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REMARKS ON SUBDIFFERENTIALS OF CONVEX FUNCTIONALS

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Abstract: Differentiability properties of convex functions and their subdifferentials are studied.

Key words: Convex functions, differentiability, Banach spaces.

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The present paper contains some remarks on the subdifferential ∂f of a convex functional f defined on a real Banach space. Theorem 1 deals with strict monotonicity of ∂f. Theorem 2 characterizes the uniform differentiability of convex functionals using their conjugate functionals. Theorems 3, 4, 5 are concerned with the uniform continuity of ∂f. The main results and proofs of this note have been encouraged by the works [1, 2], where A. Asplund and R.T. Rockafellar have generalized in [1] the results concerning the continuity of the spherical mappings proved in [2] to the case of subdifferentials of convex functionals.

Throughout this paper X, X^* will denote a real Banach space and its normed conjugate space respectively, unless explicitly stated otherwise. We shall write $\langle x, u^* \rangle$ for the value of $u^* \in X^*$ at $x \in X$. The system of all subsets of a given set $M \subset X$ is denoted by 2^M and its boundary by $Fr\ M$. A set-valued mapping φ of M into 2^{X^*} is said to be strictly monotone on M if

$$\langle x - y, u^* - v^* \rangle > 0$$

whenever $x, y \in M$, $x \neq y$, $u^* \in \varphi(x)$, $v^* \in \varphi(y)$. Let R denote the set of all real numbers. An element $u^* \in X^*$ is said to be subgradient of the functional $f: M \subset X \longrightarrow R$ at $x \in M$ if

$$f(y) - f(x) \geq \langle y - x, u^* \rangle$$

for each y in M. We denote by $\partial f(x)$ the set of all subgradients of f at x. The set-valued mapping $\partial f:$ $: x \longrightarrow \partial f(x)$ of M into 2^{X^*} is called the subdifferential of f. If $\partial f(x) \neq \emptyset$, f is said to be subdifferentiable at x. For a functional $f: M \longrightarrow R$, $N \subset M$ and $u^* \in X^*$ we shall use the following notations:

$$R(N, \partial f) = \bigcup_{x \in N} \partial f(x),$$

$$(\partial f)^{-1}(N, u^*) = \{x \in N: u^* \in \partial f(x)\}.$$

Furthermore, for any functional $f: M \longrightarrow R$ we define

$$M^* = \{u^* \in X^* : \sup_{x \in M} [\langle x, u^* \rangle - f(x)] < +\infty\},$$

$$f^*(u^*) = \sup_{x \in M} [\langle x, u^* \rangle - f(x)] \quad \text{for all } u^* \text{ in } M^*.$$

If $M^* \neq \emptyset$, then the functional $f^*: M^* \longrightarrow R$ is called the conjugate of $f: M \longrightarrow R$.

We say that a functional $f: M \longrightarrow R$ is convex if M is a convex set and

$$f(tx + (1 - t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in M$ and $0 \leq t \leq 1$. A convex functional $f: M \longrightarrow$

$\longrightarrow R$ is said to be closed if the (convex) set

$$\operatorname{epi} f = \{(x,t) \in X \times R : x \in M,\ f(x) \leq t\}$$

is closed in the space $X \times R$.

Suppose now $f: M \longrightarrow R$ is a closed and convex functional. Then f has a conjugate f^* and f is a turn of the conjugate of f^* in this sense, that

$$M = \{x \in X : \sup_{u^* \in M^*} [\langle x,u^* \rangle - f^*(u^*)] < +\infty\},$$

$$f(x) = \sup_{u^* \in M^*} [\langle x,u^* \rangle - f^*(u^*)] \text{ for all } x \text{ in } M$$

(see e.g. [3]). In this case we say that $f: M \longrightarrow R$ and $f^*: M^* \longrightarrow R$ are conjugate to each other and for arbitrary $x \in M$, $u^* \in M^*$ the following relations hold:

(a) $u^* \in \partial f(x) \Longleftrightarrow \varkappa(x) \in \partial f^*(u^*) \Longleftrightarrow f(x) + f^*(u^*) = \langle x,u^* \rangle$;

(b) $R(M, \partial f) \subset M^*$,

$R(M^*, \partial f^*) \cap \varkappa(X) \subset \varkappa(M)$,

where $\varkappa : X \longrightarrow X^{**} = (X^*)^*$ is the canonical imbedding of X into X^{**}.

