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TRANSFORMATIONS DETERMINING UNIQUELY A MONOID III

WEAK DETERMINANCY

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Dedicated to Prof. Š. Schwarz to his 60^{th} birthday

Abstract: These two papers give necessary and sufficient conditions for a translation to be a member of only isomorphic Cayley's representations. In order to prove the necessity of conditions it contains a number of constructions which lead to non-isomorphic monoids. x)

Key words: Algebraic monoid, Cayley's representation, left translation, right translation, isomorphism of algebraic monoids.

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If a transformation f on a set X has a suitable form, then there exists an algebraic monoid $M = (X, .,e)$ such that f is its left translation, in other words that f is the left multiplication by $f(e)$. In this case f is called a <u>potential translation</u>. It may happen that such a monoid is unique, then we call f a <u>determining translation</u> (see [5]). The uniqueness of $M = (X,.,e)$ means that $e \in X$ and the associative binary operation " . " on X such that $f(e).x = f(x)$ are unique.

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x) At this occasion I wish to express my thanks to Pavel Goralčík, Zdeněk Hedrlín and Václav Koubek for their kind helpt and valuable suggestions during my work at this paper.

From this definition it follows that no group can have a determining translation (a determining element $a = f(e)$), because we can take $M_b = (X,.,b)$ a monoid for which also $f(b).x = f(x)$, b being an identity element. Hence one can introduce a weaker type of determinancy of monoids; the transformation f will be called a <u>weakly determining translation</u> if for every couple of algebraic monoids $M = (X,.,e)$, $M' = (X, \circ, e')$ such that f is a left translation of both M, M', there is an algebraic isomorphism φ from M onto M'.

Our aim is to describe all translations weakly determining in this sense. Through the whole paper we shall use the results of the papers [1],[2],[3] about potential translations (further shortly translations) and [5],[6] about determining translations.

Now, several notions and statements given in [4],[5],[6]. A T-monoid will be a couple (X,S), X being a set, $S \subset X^X$ satisfying the following conditions:

(1) identity transformation 1_X is an element of S;

(2) for all $f, g \in S$, it is $f\,g \in S$.

A centralizer (isocentralizer) of (X,S) is a T-monoid $(X, \mathcal{C}(S))((X, \mathcal{J}\mathcal{C}(S)))$, where

$$\mathcal{C}(S) = \{g \in X^X \mid f\,g = g\,f \text{ for all } f \in S\}$$

$$(\mathcal{J}\mathcal{C}(S) = \{g \in X^X \mid g \text{ is a bijection and } g \in \mathcal{C}(S)\}).$$

A point e is a source (exact source) of (X,S) is for every $x \in X$ there is (unique) $f \in S$ with $f(e) = x$. A T-monoid (X,F) is called a <u>Cayley's T-monoid</u> if there exists an algebraic monoid M with $F = L(M)$. There is a 1-1 correspondence between Cayley's T-monoids with a marked element e and algebraic monoids.

According to [4], a transformation which can be a member of some Cayley's T-monoid is called a translation.

Now some notions necessary for a description of a transformation $f: X \longrightarrow X$: the kernel Q_f of f is

$$Q_f = \cup \{A \mid A \subset X \ \& \ f(A) = A\}.$$

We shall call f a transformation with an increasing kernel if $f \mid Q_f$ is not injective; f has a bijective kernel, if $f \mid Q_f$ is a bijection. ($f \mid Q_f$ means the transformation $g: Q_f \longrightarrow Q_f$ defined by $g(x) = f(x)$.) For a given $x \in X$, the set $P_f(x) = \{f^m(x) \mid m \geq 0\}$ is the path of f of the element x. The elements $x,y \in X$ are E_f-equivalent if $f^m(x) = f^n(y)$ for some $m, n \geq 0$. E_f is an equivalence, its classes being components of f. Transformations with one component are connected, others are disconnected. An element $x \in X$ is a cyclic element of f, if $x \in P_f(f(x))$; the set Z_f of all cyclic elements of f forms the cycle of f. We have the kernel of the component of x : $Q_f(x) = Q_f \cap E_f(x)$, the cycle of the component of x : $Z_f(x) = Z_f \cap E_f(x)$. The order of x is the cardinality $r(x)$ of the $Z_f(x)$, if $r(x) = |Z_f(x)| < \aleph_0$. A transformation is called periodical, if every element has an order. Let x be an element with $Q_f(x) \neq \emptyset$, the height $u(x)$ of x is defined as the smallest integer with $f^{u(x)}(x) \in Q_f(x)$. An element e is a top element of f, if there is a Cayley's T-monoid (X,S), e being its exact source, with $f \in S$. Denote by T the set of all top elements of f. For a given top element e, the main component of f will be $E_f(e)$, $X \setminus E_f(e)$ will be designated by Y.

Define K as follows: if $Q_f(e) = \emptyset$, then $K = E_f(e)$;

if $Q_f(e) \neq \emptyset$, then $K = \{x \in E_f(e) \mid f^{u(x)}(x) \in P_f(e)\}$.
For $m \geq 0$, $n > 0$ set
$T_{m,n} = \{x \in E_f(e) \setminus K \mid f^{n-1}(x) \notin P_f(e) \ \& \ f^n(x) = f^{u(e)+m}(e)\}$,
for $m \geq 0$ $\quad T_{m,0} = \{f^{u(e)+m}(e)\}$.

