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TRANSFORMATIONS DETERMINING UNIQUELY A MONOID II

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Abstract: This paper is giving necessary and sufficient conditions for non-periodical translation to be determining translation, determining translation being a member of exactly one Caleye's representation of some algebraic monoid. Together with the paper [1] it shows the form of all translations determining in this sense. In order to prove the necessity of conditions it contains a number of constructions of different monoids.

Key words: Algebraic monoid, Caleye's representation, left translation, right translation.

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When dealing with the problem of obtaining an economical description of a monoid we are faced with several separate problems. One of them is to reconstruct a monoid when we know the results of multiplication of this monoid by only one element. In other words, we know one of the left translations of a monoid. In some cases such an information is sufficient to uniquely reconstruct the whole monoid.

Our aim is to describe all transformations which are determining in the sense as given above. The finite transformations having been described in the paper [1] and the theorems given in [1] for finite transformations hold also for infinite periodical ones. The proofs and constructions in no way differ from those given in [1] and are not being repeated in this paper. Thus is here described the form for all non periodical transformations. To this purpose we are constantly using the basic results established in [2], [3] and [4].

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Given an <u>algebraic monoid</u> $M = (X, e, \cdot)$ - a set X together with an associative multiplication possessing an identity element e, we can associate with every element $a \in X$ its <u>left translation</u>

(1) f_a defined by $f_a(x) = a \cdot x$ for all $x \in X$.

All the left translations of M form $L(M)$. In accord with the paper [1] we shall call an element a such that f_a determines a monoid M (its operation $\cdot$ and identity element e), a <u>left determining element</u> (or shortly, a <u>determining element</u>) and its left translation a <u>determining left translation</u> (or a <u>determining translation</u>).

If we start from M^{op} we get the system of all the right translations

(2) $\quad R(M) = \{\varphi_x \mid x \in X\}, \quad \varphi_x(y) = y \cdot x$.

Our aim is to describe all the non periodical transformations which are determining left translations of some monoid M .

A T-monoid will be a couple (X, S) such that X is a set and $S \subset X^X$ is satisfying the following conditions:

(1) identity transformation $1_X \in S$;

(2) for all $f, g \in S$, it is $f \circ g \in S$.

A centralizer of (X, S) is a T-monoid $(X, \mathscr{C}(S))$, where

$$\mathscr{C}(S) = \{ g \in X^X \mid f \circ g = g \circ f \quad \text{for all } f \in S \} .$$

A point e is a source (exact source) of (X, S) if for every $x \in X$ there exists (unique) $f \in S$ with $f(e) = x$. A T-monoid (X, F) is called a regular T-monoid, if there exists an algebraic monoid M with $F = L(M)$. There is a 1-1 correspondence between regular T-monoids with marked element e and algebraic monoids.

The necessary and sufficient conditions for a T-monoid to be regular are in the following statement:

Statement 1: The following assertions are equivalent:

(A) (X, S) is a regular T-monoid;

(B) (X, S) is a T-monoid with an exact source;

(C) (X, S) and $(X, \mathscr{C}(S))$ have a common source.

The proof of this statement is given in [2].

A transformation which can be a member of some regular T-monoid is called a translation.

We bring some notions necessary for a description of a transformation f , $f : X \longrightarrow X$. The kernel Q_f of f is

(3) $$Q_f = \bigcup \{ A \mid A \subset X \ \& \ f(A) = A \} .$$

We shall call f a transformation with an increasing kernel, if $f \mid Q_f$ is not injective, otherwise it is a transformation with a bijective kernel ($f \mid Q_f$ is a transformation $g : Q_f \longrightarrow Q_f$ such that $g(x) = f(x)$). For a given $x \in X$ the set $P_f(x) = \{ f^m(x) \mid m > 0 \}$ is the path of x . The elements $x, y \in X$ are E_f -equivalent if $f^m(x) = f^n(y)$ for some $m, n \geq 0$. E_f is an equivalence, its classes being components of f. So we have connected transformations (with one component) and disconnected ones (with more than one component). An element $x \in X$ is a cyclic element of f if $x \in P_f(f(x))$. The set of all cylic elements of f forms the cycle of f, Z_f . The kernel of a component $E_f(x)$ is $Q_f(x) = Q_f \cap E_f(x)$, the cycle of a component $E_f(x)$ is $Z_f(x) = Z_f \cap E_f(x)$. If $x \in Z_f$, then the order of an element x is the cardinality $\varkappa(x)$ of the set $Z_f(x)$ ($\varkappa(x) = |Z_f(x)|$). Let x be an element such that $Q_f(x) \neq 0$, then the height $u(x)$ of x is defined as the smallest integer with

$$f^{u(x)}(x) \in Q_f(x) \quad . \tag{4}$$

An element e is a top element of f , if e is a source of $\mathcal{C}(f)$.