A functional $f: M \longrightarrow R$ is said to be uniformly Gâteaux differentiable on a set $N \subset M$, if f has the Gateaux derivative $f'(x)$ at each $x \in N$ and

$$\lim_{t \to 0} \frac{f(x + th) - f(x)}{t} = \langle h, f'(x) \rangle$$

is uniform with respect to $x \in N$. We say that f is uniformly Fréchet differentiable on N if the Fréchet derivative $f'(x)$ exists on N and

$$\lim_{h \to 0} \frac{r(x,h)}{\|h\|} = 0$$

is uniform with respect to $x \in N$, where

$$r(x,h) = f(x+h) - f(x) - \langle h, f'(x) \rangle .$$

We start with the following theorem which is a generalization of the theorem 5.1 in [4] and the theorem 1 in [5].

Theorem 1. Let M be a nonempty convex subset of X and f a subdifferebtiable convex functional on M. Then the subdifferential ∂f of f is strictly monotone if and only if f is strictly convex.

Proof. Let $\partial f : M \longrightarrow 2^{X^*}$ be strictly monotone. If f were not strictly convex, then there would exist $x_1, x_2 \in M$ and $\lambda \in (0,1)$ such that for $x_0 = \lambda x_1 + (1-\lambda) x_2$ we have

$$f(x_0) = \lambda f(x_1) + (1-\lambda) f(x_2) . \tag{1}$$

Choose arbitrary $u_0^* \in \partial f(x_0)$ and fix it. Then

$$f(x_i) - f(x_0) \geq \langle x_i - x_0, u^* \rangle , \quad i = 1,2 . \tag{2}$$

On the other hand, it follows from (1) and (2) that

$$f(x_2) - f(x_0) = [\frac{1}{1-\lambda} f(x_0) - \frac{1}{1-\lambda} f(x_1)] - f(x_0) =$$

$$= \frac{\lambda}{1-\lambda} [f(x_0) - f(x_1)] \leq \langle \frac{\lambda}{1-\lambda} (x_0 - x_1), u_0^* \rangle =$$

$$= \langle x_2 - x_0, u_0^* \rangle ,$$

and

$$f(x_1) - f(x_0) = [\frac{1}{\lambda} f(x_0) - \frac{1-\lambda}{\lambda} f(x_2)] - f(x_0) =$$

$$= \frac{1-\lambda}{\lambda} [f(x_0) - f(x_2)] \leq \langle \frac{1-\lambda}{\lambda} (x_0 - x_2), u_0^* \rangle =$$

$$= \langle x_1 - x_o, u_o^* \rangle .$$

Hence

$$f(x_i) - f(x_o) = \langle x_i - x_o, u_o^* \rangle , \quad i = 1,2.$$

Now, for every $x \in M$, $i = 1,2$:

$$f(x) - f(x_i) = [f(x) - f(x_o)] + [f(x_o) - f(x_i)] \geqq$$

$$\geqq \langle x - x_o, u_o^* \rangle + \langle x_o - x_i, u_o^* \rangle = \langle x - x_i, u_o^* \rangle .$$

From the definition of subgradients and the last inequality it follows that $u_o^* \in \partial f(x_1) \cap \partial f(x_2)$ and then $\langle x_1 - x_2, u_o^* \rangle = 0$. This contradicts the strict monotonicity of ∂f .

Let f be strictly convex. Let $x, y \in M$, $u^* \in \partial f(x)$, $v^* \in \partial f(y)$ be any fixed elements. From the definition of subgradients we have

$$\langle y - x, u^* \rangle = 2 \langle \frac{x+y}{2} - x , u^* \rangle \leqq 2 [f(\frac{x+y}{2}) - f(x)] .$$

Hence

$$f(y) - f(x) - \langle y - x, u^* \rangle \geqq f(y) - f(\ ) - 2 [f(\frac{x+y}{2}) - f(x)] =$$

$$= 2 [\frac{f(x) + f(y)}{2} - f(\frac{x+y}{2})] > 0 .$$

Thus

$$\langle y - x, u^* \rangle < f(y) - f(x) .$$

Similarly as above one can deduce

$$\langle x - y, v^* \rangle < f(x) - f(y) .$$

Now we have

$$\langle x - y, u^* - v^* \rangle = \langle x - y, u^* \rangle - \langle x - y, v^* \rangle >$$

$$\geq [f(x) - f(y)] + [f(y) - f(x)] = 0.,$$

so that ∂f is strictly monotone. This completes the proof.

