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FIXED POINT THEOREMS FOR COMPACT AND NONEXPANSIVE MAPPINGS ON STARSHAPED DOMAINS

(Preliminary communication)

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Abstract: The wellknown fixed point theorems of Brouwer/Schauder/Tychonoff/Klee/Landsberg/Sadovsky for compact and condensing mappings respectively in admissible topological linear Hausdorff spaces and of Browder/Göhde/Kirk for nonexpansive mappings in Banach spaces are examined with regard to their correctness for merely starshaped domains.

Key words: Plane continua, starshaped sets, compact/condensing mappings, nonexpansive mappings, admissible spaces, fixed points.

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In this preprint we shall give some results on fixed points for compact/condensing and nonexpansive mappings defined on starshaped (but not necessarily convex) sets in topological linear Hausdorff spaces (t.l.h.s.). Detailed expositions including further results and the proofs of the theorems cited in this note will be published in the journals: Mathematica Balkanica; Publ. de l'Inst.Math.Beograd; Berichte der Gesellsch.für Math.und Datenverarbeitung Bonn.

Definition 1. Let E be a $\mathbb{R}$ -linear space; $X \subset E$ is said to be starshaped iff there exists $x_0 \in X$ such that $tx + (1-t)x_0 \in X$ for $x \in X$ and $t \in [0,1]$. By the wellknown so called Brouwer fixed point theorem a nonvoid compact convex subset of a finite-dimensional t.l.h.s. has the fixed point property for continuous mappings. This is, however, not true for the class of nonvoid compact starshaped subsets lying in finite-dimensional t.l.h.s. In $\mathbb{R}^3$ e.g. a counterexample is given by taking a suitable cone $(S \cup S^1)$ over the closure of a spiral $S \subset \mathbb{R}^2$ surrounding the unit sphere $S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$ infinitely many times, see [2]. For $dim(E) \in \{1,2\}$ we have, however,

Theorem 1. Let $m \in \{1,2\}$ and $\emptyset \neq X \subset \mathbb{R}^m$ be compact and starshaped. Then X has the fixed point property for continuous mappings.

The case $m = 1$ is obvious. The proof of Theorem 1 for $m = 2$ depends heavily on a deep fixed point theorem for plane continua due to Bell and Siekluoki [13].

Definition 2. Let E be a $\mathbb{R}$-t.l.h.s.; $\emptyset \neq X \subset E$ is said to be shrinkable iff $[0,1)\overline{X} \subset \mathring{X}$.

A shrinkable set is a starshaped but not necessarily convex neighborhood of 0 (obvious).

Theorem 2. Let $(E, \| \|)$ be a Banach space, $X \subset E$ be open and shrinkable. Let $f : \overline{X} \longrightarrow E$ be condensing [9], [10], [11] such that $f[\overline{X}]$ is bounded and $x \in \partial X$, $\lambda \in \mathbb{R}$ and $f(x) = \lambda x$ implies $\lambda \leq 1$. Then f has a fixed point.

The proof of Theorem 2 uses a standard fixed point theorem for condensing mappings [9],[10] and the fact X being a retract of E [7].

Theorem 3. Let E be an admissible t.l.h.s. ([7]) and $X \subset E$ be a closed starshaped neighborhood retract. Let $f : X \to X$ be compact. Then f has a fixed point. For the proof of Theorem 3 we may assume $0 \in X$ without loss of generality and then use the fact that X is a neighborhood retract with respect to a shrinkable set [7].

Theorem 4. Let $(E, \| \|)$ be a (F)-normed linear space (in the sense of Banach) such that $\|tx\| < \|x\|$ for $x \neq 0$ and $t \in [0,1)$. Let $\emptyset \neq X \subset E$ be closed and starshaped and $f : X \to X$ be compact and nonexpansive. Then f has a fixed point.
The proof of Theorem 4 uses a current approximation method and a wellknown fixed point theorem due to Edelstein [5].

Theorem 5 (Rothe-type). Let $(E, (\,,\,))$ be a Hilbert space and let $\emptyset \neq X \subset E$ be closed bounded and starshaped. Let $f : X \to E$ be nonexpansive such that $f[\partial X] \subset X$. Then f has a fixed point.
Theorem 5 generalizes a theorem due to Browder and one of the authors. The proof can be done with the aid of a fixed point theorem due to Assad [1] and an approximation method for Hilbert spaces given by Crandall and Pazy in the context of nonlinear contraction semi-group problems [4]. Theorem 5 extends - in some sense - to the spaces l_p $(1 < p < \infty)$ and to uniformly-convex Banach spaces having a weakly continuous duality mapping [3].

__Theorem 6__ (Antipodal-type). __Let__ $(E,(\,,\,))$ be a Hilbert space and let $X \subset E$ be an open bounded starshaped symmetric neighborhood of 0. Let $f: \overline{X} \longrightarrow E$ be nonexpansive such that $f(-x) = -f(x)$ for $x \in \partial X$. __Then__ f has a fixed point.

The proof of Theorem 6 uses a fixed point theorem for k-set-contractions due to Petryshyn and Fitzpatrick [8] and the method of Crandall and Pazy mentioned above. There are some variants of Theorem 6 for ℓ_p $(1 < p < \infty)$ and - using an approximation method due to Göhde/Reinermann [6],[9] - the same is true for an arbitrary uniformly-convex Banach space if X is assumed to be convex.

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