Statement 2: Let $f : X \longrightarrow X$ be a non surjective transformation, then f is a translation if and only if there exists a top element e of f and either $Q_f(e) = 0$ or f has a bijective kernel or $|f^{-1}(f^{u(e)+m}(e)) \cap Q_f| \geq 2$ for all $m > 0$.

Let $f : X \longrightarrow X$ be a surjective transformation, then

f is a translation if and only if there exists a top element e of f and either f is a permutation or there exists one to one transformation g, $g \in \mathscr{C}(f)$, with $g(e) = f(e)$, $g^{-1}(e) = 0$.

The proof is in [2],[3],[4].

Further we shall deal only with transformations which are non periodical translations, i.e. $Z_f(e) = 0$.

The main component of f, $E_f(e)$, will be the component containing a specified top element e, $X \setminus E_f(e)$ will be designated by Y.

Let us designate some subsets of X, which will be of use later: let $Q_f(e) = 0$, then $K = E_f(e)$; let $Q_f(e) \neq 0$, then

$$K = \{x \in E_f(e) \mid f^{\mu(x)}(x) \in P_f(e)\} . \tag{5}$$

$$T_{m,n} = \{x \in E_f(e) \setminus K \mid f^{n-1}(x) \notin P_f(e) \;\&\; f^{n}(x) = f^{\mu(e)+m}(e)\} , \quad m \geq 0,\ n > 0 . \tag{6}$$

Let us define a mapping $d : K \longrightarrow N$; if $Q_f(e) \neq 0$, then $d(x) = m - \mu(x)$, where m is the least integer with $f^{m}(e) = f^{\mu(x)}(x)$, if $Q_f(e) = 0$, then $d(x) = m - n$, where $f^{m}(e) = f^{n}(x)$ (it is quite obvious that d is properly defined). Such d is called the difference with regard to e.

It is easy to see that the following lemmas hold:

Lemma 1. Let $x, y \in K$; if $f^{d(x)}(y) \in P_f(e)$, then $f^{d(x)}(y) = f^{d(x)+d(y)}(e)$.

Lemma 2. Let $t \in T_{p,q}$, r an integer; then $f^{r}(t) \in T_{p,q-r}$ for $q > r$, $f^{r}(t) \in X$ for $q \leq r$ and $d(f^{r}(t)) = u(e) + p + r - q$.

We shall define a mapping h, $h : Q_f \longrightarrow Q_f$, with $h | Z_f = (f \mid Z_f)^{-1}$, $f(h(t)) = t$ for all $t \in Q_f$ and if f has an increasing kernel, then $Im\, h \cap P_f(e) = 0$. The existence of such a mapping is given in [2], [3], [4].

Following lemmas hold:

Lemma 3. Let $Q_f(e) \neq 0$, $t \in X$; then $h^{n} f^{u(e)+m}(t) \in T_{m+d(t),n}$.

Lemma 4. Let $t \in T_{p,q}$, $n \neq 0$, then for $q > u(e) + m$ it is $h^{n} f^{u(e)+m}(t) \in T_{p,q+n-u(e)-m}$; for $q \leq u(e) + m$ it is $h^{n} f^{u(e)+m}(t) \in T_{p+u(e)+m-q,n}$.

Let $x \in X \setminus Q_f$, let for k, k is an integer, be $f^{-k}(x) \neq 0$ and $f^{-(k+1)}(x) = 0$, then we shall call the integer k the grade $st(x)$ of x. (This notion is not defined for all $x \in X \setminus Q_f$.) Designate

(7) $\quad A = \{x \in X \mid st(x) = 0\} \setminus \{e\}$,

(8) $\quad L_x = \bigcup_{m=0}^{\infty} f^{-m}(x)$.