For $y \in K$ define a mapping $d_y : E_f(e) \longrightarrow N$ as follows: $d_y(x) = m_o - n_o$ where n_o is the smallest integer such that there is an integer m for which $f^m(y) = f^{n_o}(x)$, m_o is the smallest integer for which $f^{m_o}(y) = f^{n_o}(x)$. Such a mapping d_y is called the <u>difference relative to</u> y. If there will be no possibility of misunderstanding we shall designate by $d(x)$ the difference relative to a fixed top element e.

Given a translation f, we shall define a mapping $h: Q_f \longrightarrow Q_f$, with $f(h(t)) = t$ for all $t \in Q_f$, and in the case f has an increasing kernel, $\operatorname{Im} h \cap P_f(e) = \emptyset$. (1)
($\operatorname{Im} h = \{y \mid \exists x,\ h(x) = y\}$.) The existence of such a mapping is proved in [1] and [2]. Let $x \in X \setminus Q_f$, let k be an integer with $f^{-k}(x) \neq \emptyset$ and $f^{-(k+1)}(x) = \emptyset$, then we shall call the integer k the <u>grade</u> $st(x)$ of x. Designate $A = \{x \in X \mid st(x) = 0\} \setminus \{e\}$.

Now we give a type of constructions of Cayley's T-monoids with a parameter p, p being a mapping $p: Y \longrightarrow Y$ such that $p^2 = p$,
$f(p(t)) = p(f(t))$ for $t \in Y$ and
$h(p(t)) = p(h(t))$ for $t \in Y \cap Q_f$ (see (1)). (2)

<u>Construction 1</u>. Let f be a non-surjective translation with an increasing kernel, e its top element. Let $f|Q_f(e)$ be not a translation of the bicyclic semigroup, i.e. there exists $v \in Q_f \setminus P_f(e)$ for which $h^{-1}(v) = \emptyset$ (see (1)). Then

there exists a Cayley's T-monoid $(X, L(M))$, $L(M) = \{f_x \mid x \in X\}$ defined as follows:

for $x \in T_{m,n}$ and t such that $f^{u(e)+m-1}(t) = v$ it is $f_x(t) = h^{n-1}(v)$, otherwise $f_x(t) = f_x(t)$.

The translations f_x are given in Construction 1 in [6] for a fixed p.

The demonstration is analogous to the demonstration of Construction 1 in [6], using the property of v.

We will now consider the case of a connected translation f which is surjective and has an increasing kernel. By [2] there exist a top element e of f and an injection $g \in \mathcal{C}(f)$ with $g(e) = f(e)$, $g^{-1}(e) = \emptyset$. (3)

We will refer to such a couple e, g as having the property (3). As was shown in [2], to any couple e, g with Property (3), two translations h, k can be found with $g, k \in \mathcal{C}(f,h)$ and $fh = kg = 1_X$ and $k(T_{m,n}) \subseteq T_{m-1,n}$ for $m > 0$. (4)

<u>Construction 2</u>. For e, g, h, k satisfying (3) and (4) and p with the properties (2) define transformations f_x, $x \in X$, by $f_x(e) = x$,

for $x \in T_{m,n}$, $t \in T_{p,q}$ if $q > m$ or $t \in Y$ it is $f_x(t) = h^n f^m(t)$, if $q \leq m$ it is $f_x(t) = g^p k^q(x)$, for $x \in Y$ it is $f_x(t) = p(x)$.

The T-system $L(M) = \{f_x ; x \in X\}$ is a Cayley's T-monoid with $f \in L(M)$.

<u>Demonstration</u>: The system of all right translations of M is defined by $R(M) = \{g_y ; y \in X\}$ where $g_y(e) = y$ and for $t \in T_{p,q}$ and $y \in T_{r,s}$ with $s > p$ or $y \in Y$ it is $g_y(t) = h^q f^p(y)$; for $t \in T_{p,q}$ and $y \in T_{r,s}$ with $s \leq p$ it is $g_y(t) = g^r k^s(t)$; for $t \in Y$ it is $f_y(t) = p(t)$.

Evidently e is a common source of both $L(M)$ and $R(M)$, thus using Statement 1 in [4] it is sufficient to show that

$$g_y f_x(t) = f_x g_y(t) \quad \text{for all } x, y, t \in X . \tag{5}$$

Let $x \in T_{m,n}$, $y \in T_{r,s}$, $t \in T_{p,q}$, then for $s > p$ it is $f_x g_y(t) = f_x(h^q f^p(y))$ and further for $s - p + q > m$ it is $h^n f^m h^q f^p(y)$, for $s - p + q \leq m$ it is $g^r k^{s-p+q}(x)$, because $h^q f^p(y) \in T_{r,s-p+q}$. If $s \leq p$, then $f_x g_y(t) = f_x(g^r k^s(t))$ and for $q > m$ it is $h^n f^m g^r k^s(t)$, for $q \leq m$ it is $g^{p-s+r} k^q(x)$ (use $g^r k^s(t) \in T_{p-s+r,q}$).

