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THE PRODUCT OF RELATIVELY REGULAR OPERATORS

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Abstract: The product T_1T_2 of relatively regular bounded linear operators between Banach spaces is shown to be relatively regular iff the product QP is relatively regular, where Q is a projection parallel to the null space of T_1, and P a projection onto the range of T_2. The paper gives applications of this result.

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1. The Main Result

Let X and Y be complex Banach spaces, and let $B(X,Y)$ be the space of all bounded linear operators from X to Y; $B(X,X)$ is written as $B(X)$. If $T \in B(X,Y)$, $N(T)$ and $R(T)$ denote the null space and the range of T, respectively. An operator $S \in B(Y,X)$ is called a pseudo-inverse of $T \in B(X,Y)$ if it satisfies the equation

$$TST = T ; \tag{1}$$

an operator $T \in B(X,Y)$ is called relatively regular if it possesses a pseudo-inverse. It is known (Caradus [5], Nashed [9]) that T is relatively regular iff $N(T)$ is complemented in X and $R(T)$ closed and complemented in Y. If S

is a pseudo-inverse of T, ST is a projection of X parallel to $N(T)$, and TS a projection of Y onto $R(T)$, i.e., $N(ST) = N(T)$ and $R(TS) = R(T)$. If $T \in B(X,Y)$ is a relatively regular operator, we can find a pseudo-inverse S of T which satisfies, in addition to (1), the equation

$$STS = S \tag{2}$$

(cf. Caradus [5], p. 9). In this paper we call such S a <u>strict pseudo-inverse</u> of T.

Caradus [4], [5] recently initiated the study of relative regularity of the product T_1T_2 of two relatively regular operators in $B(X)$ based on the product QP of a projection $Q \in B(X)$ onto $R(T_2)$ and a projection $P \in B(X)$ parallel to $N(T_1)$, and obtained useful sufficient conditions. The present paper continues in this investigation, and gives a complete solution to the problem of the relation between relative regularity of T_1T_2 and of QP.

Let X, Y and Z be complex Banach spaces, and let $T_1 \in B(Y,Z)$ and $T_2 \in B(X,Y)$ be two relatively regular operators. Sufficient conditions for the relative regularity of T_1T_2 have been given by various authors (e.g. Atkinson [1], Caradus [4], [5], Koliha [8]). Bouldin [2], [3] has found a necessary and sufficient condition in the case when $X = Y = Z$ is a Hilbert space. We show that the relative regularity of the product T_1T_2 is equivalent to the relative regularity of the product of two projections in $B(Y)$.

<u>Theorem 1</u>: <u>Let</u> $T_1 \in B(Y,Z)$ <u>and</u> $T_2 \in B(X,Y)$ <u>be relatively regular operators with pseudo-inverses</u> S_1 <u>and</u> S_2,

respectively. Then the operator T_1T_2 *is relatively regular iff the operator* $S_1T_1T_2S_2 \in B(Y)$ *is relatively regular. If* U *is a pseudo-inverse of* T_1T_2, T_2UT_1 *is a pseudo-inverse of* $S_1T_1T_2S_2$ *, then* $S_2T_2S_2VS_1T_1S_1$ *is a pseudo-inverse of* T_1T_2 .

Proof: Let us first assume that U is a pseudo-inverse of T_1T_2 . Then

$$\begin{aligned} S_1T_1T_2S_2(T_2UT_1)\, S_1T_1T_2S_2 &= S_1T_1(T_2S_2T_2)\, U\, (T_1S_1T_1)\, T_2S_2 \\ &= S_1T_1T_2UT_1T_2S_2 \\ &= S_1T_1T_2S_2 , \end{aligned}$$

so that T_2UT_1 is a pseudo-inverse of $S_1T_1T_2S_2$. Conversely, suppose that $V \in B(Y)$ is a pseudo-inverse of $S_1T_1T_2S_2$. Let us write $P = T_2S_2$ and $Q = S_1T_1$. Then $P \in B(Y)$ is a projection onto $R(T_2)$, $Q \in B(Y)$ projection parallel to $N(T_1)$, and $QPVQP = QP$. Put

$$W = PVQ . \tag{3}$$

Then

$$PWQ = W \quad \text{and} \quad QWP = QP . \tag{4}$$

From (4) we get $Q(I - W)P = 0$. Hence $I - W$ maps $R(P) = R(T_2)$ into $N(Q) = N(T_1)$ in Y , and $T_1(I - W)T_2 = 0$. Thus

$$\begin{aligned} T_1T_2S_2WS_1T_1T_2 &= T_1PWQT_2 = T_1WT_2 \\ &= T_1T_2 - T_1(I - W)T_2 \\ &= T_1T_2 , \end{aligned}$$

which shows that $S_2WS_1 = S_2PVQS_1 = S_2T_2S_2VS_1T_1S_1$ is a pseudo-inverse of T_1T_2 .

