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ON PRODUCTS OF BINARY RELATIONAL STRUCTURES

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Abstract: In [2], R. Mc Kenzie considered cardinal multiplication of structures with a reflexive relation. He put a problem whether there exists a countable reflexive binary structure G such that G is not isomorphic to G^2 while G^n is isomorphic to G for a given $n > 2$. We construct such a structure G and give some stronger results in this direction. For example, any countable reflexive binary structure can be embedded into $2^{\aleph_0}$ of non-isomorphic structures with the above property.

Key words: Binary relational structure, product, cardinal multiplication, representation of semigroups.

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1. Conventions and notation. In the present note, a structure is always a binary relational structure, i.e. a pair (X,R), where X is a set, $R \subset X \times X$. The cardinality card G of a structure $G = (X,R)$ is defined as card X. A structure G is said to be reflexive (or transitive) if R has this property. We say that $G = (X,R)$ can be embedded into $G' = (X',R')$ if there exists a one-to-one mapping $\varphi : X \longrightarrow X'$ such that $(x,y) \in R$ iff $(\varphi(x), \varphi(y)) \in R'$. If φ is also a mapping onto X', we say that G and G' are isomorphic and denote it by $G \simeq G'$. Given $G = (X,R)$ and $G' = (X',R')$, the product $G \times G'$ is defined as the structure $(X \times X', S)$, where $((x,x'),(y,y')) \in S$ iff $(x,y) \in R$ and

$(x',y') \in R'$. The operation $\times$ (denoted also by Π for infinite collections) is called a <u>cardinal multiplication</u> in [2]. As usual, we define $G^1 = G$, $G^{n+1} = G \times G^n$.

2. Given a structure G, let us define an equivalence $\sim$ on the set of all natural numbers by $n \sim m$ iff $G^n \simeq G^m$. Clearly, $\sim$ is a congruence with respect to the addition of natural numbers. The aim of the present note is to prove the following theorem.

<u>Theorem</u>. For any congruence $\sim$ on the additive semigroup of all natural numbers and for any structure G there exists a set $\mathcal{H}$ of non-isomorphic structures such that card $\mathcal{H} = 2^{\aleph_0}$ and

(a) for every $H \in \mathcal{H}$, $H^m \simeq H^n$ iff $m \sim n$,

(b) for every $H \in \mathcal{H}$

card $H = \aleph_0 \cdot$ card G and G can be embedded into H.

Moreover, if G is reflexive or transitive or antisymmetric, then every $H \in \mathcal{H}$ has the same property.

Note. A countable structure H such that $H^m \simeq H^n$ iff $m \sim n$ is constructed in [4]. In the present paper, we use the methods of [4] and a modification of some methods of [3].

3. Let S be a semigroup. Denote by $g\ell S$ (see [1]) the semigroup of all subsets of S, where the operation is defined by

$$A \cdot B = \{a \cdot b \mid a \in A,\ b \in B\}.$$

Denote by N the additive semigroup of all non-negative integers and by N^N the semigroup of all functions $f: N \to N$ where the operation + is defined by $(f + g)(n) =$

$= f(n) + g(n)$ for all $n \in N$. Denote by $\mathbb{O}$ the function $f \in N^N$ with $f(n) = 0$ for all $n \in N$. Following [4], a semigroup S is called <u>$(\aleph_0, \aleph_0)$-embeddable</u> iff there exists a monomorphism $\varphi : S \longrightarrow g\ell\, N^N$ such that $\mathbb{O} \notin \varphi(s)$ and card $\varphi(s) = \aleph_0$ for all $s \in S$, the monomorphism φ is called an <u>$\aleph_0$-embedding</u>.

4. Let (S,+) be a commutative semigroup, $\mathcal{C}$ a class of structures. We say that a mapping $r: S \longrightarrow \mathcal{C}$ is a <u>representation of</u> S <u>by products in</u> $\mathcal{C}$ if $r(s_1 + s_2)$ is always isomorphic to $r(s_1) \times r(s_2)$ and $r(s_1)$ is not isomorphic to $r(s_2)$ whenever $s_1 \neq s_2$.
Given a structure G, denote by $\mathcal{C}(G)$ the class of all structures H such that card H $= \aleph_0 \cdot$ card G, G can be embedded into H and H is reflexive or transitive or antisymmetric whenever G has this property. In the rest of the paper, we prove the following proposition.