Now we introduce the concept which is a generalization of rotundity defined in [1]. Let g be a functional defined on a subset D of a Banach space Y, and let τ be a locally convex topology in Y. Suppose $A^* \subset Y^*$ is a subset such that $A^* \subset R(D, \partial g)$.

<u>Definition</u>. We shall say that $g: D \longrightarrow R$ is τ-uniformly rotund on the set $N \subset D$ in the direction A^* if for any open τ-neighborhood V of the origin in Y there is $\delta > 0$ such that for every $u^* \in A^*$ and $u \in (\partial g)^{-1}(N, u^*)$ the following implication is valid:

$$v \in Y, u + v \in D, g(u + v) - g(u) - \langle v, u^* \rangle < \delta \Longrightarrow v \in V.$$

The next theorem will show that just introduced concept is not empty. Before stating this theorem, we give an example of a uniformly rotund functional. As follows, a Banach space X is always identified with the range under the canonical mapping $\varkappa : X \longrightarrow X^{**}$. It is worth to say that a functional g defined on the conjugate space X^* is τ-uniformly rotund on a set $N^* \subset X^*$ in the direction $A \subset X$.

<u>Example</u>. Let us consider a functional f^* defined on the set

$$M^* = \{u^* \in X^* : \| u^* \| \leqq 1\} \subset X^*$$

by the following prescription

$$f^*(u^*) = 0 \text{ for all } u^* \in M^*.$$

We demonstrate that $f^* : M^* \longrightarrow R$ is norm uniformly rotund (i.e. uniformly rotund with respect to the norm topology) on M^* in the direction $S = \{x \in X : \| x \| = 1\}$ if and only if the Banach space X^* is uniformly convex (i.e. for a given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\| u^* - v^* \| \geqq \varepsilon$ for $u^*, v^* \in X^*$ with $\| u^* \| \leqq 1$ and $\| v^* \| \leqq 1$ implies $1 - \frac{1}{2} \| u^* + v^* \| \geqq \delta(\varepsilon)$).

Let $f : X \longrightarrow R$ be the functional such that $f(x) = \| x \|$ for all x in X. Then $f : X \longrightarrow R$ and $f^* : M^* \longrightarrow R$ are conjugate to each other. In virtue of this one can find such $S \subset R(X, \partial f)$ that the following relations are true for any x in S:

$$u^* \in (\partial f^*)^{-1} (M^*, x) \Longleftrightarrow u^* \in \partial f(x) ,$$

$$u^* \in (\partial f^*)^{-1} (M^*, x) \Longrightarrow \| u^* \| = 1 , \langle x, u^* \rangle = 1 .$$

Suppose that f^* is norm uniformly rotund on M^* in the direction S. If X^* were not uniformly convex, then there would exist $\varepsilon_0 > 0$ such that for every $\delta > 0$ there would be $u_1^*(\delta)$, $u_2^*(\delta)$ in M^* such that

$$\| u_1^*(\delta) - u_2^*(\delta) \| \geqq \varepsilon_0 , \; 1 - \frac{1}{2} \| u_1^*(\delta) + u_2^*(\delta) \| < \delta .$$

Now, from the uniform rotundity of f^* it follows that there is $\delta_0 > 0$ such that for every $x \in S$ and $u^* \in (\delta f^*)^{-1} (M, x)$ the following implication is valid:

$$v^* \in X^* , \; u^* + v^* \in M^* , \; f^*(u^* + v^*) - f^*(u^*) - \langle x, v^* \rangle = - \langle x, v^* \rangle \delta_0 \Longrightarrow \| v^* \| < \frac{\varepsilon_0}{2} .$$

Set $u_1^* = u_1^*(\frac{\delta_0}{2})$, $u_2^* = u_2^*(\frac{\delta_0}{2})$. Then

$$\| u_1^* - u_2^* \| \geq \varepsilon_0 , \; 1 - \frac{1}{2} \| u_1^* + u_2^* \| < \frac{\delta_0}{2} .$$

Since $\| u^* \| = \sup_{y \in S} \langle y, u^* \rangle$ for each $u^* \in X^*$, there exists $x \in S$ such that

$$1 - \frac{1}{2} \langle x, u_1^* + u_2^* \rangle < \frac{\delta_0}{2} .$$

Hence we have that

$$[1 - \langle x, u_i^* \rangle] < \delta_0 , \quad i = 1,2 .$$

Choose arbitrary (but fixed) $u^* \in (\partial f^*)^{-1} (M^*, x)$ and set $v_i^* = u_i^* - u^*$, $i = 1,2$. Then $u^* + v_i^* \in M^*$ for $i = 1,2$, and consequently

$$- \langle x, v_i^* \rangle = - \langle x, u_i^* - u^* \rangle = 1 - \langle x, u_i^* \rangle < \delta_0 ,$$

$i = 1,2$.