Lemma 5. Let $x, y \in X$, $st(x) = r$, $st(y) = s$; then there exists a mapping φ, $\varphi : L_x \longrightarrow L_y$ with $\varphi(x) = y$

(9) $\quad \varphi(f(t)) = f(\varphi(t))$ for all $t \in L_x \setminus \{x\}$

if and only if $r \leq s$.

We shall designate by $|\mathcal{B}|$ the cardinality of the set $\mathcal{B}$.

Now we give a number of constructions of regular $\mathcal{T}$-monoids which will be of use in the next.

<u>Construction 1</u>. Let f be a non surjective translation or a bijection, then for every top element e, e satisfies the condition from Statement 2, there exists a regular $\mathcal{T}$-monoid $(X, L(M))$ with $f \in L(M)$; $L(M) = \{f_x ; x \in X\}$, where

(10) $$f_x(e) = x$$

(11) for $x \in K$ it is $f_x(t) = f^{d(x)}(t)$,

(12) for $x \in T_{m,n}$ it is $f_x(t) = h^n f^{u(e)+m}(t)$,

for $x \in Y$ it is $f_x(t) = p(x)$;

(13) where $p: Y \longrightarrow Y$ is a transformation such that

$p^2 = p$,

$f(p(t)) = p(f(t))$ for $t \in Y$, $h(p(t)) = p(h(t))$ for $t \in Y \cap Q_f$.

Demonstration: Let us define $R(M) = \{g_y \mid y \in X\}$, $g_y(t) = f_t(y)$. e is a common source of both $L(M)$ and $R(M)$. It is sufficient to show that $f_x \circ g_y = g_y \circ f_x$ for all $x, y \in X$.

1. Let $x \in K$, $y \in X$, then if $t \in K$, it is $f_x \circ g_y(t) = f^{d(x)+d(t)}(y)$, $g_y \circ f_x(t) = g_y(f^{d(x)}(t)) = f^{d(x)+d(y)}(y)$ by Lemma 1. If $t \in T_{p,q}$, then $f_x \circ g_y(t) = f_x(h^q f^{u(e)+p}(y)) = f^{d(x)} h^q f^{u(e)+p}(y)$. It is $h^{q-d(x)} f^{u(e)+p}(y)$ for $q > d(x)$, $f^{d(x)+u(e)+p-q}(y)$ for $d(x) \leq q$. $g_y \circ f_x(t) = g_y(f^{d(x)}(t))$, by Lemma 2 it is $h^{q-d(x)} f^{u(e)+p}(y)$

for $q > d(x)$ and $f^{u(e)+p+d(x)-q}(y)$ for $q \leq d(x)$. If $t \in Y$, then $f_x \circ g_y(t) = f^{d(x)}(p(t)) = p(f^{d(x)}(t))$ and $g_y \circ f_x(t) = g_y(f^{d(x)}(t)) = p(f^{d(x)}(t))$ because $f^{d(x)}(t) \in Y$.

2. Let $x \in T_{m,n}$, $y \in X$, then if $t \in K$, then $f_x \circ g_y(t) = f_x(f^{d(t)}(y)) = h^n f^{u(e)+m+d(t)}(y)$ and $g_y \circ f_x(t) = g_y(h^n f^{u(e)+m}(t))$. Using Lemma 3 we get $g_y \circ f_x(t) = h^n f^{u(e)+m+d(t)}(y)$. If $t \in T_{p,q}$, then $f_x \circ g_y(t) = f_x(h^q f^{u(e)+p}(y)) = h^n f^{u(e)+m} h^q f^{u(e)+p}(y)$ and it is $h^{n+q-u(e)-m} f^{u(e)+p}(y)$ for $q > u(e)+m$ and $h^n f^{2u(e)+m+p-q}(y)$ for $q \leq u(e)+m$. $g_y \circ f_x(t) = g_y(h^n f^{u(e)+m}(t))$, using Lemma 4 we get for $q > u(e)+m$ $h^{q+n-u(e)-m}(f^{u(e)+p}(y))$ and $h^n f^{2u(e)+p+m-q}(y)$ for $q \leq u(e)+m$. If $t \in Y$, then $f_x \circ g_y(t) = f_x(p(t)) = h^n f^{u(e)+m}(p(t))$ and $g_y \circ f_x(t) = p(h^n f^{u(e)+m}(t)) = h^n f^{u(e)+m}(p(t))$, because $h^n f^{u(e)+m}(t) \in Y$.