<u>Proposition</u>. For every structure G and for every $(\aleph_0, \aleph_0)$-embeddable semigroup S there exists a set $\mathcal{R}$ of representations of S by products in $\mathcal{C}(G)$ such that card $\mathcal{R} = 2^{\aleph_0}$ and for every $s, s' \in S$, $r, r' \in \mathcal{R}$, $r \neq r'$, $r(s)$ is not isomorphic to $r'(s')$.
The theorem follows from this proposition because every semigroup on one generator is $(\aleph_0, \aleph_0)$-embeddable, by [4].

5. Denote by L the set of all odd $n \in N$.

<u>Lemma</u>. For any $(\aleph_0, \aleph_0)$-embeddable semigroup S there exists an $\aleph_0$-embedding $\varphi : S \longrightarrow g\ell\, N^N$ such that for every $s \in S$ there exists $f_s \in \varphi(s)$ with $f_s(n) \neq 0$ for all

$n \in L$.

<u>Proof</u>. For any $f \in N^N$ define a set $A_f \subset N^N$ by

$g \in A_f$ iff $g/_L$ is constant and $g(2n) = f(n)$ for all $n \in N$.

If $\psi : S \longrightarrow g\mathcal{L} N^N$ is an $\aleph_o$-embedding, then φ defined by

$$\varphi(s) = \bigcup_{f \in \psi(s)} A_f$$

is an $\aleph_o$-embedding with the required property.

6. Let Q be a subset of N^N such that

(1) if $q \in Q$, then q is one-to-one, $q(L) \subset L$ and $q(n) = n$ for all $n \in N \setminus L$.

(2) card $Q = 2^{\aleph_o}$,

(3) if q, $q' \in Q$ are distinct, then neither $q(L) \subset q'(L)$ nor $q'(L) \subset q(L)$.

If $f \in N^N$, put $L_f = \{ n \in L \mid f(n) \neq 0 \}$.

Denote by P the set of all pairs (f,q), where $f \in N^N \setminus \{0\}$, $q \in Q$. Define an equivalence $\equiv$ on P as follows.

$(f,q) \equiv (f',q')$ iff $f(n) = f'(n)$ for all $n \in N \setminus L$ and there exists a bijection σ of L_f onto $L_{f'}$ such that $q(n) = q'(\sigma(n))$, $f(n) = f'(\sigma(n))$ for all $n \in L_f$.

<u>Observation</u>. a) For any $q \in Q$, $(f_1 q) \equiv (f_2,q)$ implies $f_1 = f_2$.

b) If $f(n) \neq 0$ for all $n \in L$ and $(f,q) \equiv (f',q')$ for some f', then $q = q'$.

7. Let a structure G = (X,R) be given, let us suppose $X \cap N = \emptyset$. For any $n \in N$ put $Z_n = \{0,\dots,m + 4\}$, $X_n =$

$= X \cup Z_n$, $R_n = R \cup \{(z,z) \mid z \in Z_n\} \cup \{(0,n+3),$
$(0,n+4)\} \cup \{(i,j) \mid i,j \in \{0,\dots,n+2\},\ i \leq j\} \cup$
$\cup \{(y,x) \mid x \in X,\ y \in \{0, n+3,\ n+4\}\}$.
Put $G_n = (X_n, R_n)$.

Observation. If G is reflexive or transitive or antisymmetric, then G_n has the same property.

8. Given $p = (f,q) \in P$, denote $G_p = \prod_{n=0}^{\infty} (G_{q(n)})^{f(n)}$ (where by G_n^0 we mean $(\{0\}, \{(0,0)\})$ for all $n \in N$). Denote $G_p = (X_p, R_p)$. Thus, $X_p = \prod_{n=0}^{\infty} (X_{q(n)})^{f(n)}$. Denote by T_p the set of all (i,n), where $n \in N$, $i = 1,\dots,f(n)$. For any $t = (i,n) \in T_p$, denote by $\pi_t: X_p \longrightarrow X_{q(n)}$ the t-th projection. Put

$Y_p = \{x \in X_p \mid \pi_t(x) = 0$ except a finite number of t's$\}$,

$S_p = (Y_p \times Y_p) \cap R_p$,

$H_p = (Y_p, S_p)$.

For every $t = (i,n) \in T_p$ denote by B_t the set of all $y \in Y_p$ such that $\pi_{t'}(y) = 0$ whenever $t' \neq t$ and $\pi_t(y) \in \{0,\dots,q(n)+2\}$.