From this it follows that

$$\| v_i^* \| = \| u_i^* - u^* \| < \frac{\varepsilon_0}{2} , \quad i = 1,2 .$$

We have now

$$\| u_1^* - u_2^* \| \leq \| u_1^* - u^* \| + \| u_2^* - u^* \| < \varepsilon_0 ,$$

which is a contradiction.

Let X^* be uniformly convex and suppose that $f^* : M^* \longrightarrow R$ is not norm uniformly rotund on M^* in the direction S . This denotes that there exists $0 < \varepsilon_0 < 2$ so that for any $\delta > 0$ there exists $x(\delta) \in S$, $u^*(\delta) \in (\partial f^*)^{-1} (M^*, x(\delta))$ and $v^*(\delta) \in X^*$ such that

$$u^*(\delta) + v^*(\delta) \in M^*, \quad - \langle x(\delta), v^*(\delta) \rangle < \delta ,$$

but $\| v^*(\delta) \| > \varepsilon_0$.

Let $\delta_0 = \delta(\varepsilon_0)$, where δ is the modulus of convexity of X^* :

$$\delta(\varepsilon) = \inf_{\substack{u^*, v^* \in M^* \\ \|u^* - v^*\| \geq \varepsilon}} [1 - \frac{1}{2} \| u^* + v^* \|], \quad 0 < \varepsilon \leqq 2.$$

For $x_0 = x(\delta_0)$, $u_0^* = u^*(\delta_0)$ and $v_0^* = v^*(\delta_0)$ we have now that

$$- \langle x_0, v_0^* \rangle = 2[\langle x_0, u_0^* - \frac{(u_0^* + v_0^*) + u_0^*}{2} \rangle] =$$

$$= 2[1 - \langle x_0, \frac{(u_0^* + v_0^*) + u_0^*}{2} \rangle] \geqq$$

$$\geqq 2[1 - \frac{1}{2} \| (u_0^* + v_0^*) + u_0^* \|] \geqq$$

$$\geqq 2\delta(\|(u_0^* + v_0^*) - u_0^*\|) \geqq 2\delta_0 > \delta_0 ,$$

which is impossible. Thus f^* is norm-uniformly rotund on M^* in the direction S .

Consider now the functional $f: M \longrightarrow R$, which is continuous, closed and convex. If $\mathrm{Int}\, M \neq \emptyset$ ($\mathrm{Int}\, M$ denotes the interior of M in the norm topology), then f is subdifferentiable on $\mathrm{Int}\, M$ (see e.g. [6 , p.91]). Furthermore, $\mathrm{Int}\, M$ contains every subset N such that $N \subset M$ with $\mathrm{dist}\,(\mathrm{Fr}\, N, \mathrm{Fr}\, M) > 0$, and so $N \subset R\,(M^*, \partial f^*)$, where $f^*: M^* \longrightarrow R$ is a conjugate function of $f: M \longrightarrow R$.

<u>Theorem 2</u>. Let M be a convex set in a Banach space X with $\mathrm{Int}\, M \neq \emptyset$ and N be a subset of M such that

dist (Fr N, Fr M) > 0. Let $f: M \to R$ be a continuous, closed and convex functional.

Then $f: M \to R$ is uniformly Gâteaux (Fréchet) differentiable on the set N if and only if its conjugate functional $f^*: M^* \to R$ is w^*-uniformly rotund (norm-uniformly rotund) on M^* in the direction N.

The proof of this theorem is based on a similar argument to that of Theorem 1 [1]. We shall need the following assertion.