3. The case $x \in Y$, $y \in X$ is evident because of $f_x(t) = p(x)$ and Condition (13).

Hence Construction 1 has been confirmed.

It is evident that p can be chosen the identity transformation 1_Y. Now we shall show several transformations p, $p: Y \longrightarrow Y$ with Property (13), $p \neq 1_Y$.

Lemma 6. Let $f|Y$ be a non-injective transformation, then there exists $p: Y \longrightarrow Y$, $p \neq 1_Y$, satisfying Condition (13).

Proof: If $f|Y$ is not an injective transformation, then there exist $x, y \in Y$, $x \neq y$, with $f(x) = f(y)$ and one of the following possibilities holds:

a) in the case $st(x)$ and $st(y)$ are defined and $st(x) \geq$ $\geq st(y)$ choose $p : p | L_y = \varphi$ from Lemma 5 and $p(x) = x$ otherwise;

b) for $x \in Z_f$ put $p(x) = f^{kn(x)-m}(x), kn(x) - m \geq 0$, for $f^m(x) = x$, $p(x) = x$ otherwise;

c) for $Z_f(x) = 0$ and $x \in Q_f$ choose an infinite sequence $\{x_i\}_{i=0}$ with $x_0 = f(x)$, $x_k \in f^{-k}(x_0) \cap Q_f$ and $f(x_k) = x_{k-1}$ and put $p(x) = x_k$ for $x \in L_y$, $f^{k-1}(x) = y$, $p(x) = x$ otherwise.

Lemma 7. Let $f|Y$ be a disconnected injective transformation, each component of which has a non empty kernel. If one of the components of $f|Y$ is infinite or if for some $x_1, x_2 \in Y$ $n(x_1)$ divides $n(x_2)$ and $x_1 \notin E_f(x_2)$, then there exists $p : Y \longrightarrow Y$, $p \neq 1_Y$, with Property (13).

Construction 2. Let f be a translation as in Construction 1 with $f(x) \in P_f(e)$ for every $x \in K$. Suppose that one of the following conditions holds:

(1) there exist $x_1 \in (K \setminus P_f(e)) \cup \{f(e)\}$, $x_2, x_3 \in$ $\in K \setminus P_f(e)$ with $f(x_3) = f^{d(x_1)+1}(x_2)$, $x_3 \neq f^{d(x_1)}(x_2)$.

(2) There exist $x_1 \in K \setminus P_f(e)$, $x_2 \in E_f(e) \setminus (Q_f \cup K)$

(14) with $f^{-1}(f^{d(x_1)+1}(x_2)) \setminus Q_f \neq 0$;

i.e. there exists x_3 such that $x_3 \in X \setminus (Q_f \cup K)$ with $f(x_3) = f^{d(x_1)+1}(x_2)$ or $f^{u(e)}(e) = f^{d(x_1)+1}(x_2)$.

Then $f \in L(M')$, where $L(M') = \{f'_x \mid x \in X\}$;

(15) $f'_x(t) = f_x(t)$ for all $x \neq x_2$ or $t \neq x_1$,

(16) $\quad f'_{x_2}(x_1) = x_3$.

Here the transformations f_x are those used in Construction 1. Using Lemma 1, Lemma 2 and Conditions (1), (2) it can be verified that $L(M')$ and $R(M')$ ($R(M') = \{g'_y \mid g'_y(t) = f'_t(y)\}$) are the systems of all left and all right translations of some algebraic monoid, i.e. pointwise commute and have a common source.

Construction 3. Let f be a translation with an increasing kernel. Let there exist $x_0 \in X \setminus P_f(e)$ with

(17) $$f^2(x_0) \in P_f(e) \cap Q_f \quad .$$

Then $f \in L(M')$ for $L(M') = \{f'_x \mid x \in X\}$, where

(18) $\quad f'_x(t) = f_x(t) \quad$ for $x \neq x_0$ or $t = e$,

(19) $\quad f'_{x_0}(t) = hf^{d(x_0)+1}(t)$ for $t \neq e$.

Translations f_x are taken from Construction 1.