For $q > m$, $g_y(f_x(t)) = g_y(h^n f^m(t))$ and if $s \leq p$ it is $g^r k^s h^n f^m(t) = f_x g_y(t)$ as $f, h \in \mathcal{C}(g,k)$. If $s > p$, then evidently $s - p + q > m$ and hence $g_y f_x(t) = h^{q-m+n} f^p(y) = h^n f^m h^q f^p(y) = f_x g_y(t)$ (use $m < q$). For $q \leq m$ we have $f_x(t) = g^p k^q(x) \in T_{m-q+p,n}$, hence for $s \leq p$ it is equal to $g^r k^s g^p k^q(x) = g^{p-s+r} k^q(x)$, so the equality (5) holds. If $s > p$ then we have two possibilities: for $s > p + m - q$ it is $g_y f_x(t) = h^n f^{m-q+p}(y) = h^n f^m h^q f^p(y)$ (use $m \geq q$), for $s \leq p + m - q$ it is $g_y f_x(t) = g^r k^s g^p k^q(x) = g^r k^{s-p+q}(x)$.

Let $t \in Y$, $x \in T_{m,n}$, then $f_x g_y(t) = f_x(p(t)) = h^n f^m(p(t))$, as $p(t) \in Y$; $g_y f_x(t) = g_y(h^n f^m(t)) = p(h^n f^m(t))$, as $h^n f^m(t) \in Y$, using the properties (2), we get the equality (5).

Finally, if $x \in Y$, then $f_x g_y(t) = p(x)$, $g_y f_x(t) = p(p(x)) = p(x)$ (see (2)) as $f_x(t) \in Y$.

Hence the equality (5) holds for every $x, y, t \in X$ and $L(M)$ is a Cayley's T-monoid.

We give now some lemmas about two isomorphic monoids.

Lemma 1. Let $M = (X,.,1)$, $M' = (X,*,1)$ be two isomorphic algebraic monoids with the same underlying set X , let $\varphi : X \longrightarrow X$ carry an isomorphism of M onto M'. Let $L(M) = \{f_x ; x \in X\}$, $R(M) = \{g_x ; x \in X\}$, $L(M') = \{f_x ; x \in X\}$, $R(M') = \{g_x ; x \in X\}$ denote the systems of left and right translations of M and M' , respectively. Then the translations of M are carried by φ to those of M according to the rules

$$\varphi f_x = f'_{\varphi(x)} \varphi \quad , \quad \varphi g_x = g'_{\varphi(x)} \varphi \ .$$

Proof is evident.

In the following assume f being a connected surjective translation with an increasing kernel.

Lemma 2. Let $L(M)$ be a Caley's T-monoid from Construction 2 with e,g,h,k ,then for every $x \in T_{m,n}$ it is $\operatorname{Im} f_x = h^n(X \setminus P_f(e)) \cup \bigcup_{i=1}^{m} P_g(k^i(x))$.

Proof is easy.

Corollary 1. The only translations in $L(M)$ of Construction 2 which are both surjections and retractions are of the form $f^k = f_{f^k(e)}$, $k \geq 0$.

Lemma 3. For e_i, g_i, h_i, k_i, $i = 1,2$ satisfying Conditions (3) and (4) if $\varphi \in \mathcal{C}(f)$ is a bijection of X such that $\varphi(e_1) = e_2$, $\varphi g_1 = g_2 \varphi$, $\varphi h_1 = h_2 \varphi$, $\varphi k_1 =$

$$= k_2 \varphi \ , \quad (6)$$

then the algebraic monoids M_1, M_2 from Construction 2 (M_i with e_i, g_i, h_i, k_i) are isomorphic as algebraic monoids.

Proof: Define an isomorphism $\psi : M_1 \longrightarrow M_2$ by $\psi(f_x) = f'_{\varphi(x)}$ ($f_x \in L(M_1)$, $f'_x \in L(M_2)$). For $f_x = 1_X$ it is $f'_{\varphi(x)} = 1_X$ (use: $f_x = 1_X$ iff $x = e_1$, $f'_{\varphi(x)} = 1_X$ iff $\varphi(x) = e_2$).

The only fact we must show is the equality $\psi(f_x \circ f_y) = \psi(f_x) \circ \psi(f_y)$. Take $x \in T_{m,n}$, $y \in T_{p,q}$. Assume $q > m$, then $\varphi(f_x(y)) = \varphi(h_1^n f^m(y)) = h_2^n f^m(\varphi(y))$. The bijection φ maps the set $T_{m,n}$ onto $T'_{m,n}$, where $T'_{m,n}$ are defined relative to e_2, and thus it is $f'_{\varphi(x)}(\varphi(y)) = h_2^n f^m(\varphi(y))$. Assume $q \leq m$, then $\varphi(f_x(y)) = \varphi(g_1^p k_1^q(x)) = g_2^p k_2^q(\varphi(x)) = f'_{\varphi(x)}(\varphi(y))$. Hence $\varphi(f_x(y)) = f'_{\varphi(x)}(\varphi(y))$ and we have $\psi(f_x f_y) = \psi(f_{f_x(y)}) = f'_{\varphi(f_x(y))} = f'_{f'_{\varphi(x)}(\varphi(y))} = f'_{\varphi(x)} f'_{\varphi(y)} = \psi(f_x) \circ \psi(f_y)$.

Thus ψ is an algebraic homomorphism and as φ is a bijection, ψ is an isomorphism.

<u>Lemma 4</u>. The two algebraic monoids M_1, M_2 from Construction 2, M_1 with e,g,h,k, M_2 with e,g,f_x,k, where f_x is the left translation of M_1 by $x \in T_{0,1}$, are equal.

The proof follows from Lemma 2.

