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HEREDITARY TENSOR-ORTHOGONAL THEORIES

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Abstract: In this paper, there are determined the commutative, associative rings with unity, where every tensor-orthogonal theory for R-mod is hereditary. The corollaries give an estimate of the rings where every torsion theory is hereditary and it also shows the relation of these rings to the problem of vanishing tensor powers of modules.

Key words and phrases: Perfect ring, semiregular ring, semiartinian ring, tensor-orthogonal theory, hereditary torsion theory of simple and cosimple type, vanishing of tensor product of modules.

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Throughout the paper, R is a commutative, associative ring with unity. If $\mathfrak{M} \subseteq$ R-mod is a class of R-modules we define $\overline{\mathfrak{M}} = \{ L \mid L \in$ R-mod and $M \otimes_R L = 0$, for every $M \in \mathfrak{M} \}$. Let us recall [5] that the couple $(\mathfrak{M}, \mathfrak{L})$ of classes of R-modules is said to be a tensor-orthogonal theory ($\otimes$ -orthogonal theory) if $\overline{\mathfrak{M}} = \mathfrak{L}$ and $\overline{\mathfrak{L}} = \mathfrak{M}$. Obviously, both $\mathfrak{M}$ and $\mathfrak{L}$ are torsion classes of some torsion theories [4] and if they are both hereditary, i.e., closed under submodules, then $(\mathfrak{M}, \mathfrak{L})$ is called hereditary $\otimes$ -orthogonal theory. Similarly as in [4], we can introduce the operators + and $*$ by $\mathfrak{M}^+ = \{ X \mid X \in$ R-mod and Hom $(X,M) = 0$, for every $M \in \mathfrak{M}\}$ and $\mathfrak{M}^* =$ $= \{X \mid X \in$ R-mod and Hom $(M,X) = 0$, for every $M \in \mathfrak{M}\}$, respectively. A $\otimes$ -orthogonal theory is said to be trivial if one of its classes is R-mod. If max (R) is the set of maximal ideals of R , we put $\mathcal{S} = \{R/P \mid P \in \max(R)\}$ - a complete representative set of pairwise non-isomorphic simple modules.

A torsion theory $(\mathcal{T}, \mathcal{F})$ is of simple or cosimple type if there is a subset $\mathcal{A} \subseteq \mathcal{S}$ such that $\mathcal{A}^{*+} = \mathcal{T}$ or $\mathcal{A}^+ = \mathcal{T}$, respectively. A sequence $a_1, a_2, \ldots$ of elements of R is said to be T-nilpotent if $a_1 \ldots a_n = 0$, for some finite n . An ideal I of R is said to be T-nilpotent if every sequence of elements from I is so . Similarly, I is weakly T-nilpotent if every sequence $a_1, a_2, \ldots$ from I of bounded order i.e., there is a natural number $k \geq 1$ such that $a_i^k = 0$, $i = 1, 2, \ldots$ is T-nilpotent. Suppose that the Jacobson's radical $J(R)$ of R is T-nilpotent. Then R is said to be semiregular if $R/J(R)$ is regular (Von Neumann). Moreover, if $R/J(R)$ is a completely reducible ring (we stick to the notation of [7]) then R is called perfect. Similarly, if $J(R)$ is weakly T-nilpotent and $R/J(R)$ is regular then R is called weakly semiregular. Further, R is said to be semiartinian ring if every non-zero R-module possesses a non-zero socle. An R-module M is said to be vanishing if the tensor product of some finite number of copies of M is 0 . The smallest integer n with $\overset{n+1}{\otimes}_R M = 0$ is denoted by $d_R(M)$, and we set $\Delta(R) =$ $= \sup d_R(M)$, the supremum taken over all the vanishing modules.

The following three propositions are straightforward.

Proposition 1. Let P be a prime ideal of R. Then P is a minimal prime iff for every $p \in P$ there exist a natural number $n \geq 1$ and $y \in R \setminus P$ such that $p^n y = 0$.

Proposition 2. Let $\mathcal{A}$ be a subset of $\mathcal{S}$. Then $\mathcal{A}^+ = \overline{\mathcal{A}} = \{M \mid M \in$ R-mod and $PM = M$, for each $P \in \max(R)$ with $R/P \in \mathcal{A}\}$.

Proposition 3. Let $(\mathcal{M}, \mathcal{L})$ be a $\otimes$-orthogonal theory for R-mod. Then there is the uniquely determined subset $\mathcal{A}$ of $\mathcal{S}$ such that $\mathcal{A}^{*+} \subseteq \mathcal{M} \subseteq \{\mathcal{S} \setminus \mathcal{A}\}^+$ and $\{\mathcal{S} \setminus \mathcal{A}\}^{*+} \subseteq \mathcal{L} \subseteq \mathcal{A}^+$.

Theorem 4. Let R be a commutative ring. Then the following are equivalent:

(i) Every $\otimes$-orthogonal theory for R-mod is hereditary.

(ii) Every non-zero R-module has a proper maximal submodule.

(iii) R is semiregular.

(iv) Every torsion theory for R-mod of cosimple type is hereditary.

Proof. (i) $\Longrightarrow$ (ii). Since $\overline{\mathcal{S}}$ is hereditary, $\overline{\mathcal{S}} = \{0\}$ by Proposition 2.

(ii) $\Longleftrightarrow$ (iii). Use the results of [3].