Lemma 1 (see [1, p. 448]). Let $f: M \to R$ and $f^*: M^* \to R$ be convex functionals conjugate to each other. Let $x \in M$ and $u^* \in M^*$ be such that

$$\langle x,u^* \rangle - f(x) - f^*(u^*) = 0 .$$

For any $\delta > 0$ let

$$M_\delta(x,u^*) = \{y \in X: x + y \in M , f(x + y) - f(x) - \langle y,u^* \rangle \leqq \delta\},$$

$$M^*_\delta(x,u^*) = \{v^* \in X^* : u^* + v^* \in M^*, f^*(u^* + v^*) - f^*(u^*) - \langle x,v^* \rangle \leqq \delta \} .$$

Then, for any $\delta > 0$,

$$^\circ[M^*_\delta(x,u^*)] \subset \delta^{-1} M_\delta(x,u^*) \subset 2\, ^\circ[M^*_\delta(x,u^*)] ,$$

where by $^\circ[E^*]$ we denote the polar X of a set $E^* \subset X^*$ in X:

$$^\circ[E^*] = \{x \in X: \langle x,u^* \rangle \leqq 1 , \forall u^* \in E^*\}.$$

Proof of Theorem 2. We do the proof only for the case of Gâteaux differentiability. The proof of the case of

Fréchet differentiability is similar.

By convexity and continuity of f it follows that f is uniformly Gâteaux differentiable on the set N if and only if, for every $h_o \in X$ and $\varepsilon > 0$, there exists a $t = (t(\varepsilon, h_o)) > 0$ such that

$$(3) \qquad N + th_o \subset M ,$$

$$\left[\frac{f(x + th_o) - f(x)}{t} - \langle h_o, u^* \rangle \right] < \varepsilon, \qquad \forall x \in N, \; u^* \in \partial f(x) .$$

Let $f \colon M \longrightarrow R$ be uniformly Gâteaux differentiable on N and let $V(\varepsilon, h_o) = \{u^* \in X^* : \langle h_o, u^* \rangle \leqq \varepsilon\}$ be a w^*-neighborhood of the origin in X^* $(h_o \in X, \varepsilon > 0)$. Let $t = t\left(\frac{\varepsilon}{2}, h_o\right) > 0$ be the number such that (3) holds with $\frac{\varepsilon}{2}$. For any $x \in N$, $u^* \in \partial f(x)$ set

$$A(x,u^*) = \{h \in X : x + th \in M, \left[\frac{f(x+th) - f(x)}{t} - \langle h, u^* \rangle\right] < \frac{\varepsilon}{2}\} .$$

Then

$$h_o \in \bigcap_{\substack{x \in N \\ u^* \in \partial f(x)}} A(x,u^*) .$$

Since $A(x,u^*) = \{ h \in X : x + th \in M,$

$$[f(x + th) - f(x) - \langle th, u^* \rangle < \frac{t\varepsilon}{2} \} = t^{-1} M_{t\frac{\varepsilon}{2}} (x,u^*),$$

where $M_{\delta}(x,u^*)$ is the set introduced in Lemma 1, we have that

$$2\varepsilon^{-1} h_0 \in 2\varepsilon^{-1} t \bigcap_{\substack{x \in N \\ u \in \partial f(x)}} M_{t\frac{\varepsilon}{2}}(x,u^*) .$$

According to Lemma 1 there is a $\delta > 0$ so that

$$2\varepsilon^{-1} h_0 \in 2 \bigcap_{\substack{x \in N \\ u^* \in \partial f(x)}} {}^{0}[M^*_{\delta}(x,u^*)] =$$

$$= 2\ {}^{0}[\bigcup_{\substack{x \in N \\ u^* \in \partial f(x)}} M^*_{\delta}(x,u^*)] .$$

By taking polars we get

$$V(\varepsilon, h_0) \supset \bigcup_{\substack{x \in N \\ u^* \in \partial f(x)}} M^*_{\delta}(x,u^*) = \bigcup_{\substack{x \in N \\ u^* \in (\partial f^*)^{-1}(M,x)}} M^*_{\delta}(x,u^*) .$$

From this it follows that, for every w^*-neighborhood of the form $V(\varepsilon, h_0)$, there exists a $\delta > 0$ such that the following implication holds for each $x \in N$ and $u^* \in (\partial f^*)^{-1}(M^*,x)$:

$$v^* \in X^*,\ u^* + v^* \in M^*,\ [f^*(u^* + v^*) - f^*(u^*) - \langle x, v^* \rangle] <$$
$$< \delta \Longrightarrow v^* \in V(\varepsilon, h_0) .$$

Because the family of all finite intersections of neighborhood of the form $V(\varepsilon, h_0)$ is a base at o for the weak* topology, f is w^*-uniformly rotund on M in the direction N.