Remark. Let S_1 and S_2 be strict pseudo-inverses of T_1 and T_2, respectively. Then each pseudo-inverse V of $S_1T_1T_2S_2$ yields the pseudo-inverse S_2VS_1 of T_1T_2. It is readily verified that in this case S_2VS_1 is a strict pseudo-inverse whenever V is a strict pseudo-inverse.

Assume that $T_1 \in B(Y,Z)$ and $T_2 \in B(X,Y)$ are relatively regular, so that the spaces $N(T_1)$ and $R(T_2)$ are closed and complemented in Y. Let $P \in B(Y)$ be a projection onto $R(T_2)$, $I - Q \in B(Y)$ a projection onto $N(T_1)$. Then we can find pseudo-inverses S_1 and S_2 of T_1 and T_2, respectively, such that $T_2S_2 = P$ and $S_1T_1 = Q$ (Caradus [5]). Then, according to Theorem 1, T_1T_2 is relatively regular iff the product QP of the projections Q, $P \in B(Y)$ is relatively regular. This proves the following result.

Theorem 2: *Let $T_1 \in B(Y,Z)$ and $T_2 \in B(X,Y)$ be relatively regular operators. Then the following conditions are equivalent:*

(i) There exists a projection $P \in B(Y)$ onto $R(T_2)$ and a projection $Q \in B(Y)$ parallel to $N(T_1)$ such that the product QP is relatively regular in $B(Y)$.

(ii) For each projection $P \in B(Y)$ onto $R(T_2)$ and each projection $Q \in B(Y)$ parallel to $N(T_1)$ the product QP is relatively regular in $B(Y)$.

(iii) The product T_1T_2 is relatively regular in $B(X,Z)$.

If (i) is satisfied with a pseudo-inverse V of QP and if S_1, S_2 are strict pseudo-inverses of T_1, T_2 such that $T_2S_2 = P$ and $S_1T_1 = Q$, then the operator

(5) $$U = S_2 V S_1$$

is a pseudo-inverse of $T_1 T_2$, strict whenever V is strict.

2. <u>Special cases</u>

In this section we assume that $T_1 \in B(Y,Z)$ and $T_2 \in \in B(X,Y)$ are relatively regular operators with strict pseudo-inverses S_1 and S_2 , respectively. Unless stated otherwise, we assume that P and Q are the projection operators $P = T_2 S_2$ and $Q = S_1 T_1$.

(I) If T_1 (resp. T_2) is regular, $T_1 T_2$ is relatively regular with a strict pseudo-inverse $S_2 S_1$, where $S_1 = T_1^{-1}$ (resp. $S_2 = T_2^{-1}$) . This follows from (5) when we observe that under our assumptions $Q = I$ and $V = P$ (resp. $P = I$ and $V = Q$).

(II) If QP is a projection, $T_1 T_2$ is relatively regular with a strict pseudo-inverse $S_2 QPS_1 = S_2 S_1 T_1 T_2 S_2 S_1$. Indeed, the projection QP is relatively regular with pseudo-inverse QP .

(III) If Q and P commute, $T_1 T_2$ is relatively regular with a strict pseudo-inverse $S_2 S_1$. If $QP = PQ$, then QP is a projection, and $S_2 QPS_1 = S_2 PQS_1 = S_2 T_2 S_2 S_1 T_1 S_1 = S_2 S_1$ is a pseudo-inverse of $T_1 T_2$ by (II). This result has been obtained by Caradus [4]; cf. also [5], p. 36. As a corollary, we obtain the following result (Caradus [5], p. 37): If T_1 and T_2 are relatively regular with either $N(T_1)$ finite dimensional or $R(T_2)$ finite codimensional, then $T_1 T_2$ is relatively regular. This contains as special case a theorem due to Atkinson [1] on the product of semi-Fredholm operators.