Before we give the second type of constructions, we introduce some conventions: for a given $x \in X$ denote by $\mathcal{L}_x$ the graph $\mathcal{L}_x = \langle L_x, R_x \rangle$, where $L_x = \{y \in X;\ \exists n \geq 0,\ f^n(y) = x\}$, $R_x = \{\langle u,v \rangle;\ u,v \in L_x,\ f(u) = v\}$. The n-th <u>level</u> $\mathcal{H}_n$ will be the graph $\mathcal{H}_n = \langle H_n, R_n \rangle$, where $H_n = L_{f^n(e)} \setminus L_{f^{n-1}(e)}$, $R_n = (H_n \times H_n) \cap R_{f^n(e)}$.

<u>Construction 3</u>. Let $f: X \longrightarrow X$ be a connected surjective translation with an increasing kernel. Let there exist e, g with (3) and $x_0 \in T_{s_0,1}$, $s_0 > 1$, such that $g^{-1}(x_0) = \emptyset$ and $\mathcal{L}_e$ can be embedded into $\mathcal{L}_{x_0}$. Denote $B = \bigcup_{i \geq 0} L_{g^i(x_0)}$, $A = X \setminus B$. There exist translations h,k such that $g, k \in \mathcal{C}(h,f)$, $fh = kg = 1_X$, $k^{-1}(B) \subset B$ and k has the fol-

lowing form: for $y \in X \setminus L_{x_0}$, $y \in T_{m,n}$, $m > 0$ it is $k(y) \in T_{m-1,n}$, $m = 0$ it is $k(y) \in T_{0,n+1}$; for $y \in L_{x_0}$, $y \in T_{s_0,n}$ it is $k(y) \in T_{s_0-2,n-1}$. Define translations f_x as follows: $f_x(e) = x$; for $x \in T_{m,n} \cap A$, $t \in T_{p,q}$ it is

$f_x(t) = h^n f^m(t)$ if either $t \in A$ and $m < q$ or $t \in B$ and $m < q - 1$,

$f_x(t) = g^p k^q(x)$ if $t \in A$ and $q \leq m$,

$f_x(t) = g^{p-1} k^{q-1}(x)$ if $t \in B$ and $q - 1 \leq m$,

for $x \in T_{m,n} \cap B$, $t \in T_{p,q}$ it is

$f_x(t) = h^{n-1} f^{m-1}(t)$ if either $t \in A$ and $m - 1 < q$ or $t \in B$ and $m < q$,

$f_x(t) = g^p k^q(x)$ if $t \in A$ and $q \leq m - 1$,

$f_x(t) = g^{p-1} k^{q-1}(x)$ if $t \in B$ and $q \leq m$.

Then $L(\bar{M}) = \{f_x;\ x \in X\}$ is a Cayley's T-monoid and $f \in L(\bar{M})$.

Demonstration: Obviously $L(\bar{M})$ has an exact source, so we show that it is a T-monoid (use Statement 1 in [4]).

We will introduce the equivalence $\sim$ on X by $x \sim y \Longleftrightarrow (x = y)$ or $((\exists\, n, \exists\, z \in L_e)\ (g^n g_{x_0}(z) = x\ \&\ \&\ g^{n+s_0-1}(z) = y))$. Evidently $|\{z;\ z \sim y\}| \leq 2$ for all $y \in X$. Further the transformations $f/\sim$, $h/\sim$, $g/\sim$, $k/\sim$ are correctly defined and if we denote by $[x]$ the class of $\sim$ containing x, then $[e]$, $f/\sim$, $g/\sim$, $h/\sim$, $k/\sim$ have the properties (3) and (4). Designate by $\tilde{M}$ the monoid from Construction 2 containing $[e]$, $f/\sim$, $g/\sim$, $h/\sim$, $k/\sim$. We show $\bar{M}/\sim = \tilde{M}$. Take $f_x \in L(M)$, for $t_1 \sim t_2$ it must hold $t_1 \in A$, $t_2 \in B$ and moreover if $t_1 \in T_{p,q}$, then $t_2 \in T_{p+1,q+1}$. Thus for $x \in A$ if $q > m$ it is $[f_x(t_1)] = [h^n f^m(t_1)] =$

$= (h/\sim)^n(f/\sim)^m([t_1]) = (h/\sim)^n(f/\sim)^m([t_2]) =$
$= [h^m f^m(t_2)] = [f_x(t_2)]$, if $q \leq m$ then $[f_x(t_1)] =$
$= [g^p k^q(x)] = [f_x(t_2)]$. Analogously for $x \in B$.

The demonstration will be finished if we show that for every $x,y,t \in X$ $\quad f_x f_y(t) \in B$ iff $f_{f_x(y)}(t) \in B$. (7)

At first let us point out some assertions which follow from the definition of $L(\overline{M})$. For given $x \in T_{m,n}$, $y \in T_{r,s}$, $t \in T_{p,q}$ it holds:

I. If $x \in A$ then $(f_x(t) \in B) \Longleftrightarrow (t \in B \ \&\ q - 1 > m)$.

II. If $x \in B$ then $(f_x(t) \in B) \Longleftrightarrow ((t \in B \ \&\ q > m) \vee (t \in A \ \&\ q \leq m - s_0) \vee (t \in B \ \&\ q - 1 \leq m - s_0))$.

