $H(A,H(B,C))$ and $H(B,H(A,C))$; also for all A and B, $S_{AB2}: H(A,B^*) \longrightarrow H(B,A^*)$ is an isomorphism.

On the other hand, even if both B and C are complete, $H(A,H(B,C))$ and $H(B,H(A,C))$ need not be isomorphic (naturally or otherwise).

Proposition 2: For A the six-element p.o. set in Fifure 1, $H(A,H(3\ 3)) \not\cong H(3, H(A,3))$

Proof: Since 3 is complete, the map S_{A33}:
$: H(A,H(3,3) \longrightarrow H(A,3))$ is an order embedding.
Both these p.o. sets are finite; we will establish the claim by showing that they have different cardinalities.

First of all, the elements of $H(3,3)$ are in one-one correspondence with the pairs $(x,y) \in 3 \times 3$ with $x \not\leq y$; these can easily be listed an then one sees that $H(3,3)$ is the six-element lattice diagrammed in Figure 2.

Now the left adjoint maps $h: H(3,3) \longrightarrow A$ are all those maps which preserve all joins in $H(3,3)$, i.e. all order-preserving maps $h: H(3,3) \longrightarrow A$ with $h(0) = 0$, and $h(s) \vee h(t) = h(u)$. Now for $x, y \in A$, $x \vee y$ exists iff $x \leq y$, $y \leq x$, or $\{x,y\} = \{c,d\}$. Consequently an order-preserving map h:
$: H(3,3) \longrightarrow A$ with $h(0) = 0$ is a left adjoint iff either $h(s) = h(u)$, $h(t) = h(u)$, or $\{h(s),h(t)\} = \{c,d\}$ and $h(u) = 1$.

If $\{h(s),h(t)\} = \{c,d\}$ then $h(1) \geq h(u) = 1$ so $h(1) = 1$; also $h(r) \leq h(s)$ and $h(r) \leq h(t)$ and so $h(r) \in \{0,a,b\}$. On the other hand, each map $h: H(3,3) \to A$ with $\{h(s),h(t)\} = \{c,d\}$, $h(u) = h(1) = 1$, $h(0) = 0$ and $h(r) \in \{0,a,b\}$ is a left adjoint map. There are 6 such maps.

The remaining left-adjoint maps $H(3,3) \to A$ fall into three types; those with $h(s) = h(t) = h(u)$, those with $h(s) = h(u) \neq h(t)$ and those with $h(t) = h(u) \neq h(s)$, and moreover there is no overlap among these types. There are as many maps of the first type as there are order-preserving maps of the three-element chain into A, i.e. as many as there are triples $(a_1,a_2,a_3) \in A^3$ with $a_1 \leq a_2 \leq a_3$; these can easily be counted, and there are 44 of them.

There are as many maps of the second kind as there are order-preserving maps of the four element chain $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$ into A which do not identify $\frac{1}{3}$ and $\frac{2}{3}$, i.e. as many as there are quadruples $(a_1,a_2,a_3,a_4) \in A^4$ with $a_1 \leq a_2 \leq a_3 \leq a_4$. These are also easily counted; there are 41 of them.

There are as many maps of the third type as there are of the second, and hence $|H(H(3,3)A)| = 6 + 44 + 41 = 132$.

Now, to count the elements of $H(H(A,3),3)$: Since A and 3 are self-dual, $H(A,3) \cong H(3,A)$ and hence $H(A,3)$ is isomorphic to the set of all pairs $(x,y) \in A^2$ with $x \leq y$ and with the point-wise ordering. It is easy to list these pairs, there are 19 of them. $H(3,A)$ is diagrammed in Figure 4. Again, the elements of $H(H(A,3),3)$ are in one-one correspondence with the ordered pairs $(x,y) \in H(3,A)^2$ with $x \leq y$; using Figure 4, these can easily be counted. There are 138 altogether.

This establishes the point.

§ 3. Bimorphisms and tensor products. For A, B ∈ PA, we define $A \otimes B = H(A,B^*)^*$. (Shmuely [7] defines $A \otimes B$ to be $H(A,B^d)^d$, which is of course naturally isomorphic to $H(A,B^*)^*$). Note that there is a natural isomorphism $A \otimes B \longrightarrow B \otimes A$ given by S_{AB2}^* .

