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GENERALIZED POINTWISE SYMMETRIC SPACES

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Abstract: In this paper we give an example of a Riemannian s-manifold (with a discontinuous s-structure) which does not admit any regular s-structure in the sense of A.J. Ledger (x).

Key words: Homogeneous manifolds, Riemannian manifolds, symmetric spaces.

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1. Introduction. Let (M,g) be a differentiable Riemannian manifold. An isometry s_x of (M,g) for which $x \in M$ is an isolated fixed point is called a symmetry of M at x, ([7]). An s-structure on (M,g) is a family $\{s_x : x \in M\}$ of symmetries of (M,g) (one symmetry at each point). Here the map $s: M \longrightarrow I(M)$ need not be even continuous. According to a theorem by F. Brickel, if (M,g) admits an s-structure, then the group $I(M)$ of isometries is transitive ([7]), and thus M is a homogeneous Riemannian manifold.

An s-structure $\{s_x\}$ is called regular if for every two points $x, y \in M$

(x) I wish to thank to A.W. Deicke, who provided the basic "model", and also to A. Gray and H. Samelson, who kindly answered my questions concerning the transformation groups on spheres.

$$s_x \circ s_y = s_z \circ s_x \,, \quad z = s_x(y) \qquad \text{(Cf. [3]).}$$

If $\{s_x\}$ is regular, then the map $s\colon M \longrightarrow I(M)$ is always differentiable (cf. [5], Theorem 1).

An s-structure $\{s_x\}$ is called <u>of order</u> k if $(s_x)^k =$ = identity for all $x \in M$, and k is the least integer of this property. Following A.W. Deicke, if (M,g) admits an s-structure, then it always admits an s-structure of finite order. Further, if (M,g) admits a regular s-structure then (M,g) admits a regular s-structure of finite order. (Cf.[5], Lemma 3 and Theorem 2).

<u>A generalized symmetric Riemannian space</u> is a Riemannian manifold (M,g) admitting a regular s-structure (cf [5]). Now, we shall introduce a more general

<u>Definition</u>. <u>A generalized pointwise symmetric Riemannian space</u> is a Riemannian manifold (M,g) admitting an s-structure.

<u>Order</u> of a generalized symmetric (or generalized pointwise symmetric) Riemannian space (M,g) is the minimum order of a regular s-structure on (M,g) (or the minimum order of an s-structure on (M,g), respectively).

It is easy to show that a generalized pointwise symmetric Riemannian space of order 2 is a usual Riemannian (globally) symmetric space. Moreover, the canonical s-structure consisting of geodesic symmetries is always regular (see [3]). Thus, for order 2, the concepts "pointwise symmetric" and "symmetric" are equivalent.

The existence of generalized symmetric Riemannian spaces of order greater than two is shown in [7], and many examples of such spaces (of orders 3, 4 and 6) are given in [4] and [6].

The purpose of this paper is to present a family of generalized pointwise symmetric Riemannian spaces which are not generalized symmetric. This example seems to be non-trivial as it uses the classification of compact connected Lie groups acting transitively and effectively on spheres, due to D. Montgomery, H. Samelson and A. Borel.

2. The main theorem. Consider the Hermitean manifold $(C^{2n+1}[z^1,\dots,z^{n+1}], g_\lambda)$ with the metric $g_\lambda = \sum_{i=1}^{2n+1} dz^i\, d\bar{z}^i + \lambda (\sum_{i=1}^{2n+1} z^i\, d\bar{z}^i)(\sum_{j=1}^{2n+1} \bar{z}^j\, dz^j)$ where $\lambda \neq 0$, $\lambda > -1$ is a constant. Let us consider the sphere S^{4n+1} defined by $\sum_{i=1}^{2n+1} z^i \bar{z}^i = 1$, and the real Riemannian metric $\hat{g}_\lambda$ on S^{4n+1} induced by g_λ. (Here the real coordinates are introduced putting $z^j = x^j + iy^j$, $j = 1,\dots,2n+1$.)

Theorem. For $n \geq 2$, the Riemannian manifold $(S^{4n+1}, \hat{g}_\lambda)$ is generalized pointwise symmetric of order 4 but it is not generalized symmetric.

Proof. Let us define the origin of S^{4n+1} to be the point $o = (0,\dots,0,1)$ of C^{2n+1}. The transformation of C^{2n+1} given by $(z^{2i-1})' = -\bar{z}^{2i}$, $(z^{2i})' = \bar{z}^{2i-1}$ $(i = 1,\dots,n)$, $(z^{2n+1})' = \bar{z}^{2n+1}$, induces a transformation $\tilde{s}_o$ of S^{4n+1} with a fixed point o. Clearly, $\tilde{s}_o$ is an isometry of $(S^{4n+1}, \hat{g}_\lambda)$. We can see easily that the tangent map

$(\tilde{s}_o)_{*o}$ has no nonzero fixed vectors in the tangent space $(S^{4n+1})_o$, and hence o is an isolated fixed point of $\tilde{s}_o$. Moreover, we have $(\tilde{s}_o)^4$ = identity.