(iii) $\Longrightarrow$ (iv). Suppose that $M \in$ R-mod and $PM = M$, for some maximal ideal P of R. Put $s(M) = \{m \mid m \in M$ and $(0:m) \nsubseteq P\}$. We are going to show that $M = s(M)$. Since $s(M)$ is a submodule and $s(M/s(M)) = 0$, we can assume that $s(M) = 0$, without loss of generality. Therefore, if $0 \neq m \in M$

then $(0:m) \subseteq P$ and $P/(0:m)$ is a nil-ideal of $R/(0:m)$ by Proposition 1, using the fact that every prime ideal of R is maximal ([6], ex. 12,63). Hence $P/(0:m) = J(R/(0:m))$ and consequently if $P' \neq P$ is another maximal ideal then $(0:m) \nsubseteq P'$, i.e., $P'(R/(0:m)) = R/(0:m)$. Thus $M = QM$, for every $Q \in \max(R)$ and the hypothesis yields $M = 0$. Now, we have proved that if $M = PM$ then $s(M) = M$, i.e., if $m \in M$ then $Rm = Pm$ and consequently the class $\{M \mid M \in R\text{-mod} \text{ and } PM = M\} = \{R/P\}^+$ is closed submodules.

(iv) $\Longrightarrow$ (i). Let $(\mathcal{M}, \mathcal{L})$ be a non-trivial $\otimes$-orthogonal theory for R-mod. According to Proposition 3, there is the uniquely determined subset $\mathcal{A} \subseteq \mathcal{S}$ such that $\mathcal{M} \subseteq \mathcal{A}^+$ and $\mathcal{L} \subseteq \{\mathcal{S} \setminus \mathcal{A}\}^+$. The hypothesis implies that $\emptyset \neq \mathcal{A} \neq \mathcal{S}$ and both $\mathcal{A}^+$ and $\{\mathcal{S} \setminus \mathcal{A}\}^+$ are hereditary classes. Hence if $M \in \mathcal{M}$, $L \in \mathcal{L}$ and $M' \subseteq M$, $L' \subseteq L$ are submodules then $M' \otimes L' = P(M' \otimes L')$, for every maximal ideal P of R (use Proposition 2), and consequently the hypothesis again yields $M' \otimes L' = 0$.

<u>Corollary 5</u>. If R is a semiregular ring then $\otimes$-orthogonal theories for R-mod are in one-one and onto correspondence with the subsets of $\mathcal{S}$.

<u>Proof</u>. Let $\mathcal{A} \subseteq \mathcal{S}$. Put $\mathcal{M} = \mathcal{A}^+$ and $\mathcal{L} = \{\mathcal{S} \setminus \mathcal{A}\}^+$. Since R is semiregular, $(\mathcal{M}, \mathcal{L})$ is a $\otimes$-orthogonal theory. The converse follows directly from the proof of Theorem 4 (iv) $\Longrightarrow$ (i).

<u>Corollary 6</u>. If R is a semiartinian ring then every torsion theory for R-mod is hereditary of simple and cosimple type. Conversely, if every torsion theory for R-mod

is hereditary, then R is semiregular.

<u>Proof</u>. Let $(\mathcal{T}, \mathcal{F})$ be a non-trivial torsion theory. If R is semiartinian then $\mathcal{S} \cap \mathcal{F} = \mathcal{A} \neq \emptyset$. By [8], every commutative semiartinian ring is semiregular and consequently Theorem 4 implies $\mathcal{A} \neq \mathcal{S}$. Obviously, $\mathcal{T} \subseteq \mathcal{A}^+$. Suppose that $M \in \mathcal{A}^+$ and $M \notin \mathcal{T}$. Without loss of generality we can assume that $M \in \mathcal{F}$. By Theorem 4, $\mathcal{A}^+$ is hereditary and since R is semiartinian, there is a simple submodule $S \subseteq M$ such that $S \in \mathcal{A}^+ \cap \mathcal{F}$, i.e., S is isomorphic to a module from $\mathcal{A}$, a contradiction. Hence $\mathcal{T} = \mathcal{A}^+$. The fact that $\mathcal{T} = \{\mathcal{S} \setminus \mathcal{A}\}^{*+}$ follows from [2], Theorem 1, 331. Conversely, if every torsion theory is hereditary then $\mathcal{S}^+ = \{0\}$, i.e., R is semiregular.

<u>Corollary 7</u>. The following are equivalent for any ring R :

(i) Every torsion theory for R-mod is trivial.

(ii) Every $\otimes$ -orthogonal theory for R-mod is trivial.

(iii) R is a perfect, local ring.

<u>Proof</u>. (i) $\Longrightarrow$ (ii). It is obvious.

(ii) $\Longrightarrow$ (iii). The hypothesis implies that all the simple R-modules must be isomorphic and consequently, R is a perfect, local ring by Theorem 4.

(iii) $\Longrightarrow$ (i). By [1], 287.

<u>Corollary 8</u>. If $\Delta(R) = 0$ then R is weakly semiregular. Conversely, if R is semiregular then $\Delta(R) = 0$.

<u>Proof</u>. The converse immediately follows from Theorem 4, since any vanishing module has no proper maximal submo-

dule. Now, suppose that $\Delta(R) = 0$ and R is not weakly semiregular. By [9], 356 every prime ideal of R is maximal, i.e., $R/J(R)$ is a regular ring. Since $J(R)$ is not weakly T-nilpotent there is a natural number $k \geq 2$ and a sequence $a_1, a_2, \dots$ of elements from $J(R)$ such that $a_i^k = 0$, $i = 1, 2, \dots$ and $a_1 \dots a_n \neq 0$, for each $n \geq 1$. Consider the free R-module F with countable set of free generators $x_1, x_2, \dots$ and let $N \subseteq F$ be the submodule generated by the elements $(x_i - a_i x_{i+1})$, $i = 1, 2, \dots$. Since the sequence $a_1, a_2, \dots$ is not T-nilpotent, $N \neq F$. As it is easy to see, the order bound k yields $\overset{k}{\otimes} F/N = 0$, a contradiction.

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