III. If $x,y \in A$ then $(f_x f_y(t) \in B) \Longleftrightarrow (t \in B \ \&\ q - 1 > r \ \&\ q - r + s - 1 > m)$.

IV. If $x \in A$, $y \in B$ then $(f_x f_y(t) \in B) \Longleftrightarrow ((t \in B \ \&\ q > r \ \&\ q - r + s - 1 > m) \vee (t \in A \ \&\ q \leq r - s_0 \ \&\ s - 1 > m) \vee (t \in A \ \&\ q - 1 \geq r - s_0 \ \&\ s - 1 > m))$.

V. If $x \in B$, $y \in A$ then $(f_x f_y(t) \in B) \Longleftrightarrow ((t \in B \ \&\ q - 1 > r \ \&\ q - r + s > m) \vee (t \in A \ \&\ s \leq m - s_0 \ \&\ m - s_0 > q - r + s) \vee (t \in B \ \&\ s < m - s_0 \ \&\ r \geq q - 1) \vee (t \in B \ \&\ q - 1 > r \ \&\ q - r + s - 1 \leq m - s_0))$.

VI. If $x,y \in B$ then $(f_x f_y(t) \in B) \Longleftrightarrow ((t \in B \ \&\ q > r \ \&\ q - r + s > m) \vee (t \in A \ \&\ q \leq r - s_0 \ \&\ s > m) \vee (t \in B \ \&\ q - 1 \leq r - s_0 \ \&\ s > m) \vee (t \in A \ \&\ s - 1 \leq m - s_0 \ \&\ m - s_0 \geq q - r + s) \vee (t \in B \ \&\ s - 1 \leq m - s_0 \ \&\ m - s_0 \geq q - r + s - 1))$.

Now if $x,y \in A$ then for $s > m$, $f_x(y)$ is an element of $A \cap T_{r,s-m+n}$, for $s \leq m$ of $A \cap T_{m-s+r,n}$. Comparing I and III we get the proposition (7).

If $x \in A$, $y \in B$, then for $s - 1 > m$, $f_x(y)$ is an element of $B\cap T_{r,s-m+n}$ for $s-1\leq m$ of $A\cap T_{m-s+r,n}$. Compare I, II and IV.

If $x\in B$, $y\in A$, then $f_x(y)\in A\cap T_{r,s-m+n}$, for $s>m-1$ and $f_x(y)\in A\cap T_{m-s+r-1,n-1}$, if $m-s_0<s<m-1$ or $B\cap T_{m-s+r,n}$ if $s\leq m-s_0$. As $q-1>r$ and $q-s+s-1\leq m-s_0$ we have $s\leq m-s_0$; it suffices to compare II and IV. For $q-1>r$ and $s>m-s_0$ or $s>m-s_0$ and $q-1\leq r$ compare I and V; for $q-1\leq r$ and $s\leq m-s_0$ compare II and V. In all cases we get (7).

At last if $x,y\in B$ then for $s>m$ it is $f_x(y)\in B\cap T_{r,s-m+n}$, for $m-1\geq s-1>m-s_0$, $f_x(y)\in A\cap T_{m-s+r-1,n-1}$, for $s-1\leq m-s_0$, $f_x(y)\in B\cap T_{m-s+r,n}$. And again comparing II and VI we get (7).

Hence $L(\overline{M})$ is closed under the composition.

Before we come to the last two types of constructions of Cayley's monoids we will need some definitions and lemmas: for given e,g,h,k satisfying (3) and (4) and $x\in X$ denote $K_x=\{y;\ \exists\, n,m\geq 0\ \ k^nf^m(y)=x\}$. For $y\in K_x$ set $s^x(y)=(m,n)$ if m,n are the least non-negative integers with $k^nf^m(y)=x$.

Lemmas 5 - 7 bring some properties of K_x.

Lemma 5. Let e,g,h,k have (3) and (4), let for every x with $g^{-1}(x)=0$ be $k(x)=hkf(x)$. Then for $x_1,x_2\in X$ such that $h^{-1}(x_i)=g^{-1}(x_i)=\emptyset$, $i=1,2$, $K_{x_1}\cap K_{x_2}\neq\emptyset$ it holds $K_{x_1}\subseteq K_{x_2}$ or $K_{x_2}\subseteq K_{x_1}$.

<u>Proof</u>: Let $z\in K_{x_1}\cap K_{x_2}$, $s^{x_1}(z)=(b,a)$, $s^{x_2}(z)=(c,d)$; we can suppose that $c<b$. It is sufficient to show that $d\leq a$, as it means $k^{b-c}f^{a-d}(x_2)=x_1$ and $x_2\in K_{x_1}$ implies $K_{x_2}\subseteq K_{x_1}$.

Suppose $a < d$. Take $z' = k^c f^a(z)$; we have $s^{x_1}(z') = (b - c, 0)$. For every u with $s^{x_1}(u) = (b - c, 0)$ it is $h^{-1}(u) = \emptyset$. Further $f^{d-a}(z') = x_2$, thus $g^{-1}(z') = \emptyset$ and $k(z') = hkf(z')$. Hence $hf(z')$ $k^{-1}(k(z'))$ and $s^{x_1}(hf(z')) = (b - c, 0)$, thus for $u = hf(z')$ it is not $h^{-1}(u) = \emptyset$, a contradiction.

Lemma 6. Given e,g,h,k from Lemma 5. Take an element y with $h^{-1}(y) = \emptyset$. Then $k^c(z) = y$ if and only if $z = g^c(y)$.