The group $U(2n + 1)$ of all unitary transformations of C^{2n+1} (with respect to its natural structure of a linear Hermitean space) preserves the metric g_λ and it acts transitively and effectively on S^{4n+1}. Thus $U(2n + 1)$ can be considered as a group of isometries of the Riemannian manifold $(S^{4n+1}, \hat{g}_\lambda)$.

Define an isometry $\tilde{s}_x$ of $(S^{4n+1}, \hat{g}_\lambda)$ for every $x \in S^{4n+1}$ as follows: let $A \in U(2n + 1)$ be such that $A(0) = x$, and put $\tilde{s}_x = A \circ \tilde{s}_o \; A^{-1}$. (The transformation $\tilde{s}_x$ depends, in general, on the choice of A). Then x is an isolated fixed point of $\tilde{s}_x$. Thus $(S^{4n+1}, \hat{g}_\lambda)$ is a generalized pointwise symmetric space. (This example was pointed out by A.W. Deicke.)

Let us remark that $(S^{4n+1}, \hat{g}_\lambda)$ is not locally symmetric and it is of odd dimension. Thus, the order of the space cannot be 2 or 3 and hence $k = 4$.

We shall now prove the second part of the Theorem. In the following, $SO(4n + 2)$, $U(2n + 1)$ and $SU(2n + 1)$ will always denote the transformation groups of S^{4n+1} which are induced by the corresponding transformation groups of the given real space R^{4n+2} and of the complex space C^{2n+1}.

Lemma. Let K be a connected group of isometries of $(S^{4n+1}, \hat{g}_\lambda)$ acting transitively on S^{4n+1}. Then

$K \supseteq SU(2n+1)$.

Proof. According to Montgomery - Samelson [8], and Borel [1],[2], each compact connected Lie transformation group acting transitively on S^{4n+1} is isomorphic to one of the following groups: $SO(4n+2)$, $U(2n+1)$, $SU(2n+1)$. Let G be the component of unity of the full isometry group $I(S^{4n+1}, \hat{g}_\lambda)$, then $G \supseteq U(2n+1)$. G cannot be isomorphic to $SO(4n+2)$; otherwise $\hat{g}_\lambda$ would be a metric of constant curvature. Thus $G = U(2n+1)$.

Let K be an arbitrary connected and transitive group of isometries of $(S^{4n+1}, \hat{g}_\lambda)$; then $K \subseteq U(2n+1)$. If K is isomorphic to $U(2n+1)$, then $K = U(2n+1)$ and Lemma is proved. Let now K be isomorphic to $SU(2n+1)$. Then the Lie algebra $\underline{k}$ is isomorphic to $\underline{su}(2n+1)$, and $\underline{k} \subset \underline{u}(2n+1)$. On the other hand, we have $\underline{u}(2n+1) = \underline{su}(2n+1) \oplus \mathbb{R}$ (direct sum), and the subalgebra $\underline{su}(2n+1)$ is simple. Hence it follows $\underline{k} = \underline{su}(2n+1)$, and consequently, $K = SU(2n+1)$. This completes the proof.

Let now $\{s_x\}$ be a regular s-structure on $(S^{4n+1}, \hat{g}_\lambda)$, and let K denote the component of unity of the automorphism group of the Riemannian s-manifold $(S^{4n+1}, \hat{g}_\lambda, \{s_x\})$. (Here, by automorphisms we mean isometries $A \in G$ such that $A\, s_x = s_{A(x)} \circ A$ for all $x \in M$.) According to [3], Theorem 5.6, K is a closed subgroup of G acting transitively on M . According to the Lemma, $K \supseteq SU(2n+1)$. For the stability group K_o of K at the origin o we have $K_o \supseteq SU(2n)$ (= the subgroup of $SU(2n+1)$ leaving all points $(0, \ldots$

$\ldots,0,e^{i\varphi})$ of S^{4n+1} fixed). The transformation s_0 commutes with each element of K_0 and particularly, it commutes with each element of $SU(2n)$.

Consider the tangent space $(S^{4n+1})_0$. It is generated by the vectors

$e_i = \left(\frac{\partial}{\partial x^i}\right)_0$, $f_j = \left(\frac{\partial}{\partial y^j}\right)_0$, where $i = 1,\ldots, 2n$, $j = 1,\ldots$ $\ldots, 2n + 1$..
Here f_{2n+1} is orthogonal to the 4n-dimensional subspace V generated by e_i, f_i for $i = 1,\ldots, 2n$.