9. Lemma. $\{B_t \mid t \in T_p\}$ is just the set of all subsets B of Y_p such that

(α) if $x,y \in B$ then either $(x,y) \in S_p$ or $(y,x) \in S_p$;

(β) if $x \in B$ and $(y,x) \in S_p$, then $y \in B$;

(γ) B is maximal with respect to (α) and (β);

(δ) card $B \geq 3$.

Proof. Each B_t clearly fulfils (α), (β) and (δ), let us prove (γ). If $B \supset B_t$ and B fulfils (α),(β), then either $B = B_t$ or B contains x not in B_t. Denote by a the point of B_t such that $\pi_t(a) = 1$, $\pi_s(a) = 0$ for all $s \in T_p \setminus \{t\}$. Let us recall that $t = (i,n)$, $p = (f,q)$. Since x is not in B_t, either $\pi_t(x) \in X \cup \{q(n) + 3, q(n) + 4\}$ or $\pi_{t'}(x) \neq 0$ for some $t' \in T \setminus \{t\}$. In the first case, neither (a,x) nor (x,a) is in S_p which contradicts (α). In the second case, define b by $\pi_{t'}(b) = \pi_{t'}(x)$, $\pi_s(b) = 0$ for all $s \in T \setminus \{t'\}$. Since $(b,x) \in S_p$, b is in B, by (β). But neither (a,b) nor (b,a) is in S_p.

Let $B \subset Y_p$ fulfil (α),(β),(γ),(δ). We have to show that $B \subset B_t$ for some $t \in T_p$. Then, by (γ), $B = B_t$. Thus, let us suppose that there exist $x_i \in B$, $\pi_{t_i}(x_i) \neq 0$ for $i = 1,2$, $t_1 \neq t_2$. Define y_i by $\pi_{t_i}(y_i) = \pi_{t_i}(x_i)$, $\pi_s(y_i) = 0$ for all $s \in T \setminus \{t_i\}$. Since $(y_i,x_i) \in S_p$, $y_i \in B$, by (β). But neither (y_1,y_2) nor (y_2,y_1) is in S_p, which contradicts (α). Consequently, there exists $t = (i,n) \in T_p$ such that $\pi_s(x) = 0$ for all $x \in B$ and all $s \in T_p \setminus \{t\}$. Let us suppose that $\pi_t(x) \in X \cup \{q(n) + 3, q(n) + 4\}$ for some $x \in B$. Define a,b,c by $\pi_t(a) = q(n) + 3$, $\pi_t(b) = q(n) + 4$, $\pi_t(c) = 1$, $\pi_s(a) = \pi_s(b) = \pi_s(c) = 0$ for all $s \in T \setminus \{t\}$. By (β) and (δ), two of the points a,b,c are in B. But this contradicts (α).

10. <u>Lemma</u>. Let $p,p' \in P$ be given. If $H_p \simeq H_{p'}$, then $p \equiv p'$.

<u>Proof</u>. Denote $p = (f,q)$, $p' = (f',q')$. Let us recall

that the functions q, q' are one-to-one. By 9, for every $n \in N$, the pair $(f(n), q(n))$ is characterized as follows. $f(n)$ is the number of distinct subsets B of H_p satisfying $(\alpha), (\beta), (\gamma)$, and card $B = q(n) + 3$. This is preserved by an isomorphism. Hence, $f(n) = f'(n)$ for all $n \in N \setminus L$. If $n \in L$ and $f(n) \neq 0$, then there exists unique $\sigma(n) \in L$ such that $(f(n), q(n)) = (f'(\sigma(n)), q'(\sigma(n)))$. Clearly, $\sigma : L_f \to L_{f'}$ is a bijection.

11. Let us recall that a <u>cardinal sum</u> of a collection $\{(X_\alpha, R_\alpha) \mid \alpha \in A\}$ of structures is a structure $G = (X, R)$ defined as follows. $X = \bigcup_{\alpha \in A} \{\alpha\} \times X_\alpha$, $(x, y) \in R$ iff $(x', y') \in R_\alpha$, $x = (\alpha, x')$, $y = (\alpha, y')$ for some $\alpha \in A$. We denote G by $\sum_{\alpha \in A} G$, where $G_\alpha = (X_\alpha, R_\alpha)$.