The sufficiency can be proved quite analogously.

<u>Corollary</u> (Šmulian [7]). A Banach space X is uniformly Fréchet smooth if and only if its conjugate space X^* is uniformly convex.

<u>Proof</u>. The assertion follows immediately from Theorem 3 and from our example.

Similarly one can formulate the necessary and sufficient condition for uniform Fréchet smoothness of X.

Definition (Cudia [2]). Let E_1, E_2 be topological linear spaces and let $\varphi : D(\varphi) \subset E_1 \longrightarrow 2^{E_2}$ be a multivalued mapping. We say that φ is uniformly lower semicontinuous on a set $M \subset D(\varphi)$, if for every neighborhood W of o in E_2 there exists a neighborhood V of o in E_1 such that

$$\varphi(y) \cap [u + W] \neq \emptyset$$

whenever $x \in M$, $y \in x + V$, $u \in \varphi(x)$.

Theorem 3. Let X be a Banach space, $M \subset X$ a convex nonvoid subset of X with $\operatorname{Int} M \neq \emptyset$, $f: M \longrightarrow R$ a subdifferentiable convex functional on M. If the multivalued mapping $\partial f: M \longrightarrow 2^{X^*}$ is uniformly lower semicontinuous on M from the norm topology relativized to M into the weak* topology on X^*, then f is uniformly Gâteaux -differentiable on each subset N of M such that $\operatorname{dist}(\operatorname{Fr} N, \operatorname{Fr} M) > 0$.

Proof. Let N be any subset of M with $\operatorname{dist}(\operatorname{Fr} N, \operatorname{Fr} M) > 0$. By translation to the set $\operatorname{Int} M$ we may assume that M is open. Being f subdifferentiable on M, f is lower semicontinuous on M. From this and from the completeness of X it follows that f is continuous on M (cf. [3, § 2.10]). Hence the functional f is uniformly Gateaux differentiable on N if and only if the relation (3) holds.

Let $o \neq h_o \in X$ and $\varepsilon > 0$ be arbitrary. We want to find a $0 < t = t(\varepsilon, h_o)$ so that (3) may hold. According to the definition of the uniform lower semicontinuity to the weak* neighborhood

$\overline{W} = \{ w^* \in X^* : |\langle h_0, w^* \rangle| \leqq \frac{\varepsilon}{4} \}$ of o in X^* there corresponds a $\delta > 0$ such that

(4) $\partial f(y) \cap [u^* + \overline{W}] \neq \emptyset$

whenever $x \in N$, $\| x - y \| < \delta$, $u^* \in \partial f(x)$.

Let now $x, y \in N$ be arbitrary and such that $\| x - y \| < \delta$. Since ∂f is weak* compact (see [8]) , there are $u_1^*, \ldots, u_n^* \in \partial f(x)$ so that

$$\partial f(x) \subset \bigcup_{i=1}^{m} (u_i^* + \overline{W}) .$$

Together with (4) and the last relation it follows that

$$u_i^* \in \partial f(y) + \overline{W} , \quad i = 1,2,\ldots,n .$$

Hence

(5) $\partial f(x) \subset \partial f(y) + W$,

where

$$W = \{ w^* \in X^* : |\langle h_0, w^* \rangle| \leqq \frac{\varepsilon}{2} \} .$$

Similarly we have

(6) $\partial f(y) \subset \partial f(x) + W$.

Since dist (Fr N, Fr M) > 0 , there exists a $t = t(\delta) = t(\varepsilon, h_0)$ such that

$$0 < t < \| h_0 \|^{-1} \cdot \delta ,$$

$$N + th_0 \subset M .$$

Hence for each $x \in N$ the element $y = x + th_0$ lies in

M and $\| x - y \| < \delta$.