Let H denote the real isotropy representation of $SU(2n)$ in the tangent space $(S^{4n+1})_0$, and $S_0 = (s_{0*})_0$. All linear transformations $h \in H$, and also S_0, are orthogonal transformations of $(S^{4n+1})_0$ with respect to the scalar product $(\hat{g}_\lambda)_0$. H acts transitively on the subspace V, and all fixed vectors with respect to H are of the form λf_{2n+1}. S_0 commutes with each $h \in H$ and hence $S_0(f_{2n+1})$ is a fixed vector with respect to H. Thus $S_0(f_{2n+1}) =$ $= \pm f_{2n+1}$, and since S_0 does not admit non-zero fixed vectors, $S_0(f_{2n+1}) = - f_{2n+1}$. Also, the subspace V is invariant with respect to S_0.

Let $\underline{h}$ denote the Lie algebra of H. For every pair (r,s), $1 \leq r \neq s \leq 2n$, consider the endomorphisms B_{rs}, $C_{rs} \in \underline{h}$ defined as follows:

$B_{rs}(e_r) = e_s$, $B_{rs}(f_r) = f_s$, $B_{rs}(e_s) = - e_r$, $B_{rs}(f_s) = - f_r$,
$C_{rs}(e_r) = - f_s$, $C_{rs}(f_r) = e_s$, $C_{rs}(e_s) = - f_r$, $C_{rs}(f_s) = e_r$,

$$B_{rs}(e_i) = B_{rs}(f_i) = C_{rs}(e_i) = C_{rs}(f_i) = 0 , \qquad i \neq r, s .$$

Let S_o satisfy

$$S_o(e_i) = \sum_{j=1}^{2n} a_i^j e_j + b_i^j f_j$$
$$S_o(f_i) = \sum_{j=1}^{2n} c_i^j e_j + d_i^j f_j \qquad i = 1,\ldots, 2n .$$

From the relations $(B_{rs} \circ S_o)(e_i) = (S_o \circ B_{rs})(e_i)$

$$(B_{rs} \circ S_o)(f_i) = (S_o \circ B_{rs})(f_i) \qquad i \neq r, s$$

we get $a_i^j = b_i^j = c_i^j = d_i^j = 0$, for all i, j such that $1 \leq i \neq j$. (For this step, the inequality $n > 1$ is decisive.)

From the relations

$$(B_{rs} \circ S_o)(e_r) = (S_o \circ B_{rs})(e_r)$$
$$(B_{rs} \circ S_o)(f_r) = (S_o \circ B_{rs})(f_r)$$

we get

$$a_r^r = a_s^s , \ b_r^r = b_s^s , \ c_r^r = c_s^s , \ d_r^r = d_s^s \qquad 1 \leq r, s \leq 2n .$$

Finally, from the relation

$(C_{rs} \circ S_o)(e_r) = (S_o \circ C_{rs})(e_r)$ we get

$$a_r^r = d_s^s = a , \ b_r^r = - c_s^s = b , \qquad 1 \leq r, s \leq 2n .$$

We have obtained

$$S_o(e_j) = a e_j + bf_j$$
$$S_o(f_j) = - be_j + af_j \qquad 1 \leq j \leq 2n , \quad a^2 + b^2 = 1 .$$
$$S_o(f_{2n+1}) = - f_{2n+1} .$$

In the complex form,

$$S_o((\frac{\partial}{\partial z^j})_o) = e^{i\varphi}((\frac{\partial}{\partial z^j})_o) \quad j = 1,\ldots,2n, \ e^{i\varphi} = a + bi$$

$S_o(f_{2n+1}) = - f_{2n+1}$.

Now, let us denote by $Z_1,\dots,Z_{2n+1}$ the complex vector fields on S^{4n+1} which are tangent components of the vector fields $\frac{\partial}{\partial z^1},\dots,\frac{\partial}{\partial z^{2n+1}}$ respectively. Let ∇, R denote the Riemannian connection and the curvature tensor field of the metric $\hat{g}_\lambda$ respectively. After a long but routine calculation we derive

$$(\nabla_{Z_2} R)_o (Z_1,\bar{Z}_1,Z_{2n+1},\bar{Z}_2) \neq 0 \text{ , i.e.,}$$

$$(\nabla_{\frac{\partial}{\partial z^2}} R)_o((\frac{\partial}{\partial z^1})_o,(\frac{\partial}{\partial \bar{z}^1})_o, f_{2n+1},(\frac{\partial}{\partial \bar{z}^2})_o) \neq 0 .$$

$(\nabla R)_o$ being invariant with respect to S_o , we come to a contradiction.

Remark. For $n = 1$, the Riemannian manifold $(S^5,\hat{g}_\lambda)$ is generalized symmetric of order 4 (cf. [6]) .

R e f e r e n c e s

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