A set map $f: A \times B \longrightarrow C$ is a bimorphism for A, B, C ∈ PA, iff $f(a,-): B \longrightarrow C$ and $f(-,b): A \longrightarrow C$ are left-adjoint maps for all $a \in A$, $b \in B$.

Note that every bimorphism $f: A \times B \longrightarrow C$ has a unique extension to a bimorphism $\bar{f}: \hat{A} \times \hat{B} \longrightarrow \hat{C}$; this is Theorem 2.2 of Shmuely [7] but can easily be seen as follows: Let $\bar{f}$: : $\hat{A} \times \hat{B} \longrightarrow \hat{C}$ be defined by $\bar{f}(a,b) = \bigvee \{ f(x,y) \mid x \in A,\ y \in B,\ x \leq a,\ y \leq b \}$. Then for each $a \in \hat{A}$,

$$\begin{aligned} \bar{f}(a,-) &= \bigvee \{ \bar{f}(x,-) \mid x \in A,\ x \leq a \} \\ &= \bigvee \{ f(x,-)^\wedge \mid x \in A,\ x \leq a \} \end{aligned}$$

where for $x \in A$, $f(x,-)^\wedge$ is the extension of $f(x,-)$ to a left adjoint map $\hat{B} \longrightarrow \hat{C}$. Since $f(x,-)^\wedge$ preserves all joins for each $x \in A$, it follows that $\bar{f}(a,-)$ also preserves all joins and hence is a left-adjoint map. Similarly $\bar{f}(-,b)$ is left-adjoint for all $b \in \hat{B}$. The uniqueness of $\bar{f}$ follows from the fact that A and B are join-dense in $\hat{A}$ and $\hat{B}$ respectively.

Proposition 3: For all $A, B \in PA$, the map $\tau :$ $: A \times B \to A \otimes B$ given by $\tau(a,b)(\phi) = \phi(a)(b)$ is a bimorphism.

Proof: Since the map $A \times B \to B \times A$ given by $(a,b) \rightsquigarrow (b,a)$ is a p.o. set isomorphism, and $S_{AB2}^{*} : H(B,A^{*})^{*} \to H(A,B^{*})^{*}$ is an isomorphism, it is enough to show that $\tau(a,-)$ is a left adjoint map from B to $H(A,B^{*})^{*}$ for all $a \in A$.

First of all, for each $a \in A$, $b \in B$, $\tau(a,b) = e_{B2}(b)e_{AB^{*}}(a)$ and so by Lemma 1, $\tau(a,b)$ is a left adjoint map, i.e., $\tau(a,b) \in H(A,B^{*})^{*}$.

Now, let $a \in A$, and define $\psi : H(A,B^{*})^{*} \to B$ by

$$\psi(h) = (h^{\#}(0)(a))^{\#}(0).$$

Then for $g, h \in H(A,B^{*})^{*}$, $g \leq h$ iff $h^{\#}(0) \leq g^{\#}(0)$ iff $h^{\#}(0)(a) \leq g^{\#}(0)(a)$ iff $(g^{\#}(0)(a))^{\#}(0) \leq (h^{\#}(0)(a))^{\#}(0)$ and so ψ is order-preserving.

Moreover for $b \in B$, $h \in H(A,B^{*})^{*}$,

$$\begin{aligned} b \leq \psi(h) &\text{ iff } b \leq (h^{\#}(0)(a))^{\#}(0) \\ &\text{ iff } h^{\#}(0)(a)(b) = 0 \\ &\text{ iff } \tau(a,b)(h^{\#}(0)) = 0 \\ &\text{ iff } \tau(a,b) \leq h \end{aligned}$$

and thus $\tau(a,-)$ is a left adjoint map, with right adjoint ψ.

Corollary (Shmuely [7]): If $A, B \in PA$ are non-trivial (i.e. have at least two elements) and $H(A,B^{*})$ is complete then A and B are complete; if in addition $H(A,B^{*})^{*}$ is completely distributive, then so are A and B.

Proof: If b, $c \in B$ and $b \nleq c$ then the map $\phi : A \to B^*$ with $\phi(0) = 0$, $\phi(x) = \mu_b$ for $x \neq 0$ is a left adjoint map, and for all $0 \neq a \in A$, $\phi(a)(b) = 0$, $\phi(a)(c) = 1$. Thus for all $a \in A$ with $a \neq 0$, $\tau(a,-)$ is one-one and hence its right adjoint is a p.o. set retraction of $H(A,B^*)^*$ onto A. Since retracts of complete p.o. sets are complete (Banaschewski - Bruns [1]) this established the first point.