This Lemma 6 can be proved by an induction over n using $h^{-1}(g^n(y)) = \emptyset$ for every n.

Lemma 7. Given e,g,h,k from Lemma 5 and an element y with $g^{-1}(y) = h^{-1}(y) = \emptyset$, then $h^a g^b(x) \in K_y$ if and only if $x \in K_y$.

Proof is easy.

Construction 4. Let f be a connected surjective translation with an increasing kernel, e,g,h,k satisfying (3),(4). Assume that for every x either $g^{-1}(x) = \emptyset$ or $k(x) = hkf(x)$. Let there exist $x_1, x_2, x_3 \in X$, $x_i \in T_{m_i, n_i}$ $g^{-1}(x_i) = h^{-1}(x_i) = \emptyset$, $i = 1,2,3$, such that for $n_2 > m_1$ it is $f(x_3) = h^{n_1 - 1} f^{m_1}(x_2)$, for $n_2 \leq m_1$ it is $f(x_3) = g^{m_2} k^{n_2}(f(x_1))$ and if $x_1 = x_3$ or $x_3 \in K_{x_2}$ and $n_2 \leq m_1$ then $x_1 = x_2 = x_3$. Define translations f'_x by:

$f'_x(t) = h^a g^c(x_3)$ for $s^{x_1}(x) = (b,a)$ and $s^{x_2}(t) = (c,b)$;

$f'_x(t) = f_x(t)$ otherwise,

where f_x are translations defined in Construction 2. Then $L(M') = \{ f'_x ;\ x \in X \}$ is a Cayley's T-monoid with $f \in K(M')$.

For the proof of Construction 4 we shall need the following properties of elements x_1, x_2, x_3. In Lemmas 8 - 10 suppose assumptions of Construction 4.

<u>Lemma 8</u>. If $x_3 \in K_{x_2}$ then either $m_1 = 0$ or $x_1 = x_2 = x_3$ or $m_1 = n_1 = 1$ and $x_2 = x_3$.

Proof is easy.

<u>Lemma 9</u>. If $x_3 \notin K_{x_1}$ and $K_{x_1} \cap K_{x_2} \neq \emptyset$, then for $n_2 > m_1$ it is $m_1 + n_1 > n_2$, for $n_2 \leq m_1$ it is $m_2 + n_2 > m_1$.

<u>Proof</u>: By Lemma 5 we have $x_1 \in K_{x_2}$ or $x_2 \in K_{x_1}$.

1) Consider $n_2 > m_1$; if $x_2 \in K_{x_1}$, then $k^{m_2-m_1} f^{n_2-n_1}(x_2) = x_1$; further $f(x_3) = h^{n_1-1} f^{m_1}(x_2)$, i.e. $f^{n_1}(x_3) = f^{m_1}(x_2)$. Assuming $m_1 + n_1 \leq n_2$ we get $k^{m_2-m_1} f^{n_2-n_1}(x_2) = k^{m_2-m_1} f^{n_2-n_1-m_1} f^{n_1}(x_1) = k^{m_2-m_1} f^{n_2-m_1}(x_3)$. Thus $x_3 \in K_{x_1}$, a contradiction. If $x_1 \in K_{x_2}$, then $n_1 \geq n_2$ and $m_1 \geq m_2$, hence $m_1 + n_1 \geq n_2$. For $n_1 = n_2$ we have $k^{m_1-m_2}(x_1) = x_2$, i.e. $x_1 = g^{m_1-m_2}(x_2)$, thus $x_1 = x_2$. Assuming $m_1 = 0$ we get $x_3 \in K_{x_2} = K_{x_1}$, a contradiction.

2) Consider $n_2 \leq m_1$. If $x_2 \in K_{x_1}$, then $m_2 \geq m_1$ and thus $m_2 + n_2 > m_1 (n_2 > 0)$. Let $x_1 \in K_{x_2}$, then $k^{m_1-m_2} f^{n_1-n_2}(x_1) = x_2$. For $n_1 = n_2$ from $k^{m_1-m_2}(x_1) = x_2$ we get $x_1 = x_2$, so again $m_2 + n_2 > m_1$. Consider $n_1 > n_2$, then $f^{n_1-n_2}(x_3) = g^{m_2} k^{n_2} f^{n_1-n_2}(x_1)$ $(f(x_3) = g^{m_2} k^{n_2} f(x_1))$ and further $f^{n_1-n_2}(x_3) = g^{m_2} k^{n_2} g^{m_1-m_2}(x_2)$ (use Lemma 6). Assuming $m_2 + n_2 \leq m_1$, we get $f^{n_1-n_2}(x_3) = g^{m_1-n_2}(x_2)$ and thus $x_3 \in K_{x_1}$, a contradiction.

Lemma 10. If $x_3 \in K_{x_1}$, then either $n_2 > m_1$, $K_{x_2} \subseteq$ $\subseteq K_{x_1}$ and $n_1 + m_1 \leq n_2$ or $x_1 = x_2 = x_3$.