If there exists $x_0 \in X \setminus P_f(e)$ with $d(x_0) = u(e) - 2$, we modify Construction 3 setting

(20) $$f'_{x_0}(y) = f_{x_0}(y)$$

for $y \in X \setminus \{e\}$ with $u(y) = u(e)$.

The demonstration of Construction 3 and its modification can be easily done in the same way as in Construction 1 and using Properties (17) and (20) of f .

Construction 4. Let f be a translation with an increasing kernel. Let there exist $x_1 \in A \cup \{f(e)\}$, $x_2 \in T_{p,q} \setminus Q_f$ and $x_3, x_3 \in f^{-1}(h^{q-1}f^{u(e)+p}(x_1)) \setminus Q_f$ with

$st(x_2) \subseteq st(x_3)$. By Lemma 5 there exists a transformation φ, $\varphi : L_{x_2} \longrightarrow L_{x_3}$ with (9). Then $f \in L(M')$, where $L(M') = \{ f'_x \mid x \in X \}$

(21) $f'_x(t) = f_x(t)$ for $x \notin L_{x_2}$ or $t \neq x_1$,

(22) $f'_x(x_1) = \varphi(x)$ for $x \in L_{x_2}$.

Here again we use the translations f_x from Construction 1.

Using Lemma 3, 4 and the properties of the transformation φ it can be demonstrated that M' is an algebraic monoid.

Construction 5. Let f be a translation with $|Y| > 1$, $f|Y$ be a connected bijection. Then $f \in L(M')$, $L(M') = \{ f'_x \mid x \in X \}$

$f'_x(t) = f_x(t)$ for $x \in E_f(e)$ or $x \in Y$ and $t \in E_f(e)$,

(23) $f'_x(t) = f^{d(x,e')}(t)$ for $t \in P_f(e')$, $x \in Y$,

(24) $f'_x(t) = \bar{h}^{d(x,e')}(t)$ for $x \in Y$ and $t \in Y \setminus P_f(e')$;

where e' is a fixed element of Y, $\bar{h} = h|Y$, $d(x,e')$ is a difference of x with regard to e', the translations f_x are taken from Construction 1.

Demonstration: e is evidently a common source of both $L(M')$ and $R(M')$ ($R(M')$ is defined as obvious). The only fact we must verify is that $R(M') = \mathcal{C}(L(M'))$. It is evident that $f'_x \circ g'_y = g'_y \circ f'_x$ for $f'_x = f_x$ and $g'_y = g_y$. Because of the form of f'_x, f_x, g'_y and g_y we have to verify only the case $x \in Y$, $y \in Y$ and $t \in T_{m,n}$: $f'_x \circ g'_y(t) = f'_x(f^{d(y,e')}(t)) = h^{n-d(y,e')} \circ f^{u(e)+m}(x)$ for $d(y,e') \leq n$ and $f^{u(e)+m-n+d(y,e')}(x)$ for $n < d(y,e')$. $g'_y \circ f'_x(t) = g'_y(h^n f^{u(e)+m}(x)) = f^{d(y,e')} h^n \circ f^{u(e)+m}(x)$ for $n < d(y,e')$ and $f^{u(e)+m+d(y,e')-n}(x)$ for $n \geq d(y,e')$ (see Lemmas 1 - 4).

Construction 6. Let f be a disconnected translation with $f|Y$ being a disconnected permutation formed only by cycles of finite order. Let there exist a common divisor q, $q \neq 1$, of all $\varkappa(x)$, $x \in Y$ such that for all x there is an integer $\bar{p}(x)$ relatively prime to $\frac{\varkappa(x)}{q}$ and $\frac{\varkappa(x)\bar{p}(x_2) - q}{q}$ is an integer. Then f is a left translation of the following monoid M', $L(M') = \{f'_x \mid x \in X\}$, where

$$f'_x = f_x \quad \text{for} \quad x \in E_f(e),$$

let $x \in Z(a_j)$, then $f'_x(t) = f^{\frac{d(t,e)\varkappa_j p_j}{q}}(x)$ for $t \in E_f(e)$

$$f'_x(t) = f^{(u(e)+m-n)\frac{\varkappa_j p_j}{q}}(x) \quad \text{for} \quad t \in T_{m,n},$$

$$f'_x(t) = f^{d(t,a_i)\frac{\varkappa_j p_j}{q}}(x) \quad \text{for} \quad t \in Z(a_i),$$

where $\{a_i\}_i$ is a system of elements of Y such that for every $y \in Y$ there is a_i with $y \in Z(a_i)$ and $Z(a_i) \cap Z(a_j) = 0$ for $i \neq j$; $\varkappa_i = \varkappa(a_i)$, $\bar{p}(a_i) = p_i$; f_x are the translations from Construction 1.