Now a retract of a complete, completely distributive p.o.set by a map which preserves all joins is itself complete and completely distributive since these two conditions exactly characterize the injectives in CJSL (see Crown [3]). If $H(A,B^*)^*$ is complete and completely distributive then since the retraction $H(A,B^*)^* \to A$, being a right adjoint, preserves all meets, it follows by the dual of Crown's result that A is also complete and completely distributive.

Now, the results in § 1 establish the facts that H(,), restricted to CJSL, provides an internal hom functor for which there is a dualizer, and hence (see Banascheski-Nelson [2]) for complete A and B, $\tau : A \times B \to A \otimes B$ is a universal bimorphism on $A \times B$ in CJSL, and the functor given by $(A,B) \rightsquigarrow A \otimes B$ is a tensor multiplication relative to H. The fact that CJSL has universal bimorphisms is also contained in Shmuely [7, Lemma 1.7] and Mowatt [6]; the fact that $(A,B) \rightsquigarrow A \otimes B$ provides a tensor multiplication for the internal hom functor is due to Waterman [8].

Furthermore, for A, $B \in$ CJSL, $A \otimes B \cong H(A \otimes B,2)^d \cong \cong BiM(A,B,2)^d$ where BiM(A,B,2) is the set of all bimorphisms $A\times B \longrightarrow 2$ with the pointwise partial order, and this yields Theorem I.3 of Shmuely [7].

However, even if A and B are complete, $\tau : A\times B \longrightarrow$ $\longrightarrow A\otimes B$ is in general <u>not</u> a universal bimorphism in PA; in fact the situation is even worse than that. We will see below that there is no universal bimorphism in PA on 3×3.

For each ordinal n, let A_n be the p.o. set in Figure 3. That is, the underlying set of A_n is $(n\times 2)\cup\{0,1\}$, and the order is given by

$(i,j)<(k,h)$ iff $i<k$

and $0<(i,j)<1$ for all $(i,j)\in n\times 2$.

<u>Lemma 2</u>: <u>For infinite</u> n, <u>if</u> $f: B\longrightarrow A_n$ <u>is a left adjoint map and</u> $(0,0)$, $(0,1)\in \operatorname{im}f$ <u>then</u> $ff^{\#}$ <u>is the identity map on</u> A_n <u>and hence</u> f <u>maps onto</u> A_n.

Proof: Since $ff^{\#}f = f$ it follows that for all $x\in \operatorname{im}f$, $ff^{\#}(x) = x$, and so in particular $ff^{\#}(0) = 0$, $ff^{\#}(0,0) =$ $= (0,0)$, $ff^{\#}(0,1) = (0,1)$.

Assume for some $\lambda < n$, that $ff^{\#}(i,j) = (i,j)$ for all $i<\lambda$, $j\in\{0,1\}$ $(\lambda>0)$. Then $ff^{\#}(\lambda,0)\geq ff^{\#}(i,j)\geq$ $\geq (i,j)$ for all $i<\lambda$, $j\in\{0,1\}$ and hence either $ff^{\#}(\lambda,0)\geq(\lambda,0)$ or $ff^{\#}(\lambda,0)\geq(\lambda,1)$. But $(\lambda,0)\geq$ $\geq ff^{\#}(\lambda,0)$ and so $ff^{\#}(\lambda,0)\geq(\lambda,1)$ would imply $(\lambda,0)\geq$ $\geq(\lambda,1)$, a contradiction. Consequently $ff^{\#}(\lambda,0)\geq(\lambda,0)$ and so $ff^{\#}(\lambda,0) = (\lambda,0)$. By symmetry $ff^{\#}(\lambda,1) = (\lambda,1)$.

Thus $ff^{\#}(i,j) = (i,j)$ for all $(i,j)\in n\in\{0,1\}$. Since $ff^{\#}(1)\geq ff^{\#}(i,j)$ for all $(i,j)\in n\times\{0,1\}$, this implies that $ff^{\#}(1) = 1$ and this yields the result.

<u>Proposition 3</u>: <u>There is no universal bimorphism on</u> 3×3 <u>in</u> PA.