Let $x \in N$, $v^* \in \partial f(x + th_0)$ be arbitrary. Then by (6) we have that

$$(7) \quad \langle h_0, v^* \rangle \leqq \sup_{w^* \in \partial f(x+th_0)} \langle h_0, w^* \rangle \leqq$$

$$\leqq \sup_{w^* \in \partial f(x) + W} \langle h_0, w^* \rangle \leqq$$

$$\leqq \sup_{w^* \in \partial f(x)} \langle h_0, w^* \rangle + \frac{\varepsilon}{2} .$$

On the other hand, by definition of subgradients,

$$f(x) - f(x + th_0) \geqq \langle - th_0, v^* \rangle ,$$

or

$$t^{-1} [f(x + th_0) - f(x)] \leqq \langle - th_0, v^* \rangle .$$

Hence for each $u^* \in X^*$,

$$(8) \quad t^{-1} [f(x + th_0) - f(x)] - \langle h_0, u^* \rangle \leqq \langle h_0, v^* \rangle - \langle h_0, u^* \rangle .$$

As $u^* \in \partial f(x)$, the expression on the left of (8), again by definition of subgradients, is a non-negative number. Hence we obtain

$$\langle h_0, u^* \rangle \leqq \langle h_0, v^* \rangle \quad (\forall x \in N, \ u^* \in \partial f(x), \ v^* \in \partial f(x + th_0)) .$$

This implies the relation

$$(9) \quad \sup_{w^* \in \partial f(x)} \langle h_0, w^* \rangle \leqq \inf_{w^* \in \partial f(x+th_0)} \langle h_0, w^* \rangle , \quad \forall x \in N .$$

By (5) we have for each $u^* \in \partial f(x)$

$$(10) \quad \langle h_0, u^* \rangle \geq \inf_{w^* \in \partial f(x)} \langle h_0, w^* \rangle \geq$$

$$\geq \inf_{w^* \in \partial f(x+th)+W} \langle h_0, w^* \rangle \geq$$

$$\geq \inf_{w^* \in \partial f(x+th_0)} \langle h_0, w^* \rangle - \frac{\varepsilon}{2} \quad .$$

Together with (7), (9) and (10) this gives

$$\langle h_0, v^* \rangle \leqq \sup_{w^* \in \partial f(x)} \langle h_0, w^* \rangle + \frac{\varepsilon}{2} \leqq$$

$$\leqq \inf_{w^* \in \partial f(x+th_0)} \langle h_0, w^* \rangle + \frac{\varepsilon}{2} \leqq \langle h_0, u^* \rangle +$$

$$+ \varepsilon \left(\forall x \in N \, , \, \forall u^* \in \partial f(x) \, , \, \forall v^* \in \partial f(x + th_0) \right) .$$

Hence we have, for each $x \in N$, $u^* \in \partial f(x)$ and $v^* \in \partial f(x + th_0)$,

$$\langle h_0, w^* \rangle - \langle h_0, u^* \rangle < \varepsilon \, .$$

By the instalment of the last relation into (8) we get (3), so f is uniformly Gâteaux differentiable on N. This concludes the proof.

<u>Theorem 4</u>. Let X, M and f be the same as in Theorem 3. If $\partial f : M \longrightarrow 2^{X^*}$ is uniformly lower semicontinuous on M (in the norm topologies), then f is uniformly Fréchet differentiable on each subset N of M such that dist (Fr N, Fr M) > 0.

<u>Proof</u>. Let $N \subset M$ be given such that dist (Fr N, Fr M) > 0. We may suppose M is open. We

shall use the following lemma which is well-known as a result of Browder and Minty: Let $\varphi_0 : M \subset X \longrightarrow X^*$ be a hemicontinuous (singlevalued) mapping. Let M be an open set and $x \in M$, $u^* \in X^*$ such that

$$\langle y - x , \varphi_0(y) - u^* \rangle \geqq 0$$

for each $y \in M$. Then $u^* = \varphi_0(x)$.

Let $F(X^*)$ be systems of all nonempty closed subsets of the space X^*. Then $\partial f(x) \in F(X^*)$ for each $x \in M$. Since the set N with relativized norm topology is paracompact, every lower semicontinuous multivalued mapping $\varphi : N \longrightarrow F(X\ )$ has a continuous selection (cf. [9, Theorem 3.2"]).

Let us suppose that ∂f is uniformly lower semicontinuous on N. Then ∂f is obviously lower semicontinuous on N. From this it follows that there is a singlevalued mapping $\varphi_0 : N \longrightarrow X^*$ such that

$$\varphi_0(x) \in \partial f(x) \quad \text{for all} \quad x \in N .$$

Let $x_0 \in N$ be arbitrary. Because ∂f is monotone, the following inequality holds for each $u^* \in \partial f(x)$:

$$\langle y - x_0 , \varphi_0(y) - u^* \rangle \geqq 0 , \quad \text{for all} \quad y \in M .$$

Then $u^* = \varphi_0(x_0)$ by the mentioned lemma. The set $\partial f(x_0)$ consists of a single point. As $x_0 \in M$ has been chosen arbitrarily, ∂f is singlevalued on M. Hence and by our hypothesis ∂f is uniformly continuous. Further the proof is quite analogous to that of Theorem 3.