Demonstration: The fact that for every $x, y \in X$ $f'_x(g'_y(t)) = g'_y(f'_x(t))$ ($g'_y(t) = f'_t(y)$) will be shown only for $x \in Y$, $y \in X$, $t \in Y$. The rest is an easy computation.

Let $x \in Z(a_i)$, $y \in E_f(e)$, $t \in Z(a_j)$, then $f'_x(g'_y(t)) =$
$$= f'_x\left(f^{d(y,e)\frac{r_j n_j}{q}}(t)\right) = f^{d(x,a_j)\frac{r_i n_i}{q}}(x),$$

where $d(x,a_j) = d(t,a_j) + d(y,e)\dfrac{r_j n_j}{q}$. $\dfrac{r_j n_j}{q} = kq + 1$

(k is an integer) and thus $d(x,a_j) = d(t,a_j) + d(y,e) +$
$+ kq\,d(y,e)$, therefore $f'_x(g'_y(t)) = f^{d(t,a_j)\frac{r_i n_i}{q}}($
$$\left(f^{d(y,e)\frac{r_i n_i}{q}}(x)\right).\ g'_y(f'_x(t)) = g'_y\left(f^{d(t,a_j)\frac{r_i n_i}{q}}(x)\right) =$$
$$= f^{d(y,e)\frac{r_i n_i}{q}}\left(f^{d(t,a_j)\frac{r_i n_i}{q}}(x)\right).$$

For $y \in T_{m,n}$ and $y \in Z(a_r)$ it can be shown in the same way using the condition $\dfrac{n_i r_i - q}{q^2} = k$ for all i.

The last part contains several theorems which give us the answer as to the form of translations which are determining translations for some monoid.

Theorem 1. Let $f : X \longrightarrow X$ be a connected non-periodical translation with a bijective kernel; f is a determining translation if and only if the following conditions are fulfilled:

(i) there exists exactly one top element e in X ;

(ii) $f(x) \in P_f(e) \cup Q_f$ for all $x \in X$;

(iii) for all $x \in (X \setminus P_f(e)) \cup \{f(e)\}$ and for all

(23) $y \in X \setminus (Q_f \cup P_f(e))$ it is $|f^{-1}(f^{d(x)+1}(y))| = 1$;

for all $x \in T_{0,n} \setminus Q_f$ and $y \in X \setminus (P_f(e) \cup Q_f)$ it is

(24) $$|f^{-1}(h^{n-1} f^{u(e)}(y))| = 1 .$$

Proof: At first we show the necessity of Conditions (i) - (iii). The condition (i) is evident; if we choose two different identity elements, we get two different monoids.

In [2] there is described a commutative monoid containing a connected translation f. It is evident that if there exists $x_0 \in X$ with $f(x_0) \notin P_f(e) \cup Q_f$, the monoid from Construction 1 is not a commutative one.

Let the condition (iii) not be fulfilled, then there exists a monoid M' with $L(M')$ from Construction 2, which is different from the monoid M described in Construction 1.

Now, we are to show that these conditions are also sufficient. From the condition (i) only e may be a source of $L(M')$ and $R(M')$ of any monoid M' with $f \in L(M')$. The first step will be to show that all translations of $R(M')$ are determined on $P_f(e) \cup Q_f$. Let $R(M') = \{g'_y \mid y \in X\}$. Let $t \in P_f(e)$, then $g'_y(t) = g'_y(f^{d(t)}(e)) = f^{d(t)}(g'_y(e)) = f^{d(t)}(y)$. Let $t \in Q_f$, $t \in T_{0,n}$, i.e. $f^n(t) = f^{u(e)}(e)$, then $f^n(g'_y(t)) = g'_y(f^n(t)) = g'_y(f^{u(e)}(e)) = f^{u(e)}(y)$, hence $g'_y(t) \in f^{-n}(f^{u(e)}(y))$. Because of the commutativity of g'_y and f it is $g'_y(Q_f) \subset Q_f$ (the proof of this assertion can be found in [6]). So $g'_y(t) \in Q_f$, but $f \mid Q_f$ is an injective transformation and $g'_y(t) = h^n f^{u(e)}(y)$.