Proof: Consider $n_2 > m_1$; we have $f^{n_2-m_1}(x_3) =$ $= g^{m_2-m_1}(x_1)$ (use Lemma 6), further $f^{n_2-m_1}(x_3) =$ $= f^{n_2-m_1-1} h^{n_1-1} f^{m_1}(x_2)$. Assuming $n_2 - m_1 - 1 < n_1 - 1$ ($n_2 <$ $< m_1 + n_1$) we get $f^{n_2-m_1}(x_3) = h^{n_1+m_1-n_2} f^{m_1}(x_2) =$ $= g^{m_2-m_1}(x_3)$, hence $h^{-1}(g^{m_2-m_1}(x_3)) \neq \emptyset$ and $h^{-1}(x_3) \neq \emptyset$, a contradiction. So we have $n_2 \geq m_1 + n_1$ and further $g^{m_2-1m_1}(x_1) = f^{n_2-n_1}(x_1)$, which means $x_2 \in K_{x_1}$.

If $n_2 \leq m_1$, then $k^{m_2-n_2}(x_3) = x_1$ ($x_3 \in T_{m_1-n_2+m_2, n_1}$), i.e. $x_3 = g^{m_2-n_2}(x_1)$, so $m_2 = n_2$ and $x_1 = x_3$, hence $x_1 = x_2 = x_3$.

Demonstration of Construction 4: Evidently e is an exact source of $L(M')$ thus it suffices to show that $L(M')$ is closed under the composition. Denote $z' = f'_x(y)$; we are going to prove the equality:

$$f'_x f'_y(t) = f'_{z'}(t) \quad \text{for all} \quad x,y,t \in X . \qquad (8)$$

Assume $x \in T_{m,n}$, $y \in T_{r,s}$, $t \in T_{p,q}$. Evidently (8) holds if $f'_x f'_y(t) = f_x f_y(t)$ and $f'_{z'}(t) = f_z(t)$. Thus first the assertion:

Let $y \in K_{x_1}$, $t \in K_{x_2}$ with $s^{x_2}(t) = (c, r - m_1)$; if $x_3 \in$ $\in K_{x_2}$, $x \in K_{x_1}$ and $s = m$ then $f'_x f'_y(t) = h^{n-n_1} g^c(x_3)$; (A) if either $x_3 \notin K_{x_2}$ or $x \notin K_{x_1}$ and $m \leq s - n_1$ and for $n_2 >$ $> m_1$ it is $n_2 - m_1 + s > m$, for $n_2 \leq m_1$ it is $s > m$, then

$$f'_x f'_y(t) = h^{n-m+s-n_1} g^c(x_3) \qquad (B)$$

Otherwise $f'_x f'_y(t) = f_x f_y(t)$.

Let y,t be such elements that $f'_y(t) = f_y(t)$, then for $y \in K_{x_2}$, $x \in K_{x_1}$, $s - n_2 = m - m_1$ and $q \leq r - m_2$ we have $f'_x f'_y(t) = h^{n-n_1} g^{r-q+p-m_2}(x_3)$; (C)

for $x \in K_{x_1}$, $t \in K_{x_2}$, $q - r + s - n_2 = m - m_1$ and $q - n_2 \geq r$ it is $f'_x f'_y(t) = h^{n-n_1} g^{p-m_2}(x_3)$; (D)

otherwise $f'_x f'_y(t) = f_x f_y(t)$.

Now let us show the equality (8). First suppose $z' = z = f_x(y)$; i.e. if $x \in K_{x_1}$ and $y \in K_{x_2}$ then $s - n_2 \neq m - m_1$. The following holds $f'_{z'}(t) \neq f_z(t)$ iff for $s > m$ we have $y \in K_{x_1}$ and $m \leq s - n_1$, then $s^{x_1}(z) = (r - m_1, s - m + n - n_1)$ and for t with $s^{x_2}(t) = (c, r - m_1)$ it is $f'_{z'}(t) = h^{s-m+n-n_1} g^c(x_3)$. (E)

In the case $s \leq m$ iff $x \in K_{x_1}$ and $s \leq m - m_1$, then $s^{x_1}(z) = (r + m - s - m_1, n - n_1)$ and for t with $s^{x_2}(t) = (c, r + m - s - m_1)$ it is $f'_{z'}(t) = h^{n-n_1} g^c(x_3)$. (F)

The equality (8) must be shown for t such that $f'_x f'_y(t) \neq f_x f_y(t)$ or $f'_{z'}(t) \neq f_z(t)$. Suppose $f'_{z'}(t) \neq f_z(t)$ and $s > m$, thus $y \in K_{x_1}$, $m \leq s - n_1$ and $s^{x_2}(t) = (c, r - m_1)$; in this case compare (B) and (E). Let $s \leq m$, $f'_{z'}(t) \neq f_z(t)$, i.e. $x \in K_{x_1}$ and $s \leq m - m_1$. If $s = m$ and $s \leq m - m_1$, it means $m_1 = 0$. For $x_3 \in K_{x_2}$ compare (A) and (F), for $x_3 \notin K_{x_2}$ compare (B) and (F). Let $s < m$, $s^{x_2}(t) = (c, r + m - s - m_1)$ then it holds $r + m - s - m_1 + n_2 - n_2 = r + m - s - m_1 \geq r$ and $r + m - s - m_1 + n_2 - r + s - n_2 = m - m_1$, i.e. the assumptions of (D) hold and moreover

$h^{n-n_1}g^{p-m_2}(x_3) = h^{n-n_1}g^c(x_3)$.