<u>Remark</u>. The Browder-Minty's lemma is usually for-

mulated for operators acting on the whole of a space. From their proof one can see easily that the just formulated lemma is true.

Theorem 5. Let $f: M \subset X \longrightarrow R$ be a closed convex functional, $N \subset M$ a subset of M such that $\mathrm{dist}\,(\mathrm{Fr}\,N, \mathrm{Fr}\,M) > 0$. Suppose that f is bounded on M and uniformly Gâteaux (Fréchet) differentiable on M.

Then the derivative $f'(x): N \longrightarrow X^*$ is norm to weak* (norm to norm) uniformly continuous on N.

Proof. First we notice that the assumption $\mathrm{dist}\,(\mathrm{Fr}\,N, \mathrm{Fr}\,M) > 0$ implies the existence of $\lambda > 0$ such that $N + \lambda h \subset M$ for all $h \in X$ with $\|h\| = 1$. Hence

$$\begin{aligned}\sup_{x \in N} \|f'(x)\| &= \sup_{\substack{\|h\| \leqq 1 \\ x \in N}} \langle h, f'(x) \rangle \leqq \\ &\leqq \sup_{\substack{\|h\| \leqq 1 \\ x \in N}} \frac{1}{\lambda} \left[f(x + \lambda h) - f(x) \right] \leqq \\ &\leqq \frac{2}{\lambda} \sup_{x \in M} |f(x)| \,.\end{aligned}$$

Now, from the boundedness of f on M it follows that there is a $K > 0$ such that

$$\|f'(x)\| \leqq K \,, \text{ for all } x \in N \,.$$

Let f be uniformly Gâteaux differentiable on N. Furthermore, let W be any weak* neighborhood of o in X^*. By Theorem 2, the conjugate functional $f^*: M^* \rightarrow R$ of $f: M \longrightarrow R$ is weak* uniformly rotund on M^* in the direction N. This means that there exists a $\delta_1 > 0$ so that for each $x \in N$, $u^* = f'(x)$ the following implication holds:

$v^* \in X^*$, $u^* + v^* \in M^*$, $f^*(u^* + v^*) - f^*(u^*) - \langle x, v^* \rangle <$
$< \delta_1 \Rightarrow v^* \in W$.

For any $x, y \in M$ let

$$u^* = f'(x) \ , \quad w^* = f'(y) \ , \quad v^* = w^* - u^* \ .$$

Then u^*, $v^* \in M^*$ (see the relation (b)) and $u^* + v^* = w^* \in M^*$.

Furthermore,

$$f^*(u^*) = \langle x, u^* \rangle - f(x) \ ,$$

$$f^*(w^*) = \langle y, w^* \rangle - f(y)$$

(see the relation (a)). Hence

$$\begin{aligned} f^*(u^* + v^*) - f^*(u^*) - \langle x, v^* \rangle &= f^*(w^*) - f^*(u^*) - \\ &- \langle x, w^* - u^* \rangle = [\langle y, w^* \rangle - f(y)] - \\ &- [\langle x, u^* \rangle - f(x)] - \langle x, w^* - u^* \rangle = \\ &= \langle y - x, w^* \rangle + [f(x) - f(y)] \leqq \\ &\leqq \langle y - x, w^* \rangle + \langle x - y, u^* \rangle \leqq \\ &\leqq \max(\| u^* \|, \| w^* \|) \cdot \| x - y \| \ . \end{aligned}$$

If now $x, y \in N$ and $\| x - y \| < \delta = \frac{\delta_1}{K}$, then

$$f^*(u^* + v^*) - f^*(u^*) - \langle x, v^* \rangle < \delta_1 \ .$$

Hence $v^* = f'(y) - f'(x) \in W$ and f' is so norm to weak* uniformly continuous on N .

The proof of the case, when f is uniformly Fréchet differentiable on N , is similar.

Corollary. The subdifferential of the norm in a Banach space X is uniformly continuous on the unit sphere if and only if the normed conjugate space X^* is uniformly convex.

Proof. The assertion follows immediately from Theorems 4, 5.

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