(IV) If $\lambda = 0$ is a pole of $(\lambda I - QP)^{-1}$ of order 1, T_1T_2 is relatively regular with a strict pseudo-inverse $U = S_2(0_N \oplus (QP_R)^{-1})S_1$, where 0_N is the zero operator on $N(QP)$ and P_R the restriction of P to $R(QP)$. First of all, $X = N(QP) \oplus R(QP)$ with $R(QP)$ closed (Taylor [10], p. 306). This means that QP is relatively regular. Moreover, QP_R is a bijective operator on the Banach space $R(QP)$, and hence continuously invertible on $R(QP)$ by the open mapping theorem. The operator $V = 0_N \oplus (QP_R)^{-1}$ is a strict pseudo-inverse of QP, so that $U = S_2VS_1$ is a strict pseudo-inverse of T_1T_2.

(V) If $\lambda = 0$ is a pole of $(\lambda I - QP)^{-1}$ of order 1, and if the spectrum of QP is contained in the set $\{\lambda : |\lambda^2 - \alpha^{-1}| < \alpha^{-1}\} \cup \{0\}$ for some $\alpha > 0$, then T_1T_2 is relatively regular with pseudo-inverses $U_j = S_2V_jS_1$, $j = 1,2$, where

$$V_1 = \sum_{n=0}^{\infty} \alpha (I - \alpha (QP)^2)^n \, QP , \tag{6}$$

and where V_2 is the projection onto $R((PQ)^2)$ parallel to $N((PQ)^2)$ (Caradus [4],[5], Koliha [8]). Theorems 2 and 3 of [8] show that, for any fixed $\alpha > 0$, the two conditions on the spectrum of QP given above are necessary and sufficient for the convergence of the series (6); the sum V_1 is then a strict pseudo-inverse of QP. From the equations (3) and (4) it follows that $V_2 = PV_1Q$ is also a (strict) pseudo-inverse of QP. Then

$$V_2 = P\Big(\sum_{n=0}^{\infty} \alpha (I - \alpha (QP)^2)^n \, QP\Big)Q = -\sum_{n=0}^{\infty} \alpha \, PQ(I - \alpha (PQ)^2)^n PQ$$

$$= \sum_{n=0}^{\infty} \alpha (PQ)^2(I - \alpha(PQ)^2)^n = I - \lim_{N\to\infty} (I - \alpha(PQ)^2)^N = I - W ,$$

where $W = \lim_{N\to\infty} (I - \alpha(PQ)^2)^N$ is the projection onto $N((PQ)^2)$ parallel to $R((PQ)^2)$ (cf. [7], Theorem 4). This improves on a result of Caradus [4].

(VI) If there exists $L \in B(Y)$ such that the operator $Q(I - PLQ)P$ is relatively regular, then T_1T_2 is relatively regular. This follows from Theorem 2 and a result of Atkinson [1] that an operator $T \in B(Y)$ is relatively regular if $T - TLT$ is relatively regular for some $L \in B(Y)$.

(VII) If the operator $(QP)^2$ is Fredholm, then T_1T_2 is relatively regular. Atkinson [1] proved that operators S and T are relatively regular if ST is Fredholm. Our result follows on setting $S = T = QP$ and applying Theorem 2.

(VIII) Let Y be a Hilbert space, P the orthogonal projection of Y onto $R(T_2)$ and Q the orthogonal projection of Y onto $N(T_1)$. The operator T_1T_2 is relatively regular iff the range of QP (or equivalently of PQ) is closed. This follows from the fact that closed subspaces of Hilbert space are complemented. Bouldin's result [2] shows that the range $R(QP)$ is closed iff the subspaces

$$R(T_2) \quad \text{and} \quad N(T_1) \cap [N(T_1) \cap R(T_2)]^{\perp}$$

enclose a positive angle. A pseudo-inverse U of T_1T_2 may be obtained from Groetsch's representation theorem [6] as follows: Let A be the restriction of the operator $I - PQP \in B(Y)$ to $R(PQ)$, let Ω be an open subset of the interval $(-\infty,1]$ containing the spectrum of A , and let $\{S_\alpha\}$ be a net of continuous real functions on Ω such that $\lim_{\alpha} S_{\alpha}(x) = 1/(1 - x)$ uniformly on the spectrum of A. Then

$$U = \lim_{\alpha} S_2 S_{\alpha}(A) PQS_1$$

is the uniform operator topology of $B(X,Y)$.

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