Proof: Suppose $\phi : 3\times 3 \to B$ were a universal bimorphism in PA. Take $n > |B|$, and let $f: 3\times 3 \to A_n$ be given by $f(0,x) = f(x,0) = 0$ for all $x\in 3$, $f(1/2, 1/2) = 0$, $f(1/2,1) = (0,0)$, $f(1,1/2) = (0,1)$ and $f(1,1) = (1,0)$. Then for each $x\in 3$, $f(-,x)$ and $f(x,-)$ map 0 to 0 and preserve order and so are left adjoint maps. Thus f is a bimorphism and hence there exists a left adjoint $g: B \to A_n$ with $g\phi = f$, by the universality of ϕ. But then $(0,1)$ and $(1,0) \in \operatorname{im} g$ and so by Lemma 2, g maps onto A_n, which is impossible by the choice of n.

Another immediate consequence of the above Lemma, using an analogous argument to the one in the proof of Proposition 3, is that the category PA does not have coproducts, in fact $2 \amalg 2$ does not exist in PA. Since PA is self-dual this also shows thatit does not have products, which explains to some extent what is wrong with this category.

Of course, for any family of bounded p.o. sets, the set theoretic product of this family, with the pointwise ordering, is again a bounded p.o. set, and moreover the projection maps are easily seen to be left adjoint. We use the symbol " $\prod$ " to denote such products, keeping in mind that these are not (categorical) products in PA.

Lemma (Shmuely [7]): If A is complete then for all $B_i \in PA$, $A \otimes \prod B_i \cong \prod A\otimes B_i$.

Proof: Since for any $A_i \in PA$ $(i\in I)$, $(\prod A_i)^* \cong \prod A_i^*$, it is equivalent to prove $H(A,\prod B_i) \cong \prod H(A,B_i)$. But since A is complete, it follows that for each $f\in H(A,\prod P_i)$,

$pr_j f: A \rightarrow B_j$ ($pr_j: \prod B_i \rightarrow B_j$ the jth projection) is a left adjoint map, and that the correspondence $f \rightsquigarrow (pr_i f)_{i\in I}$ is the desired isomorphism.

Proposition 4 (Shmuely [7]): *If* A *and* B *are complete and completely distributive, so is* $A\otimes B$.

Proof: A is complete and completely distributive iff it is a retradt, in CJSL, of a power set (see Crown [3]). Since the formation of tensor products is functorial in CJSL, the fact that A and B are complete and completely distributive implies that $A\otimes B$ is retract in CJSL of $2^I \otimes 2^J$ for some sets I and J. By the above lemma, $2^I \otimes 2^J \cong \cong (2\otimes 2)^{I\times J} \cong 2^{I\times J}$ and hence $A\otimes B$ is complete and completely distributive.

Propositiom 5: *For* A, $B, C \in PA$, *if* A *and* C *are complete then* $A\otimes(B\otimes C) \cong (A\otimes B)\otimes C$.

Proof: If A and C are complete then the map S_{ACB}: : $H(A,H(C,B)) \rightarrow H(C,H(A,B))$ is an isomorphism for all B. Consequently

$$\begin{aligned}(C\otimes B)\otimes A &= H(H(C,B^*)^*,A^*)^* \\ &\cong H(A, H(C,B^*))^* \\ &\cong H(C, H(A,B^*))^* \\ &\cong H(H(A,B^*)^*,C^*)^* \\ &= (A\otimes B)\otimes C\end{aligned}$$

But then the result follows from the commutativity of $\otimes$.

Corollary (Shmuely [7]): $\otimes$ *is associative in* CJSL.

These results on associativity of $\otimes$ cannot be pushed any farther. In fact, for A the 6-element p.o. set in Figure

1, since

$$A\otimes(3\otimes 3) \cong H(H(3,3^*)^*,A^*)^* \cong H(H(3,3),A)^*$$

and $$(A\otimes 3)\otimes 3 \cong H(H(A,3^*)^*,3^*)^* \cong H(3,H(A,3))^* \cong H(3,H(3,A))^*$$

it follows from Proposition 2 that $A\otimes(3\otimes 3)$ is not isomorphic to $(A\otimes 3)\otimes 3$.

The fact that $\otimes$ is not associative in the category of all partially ordered sets and left adjoint maps is proved in Lisá [4].