<u>Proof of the Proposition</u>. Given $p \in P$, put $K_p = \sum_{k \in N} K_k$, where every K_k is a structure isomorphic to H_p (see 8). Let $(S,+)$ be an $(\aleph_0, \aleph_0)$-embeddable semigroup, let $\varphi : (S,+) \to g\ell\, N^N$ be an $\aleph_0$-embedding such that for every $s \in S$ there exists $f_s \in \varphi(s)$ with $f_s(n) \neq 0$ for all $n \in L$ (see 5). For every $q \in Q$ define

$$r_q(s) = \sum_{f \in \varphi(s)} K_{(f,q)}.$$

We show that $\mathcal{R} = \{r_q \mid q \in Q\}$ is the set of representations of $(S,+)$ by products in $\mathcal{C}(G)$ with the required properties. Clearly, $r_q(s) \in \mathcal{C}(G)$ for all $s \in S$, $q \in Q$.

a) First, we show that for $q_1 \neq q_2$, $r_{q_1}(s_1)$ is not isomorphic to $r_{q_2}(s_2)$ for any $s_1, s_2 \in S$. Let us suppose $r_{q_1}(s_1) \simeq r_{q_2}(s_2)$. The structure $r_{q_1}(s_1)$ contains a component isomorphic to $H_{(f_{s_1},q_1)}$. It must be isomorphic to a component of $r_{q_2}(s_2)$. This component is isomorphic to $H_{(g,q_2)}$ for some $g \in \varphi(s_2)$. By 10, $(f_{s_1},q_1) \equiv (g,q_2)$. Hence $q_1 = q_2$, by 6.

b) Now, we prove that $r_q(s_1)$ is not isomorphic to $r_q(s_2)$ whenever $s_1 \neq s_2$. We have $\varphi(s_1) \neq \varphi(s_2)$. Let us suppose $\varphi(s_1) \setminus \varphi(s_2) \neq \emptyset$ and choose f in this set. The structure $r_q(s_1)$ contains a component isomorphic to $H_{(f,q)}$. Let us suppose $r_q(s_1) \simeq r_q(s_2)$. Then $r_q(s_2)$ contains a component isomorphic to $H_{(f,q)}$. By the definition of r_q, this component must be isomorphic to $H_{(g,q)}$ for some $g \in \varphi(s_2)$. By 10, $(f,q) \equiv (g,q)$. Hence $f = g$, by 6. This is a contradiction.

c) Now, we show that for every $q \in Q$, $s_1, s_2 \in S$, $r_q(s_1 + s_2)$ is isomorphic to $r_q(s_1) \times r_q(s_2)$. We have $\varphi(s_1 + s_2) = \{f_1 + f_2 \mid f_1 \in \varphi(s_1),\ f_2 \in \varphi(s_2)\}$. Since every $r_q(s)$ contains $\aleph_0$ isomorphic copies of any of its components and since $H_{(f_1,q)} \times H_{(f_2,q)}$ is isomorphic to $H_{(f_1+f_2,q)}$ for any $f_1 \in \varphi(s_1)$ and $f_2 \in \varphi(s_2)$, $r_q(s_1) \times$ $\times\, r_q(s_2)$ is isomorphic to $r_q(s_1 + s_2)$.

12. Concluding remarks. One can see that the Proposition may be generalized to higher cardinalities. In [4], $(\mathfrak{m},\mathfrak{n})$-embeddable semigroups are defined, where $\mathfrak{m}$, $\mathfrak{n}$ are infinite cardinals, $\mathfrak{m} \leq \mathfrak{n} \leq 2^{\mathfrak{m}}$. Given a structure G and an $(\mathfrak{m},\mathfrak{m})$-embeddable semigroup S, we can construct $2^{\mathfrak{m}}$ non-isomorphic representations of S by products

in the class $\mathcal{C}(G, \mathfrak{m})$ of all structures H such that card H = $\mathfrak{m}$. card G, G can be embedded into H and H is reflexive or transitive or antisymmetric whenever G has this property. By [5], every commutative semigroup S is $(\mathfrak{m}, 2^{\mathfrak{m}})$-embeddable with $\mathfrak{m} = \aleph_0$. card S, so it has $2^{2^{\mathfrak{m}}}$ non-isomorphic representations by products in $\mathcal{C}(G, 2^{\mathfrak{m}})$.

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