Suppose $f'_x f'_y(t) \neq f_z(t)$; the case (B) for $s \leq m$ cannot be fulfilled (since $s \leq m$ implies $m > s - n_1$) , neither the case (C) ($z = z'$). The cases (A), (D) and (B) for $s > m$ have been shown.

Let us suppose $z \neq z'$, i.e. $x \in K_{x_1}$, $y \in K_{x_2}$ and $s - n_2 = m - m_1$, then

$z' = h^{n-n_1}g^{r-m_2}(x_3)$. For $n_2 > m_1$ we have $z' \in T_{r, n_2-m_1+n}$, for $n_2 \leq m_1$ $z' \in T_{r+m_1-n_2, n}$. It holds $z' \in K_{x_1}$ iff $x_3 \in K_{x_1}$ (see Lemma 7) and in this case for $n_2 > m_1$ it is $s^{x_1}(z') = (r - m_1, n_2 - m_1 + n - n_1)$, for $n_2 \leq m_1$ $s^{x_1}(z') = (r - n_2, n - n_1)$. Therefore $f'_{z'}(t) \neq f_z(t)$ iff $x_3 \in K_{x_1}$ and if

$n_2 > m_1$ for t with $s^{x_2}(t) = (c, r - m_1)$, $f'_{z'}(t) =$

$$= h^{n_2-m_1+n-n_1}g^c(x_3) \; ; \tag{G}$$

if $n_2 \leq m_1$ for t with $s^{x_2}(t) = (c, r - n_2)$, $f'_{z'}(t) =$

$$= h^{n-n_1}g^c(x_3) \; . \tag{H}$$

Further it can be shown: $f_{z'}(t) \neq f_z(t)$ iff $q \leq r - m_2$; in this case it is $f_{z'}(t) = g^{p-q+r-m_2}h^{n-n_1}(x_3)$. (J)

Let us show the equality (8). First consider $x_3 \notin K_{x_1}$, i.e. $z' \notin K_{x_1}$, hence $f'_{z'}(t) = f_{z'}(t)$. If for y and t we have $f'_y(t) = f_y(t)$, then (8) follows from (C) and (J), so for $K_{x_1} \cap K_{x_2} = \emptyset$ (8) holds (the assumptions of (D) are not fulfilled, as $q - r + s - n_2 = m - m_1$ implies $q = r$ and $q - n_2 < r$, the same holds about (A) and (B) as $y \notin K_{x_1}$). Let now $K_{x_1} \cap K_{x_2} \neq \emptyset$, $y \in K_{x_1}$

Assume $n_2 > m_1$, then $m = s - n_2 + m_1 > s - n_1$ (use $m_1 + n_1 > n_2$) and the assumptions of (B) are not fulfilled. If $n_2 \leq m_1$, then $s \leq m$ as $s - n_2 = m - m_1$. Then for t with $s^{x_2}(t) = (c, r - m_1)$ it is $f'_x f'_y(t) = f_z(t)$, moreover for this t we have $q = r - m_1 + n_2 > r - m_2$ (use $m_2 + n_2 > m_1$) and $f_{z'}(t) = f_z(t)$, too. Thus (8) holds for $x_3 \notin K_{x_1}$.

Consider $x_3 \in K_{x_1}$, i.e. $z' \in K_{x_1}$. Suppose $n_2 > m_1$, then $f'_{z'}(t) \neq f_{z'}(t)$ for t with $s^{x_2}(t) = (c, r - m_1)$. We know that $K_{x_2} \subseteq K_{x_1}$, hence $y \in K_{x_1}$ and for this t we have fulfilled the assumptions of (B) because $m \leq s - n_1$ (use $m_1 + n_1 \leq n_2$) and $n_2 - m_1 + s > m$ (use $n_2 > m_1$). Moreover $h^{n-m+s-n_1} g^c(x_3) = h^{n-m_1+n_2-n_1} g^c(x_3)$. Assumptions of (A) are not fulfilled as $s \neq m$. The rest is the same as in the case $z' \notin K_{x_1}$.

Consider $n_2 \leq m_1$, then $x_1 = x_2 = x_3$ and elements t with $s^{x_2}(t) = (c, r - m_1)$ and t with $s^{x_2}(t) = (c, r - n_1)$ are the same elements. Thus (8) follows from (A) and (H).

Construction 5. Given e, g, h, k having (3) and (4), k such that either $k(x) \in g^{-1}(x)$ or $k(x) = hkf(x)$. Let then exist a sequence $\{x_i\}_{i=0}^{m_0}$ such that $g^{-1}(x_i) = h^{-1}(x_i) = \emptyset$, $x_i \in T_{n_0, n_i}$, $i = 1, \dots, m_0, n_{m_0} = n_0$, $f(x_{i+1}) = h^{n_{i+1} - n_i - 1}(x_i)$, $i = 1, \dots, m_0 - 1$ and $f(x_1) = g^{n_0} k^{n_0} f(x_1)$. Define for $t \in K_{x_{m_0}}$ with $s^{x_{m_0}}(t) = (c, b)$

$f'_x(t) = h^a g^c(x_i)$ if $x \in K_{x_i} \setminus K_{x_{i+1}}$ and $s^{x_1}(x) = (b, a)$,

$f'_x(t) = f_x(t)$ otherwise,

where f_x are translations from Construction 2.

Then $L(M') = \{f_x; x \in X\}$ forms a Cayley's T-monoid containing f as a left translation.

Demonstration is analogous to the demonstration of Construction 4.

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