Note that if $f: A \to B$ in PA is an epimorphism then f maps onto B: for each $b \in B$, define $g, h: B \to 2$ by $h(x) = 0$ iff $x \le b$, $g(x) = 0$ iff $x \le ff^{\#}(b)$. Now for $a \in A$, $f(a) \le b$ iff $a \le f^{\#}(b)$ iff $f(a) \le ff^{\#}(b)$ and hence (since $ff^{\#}(b) \le b$), $hf = gf$. Since f is an epimorphism, $g = h$ and thus $b = f(f^{\#}(b))$ which establishes the point.

Note also that if $f: A \to B$ in PA maps onto B then for $b, c \in B$, $f^*(\mu_b) \le f^*(\mu_c)$ iff $\mu_b f \le \mu_c f$ iff $\mu_b \le \mu_c$ and thus f^* is an embedding; the self duality of PA via $(\ )^*$ then implies that the monomorphisms in PA are embeddings.

Since $2 \in$ CJSL, the same arguments show that epimorphisms are onto in CJSL and monomorphisms are embeddings in CJSL.

Because formation of tensor products is functorial in CJSL, one can define flatness in CJSL as one does for modules: $A \in$ CJSL is flat iff $1_A \otimes f: A\otimes B \to A\otimes C$ is a monomorphisms whenever $f: B \to C$ is a monomorphism ($1_A \otimes f = H(1_A, f^*)^*$).

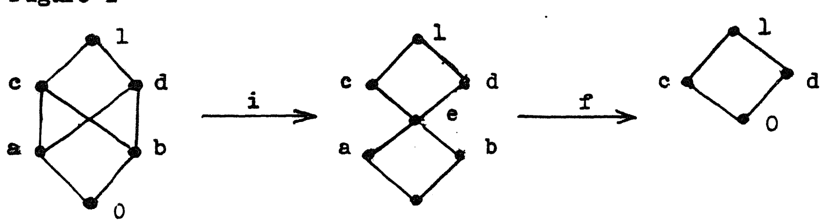
<u>Proposition 6</u>: <u>In</u> CJSL, <u>a p.o. set is flat iff it is projective</u>.

<u>Proof</u>: If A is projective and $f: B \longrightarrow C$ is a monomorphism then $f^*: C^* \longrightarrow B^*$ is onto B^* and hence by the projectivity of A, $H(1_A, f^*): H(A, C^*) \longrightarrow H(A, B^*)$ is onto, and thus $1 \otimes f = H(1_A, f^*)^*$ is a monomorphism.

Conversely, if A is flat then for all epimorphisms $f: B \longrightarrow C$, $f^*: C^* \longrightarrow B^*$ is a monomorphisms and hence $1_A \otimes f^*: A \otimes C^* \longrightarrow A \otimes B^*$ is a monomorphism. But $1_A \otimes f^* = H(1_A, f^{**})^*$ and consequently $H(1_A, f^{**})$ is onto, and this implies that $H(1_A, f)$ is onto (since $(\ )^{**}$ is an isomorphism of CJSL) and hence A is projective.

Since $\otimes$ is associative in CJSL, it follows that $A \otimes B$ is flat whenever A and B are flat and thus Proposition 6 yields a second proof of the fact that $A \otimes B$ is complete and completely distributive whenever A and B are.

Figure 1



A B C

Figure 2

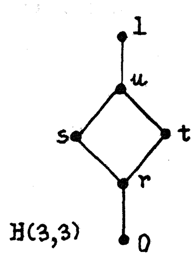


Figure 3

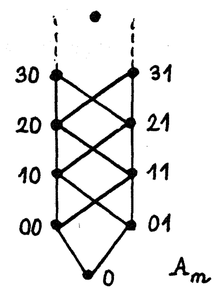
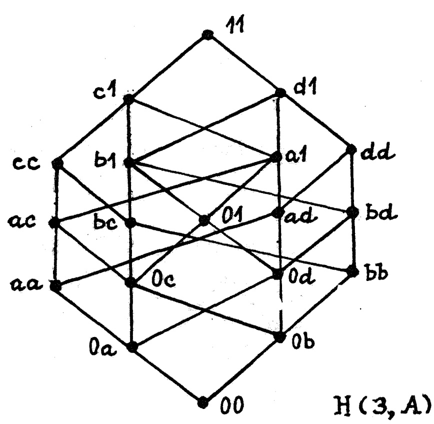


Figure 4



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McMaster University

Hamilton

Ontario, Canada

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