Further it can be easily shown that $g'_{y_0}(t) = f(t)$, where

(25) $$y_0 = f(e) \quad .$$

Using the fact verified above we can easily see that $f'_x(t) = f^{d(t)}(x)$ for $t \in K$, $f'_x(t) = h^n f^{u(e)}(x)$ for $t \in T_{0,m}$. Using Lemmas 1, 2, 3, 4 and Condition (ii), we get $f'_x(t) = f^{d(x)}(t)$ and $f'_x(t) = h^m f^{u(e)}(t)$, $x \in T_{0,m}$, if one of the elements x, t is the element of $P_f(e) \cup Q_f$.

To see that f'_x, g'_y must be equal to translations from Construction 1 it is sufficient to show the form of g'_y for $y \in X \setminus (Q_f \cup P_f(e))$, $t \in X \setminus (Q_f \cup P_f(e))$. The rest is an easy consequence of the equality $g'_y(x) = f'_x(y)$ and the condition (iii).

Thus Theorem 1 has been proved.

Theorem 2. Let $f : X \longrightarrow X$ be a connected surjective translation, then f is determining if and only if $|X| = 1$.

Proof: Bijective translation on X with $|X| > 1$ has more than one top element. An increasing transformation is a translation if and only if there exist $e \in X$ and $g : X \longrightarrow X$, e is a top element of f and $g \in \mathcal{C}(f)$, with $g(e) = f(e)$ and $g^{-1}(e) = 0$, but they are never unique.

Theorem 3. Let f be a connected translation with an increasing kernel Q_f, $X \setminus Q_f \neq 0$. Then f is a determining translation if and only if the following holds:

(i) there exists exactly one element e, with $u(e) \geq u(x)$ for all $x \in X$ and such that for all $k > 0$

$$f^{-1}(f^{u(e)+k}(e)) \cap (Q_f \setminus P_f(e)) \neq 0 .$$

(ii) The kernel Q_f is isomorphic to the connected translation of the bicyclic semigroup.

(iii) For all $x \in K$ it is $f(x) \in P_f(e)$.

(iv) For all $x \in K \setminus P_f(e)$ it is $d(x) < u(e) - 2$.

(v) For all $x \in K \setminus P_f(e)$ and $t \in (K \setminus P_f(e)) \cup \{f(e)\}$ it is $|f^{-1}(f^{d(t)+1}(x)) \cap K| = 1$; for all $x \in A \setminus K$ and $t \in K \setminus P_f(e)$ it is $|f^{-1}(f^{d(t)+1}(x)) \setminus (P_f(e) \cap Q_f)| = 1$.

(vi) For all $x \in A \cup \{f(e)\}$, $y \in T_{r,q} \setminus Q_f$ if $z \in f^{-1}(h^{q-1}f^{u(e)+r}(x)) \setminus Q_f$, then $st(z) < st(y)$.

Proof: The first step of the proof is to show that the conditions above are necessary. Condition (ii) involves existence of exactly one $h : Q_f \to Q_f \setminus P_f(e)$ with $f(h(t)) = t$ for all $t \in Q_f$. Necessity of the condition (iii) was settled by a construction of another algebraic monoid M with $f \in L(M)$ in the case $f(x) \notin P_f(e)$ for some $x \in K$ (see [4]).

Let there exist $x \in K \setminus P_f(e)$ with $f^2(x_0) \in P_f(e) \cap Q_f$. The monoid M' defined in Construction 3 is different from M from Construction 1.

Let us suppose the condition (v) is not fulfilled. We use Construction 2; if there exist $x \in K \setminus P_f(e)$ and $t \in (K \setminus P_f(e)) \cup \{f(e)\}$ with $|f^{-1}(f^{d(t)+1}(x)) \setminus (P_f(e) \cap Q_f)| > 1$, it means that there exist x_1, x_2, x_3 with the property (1) of Construction 2; if there exist $x \in A$ and $t \in K \setminus P_f(e)$ with $|f^{-1}(f^{d(t)+1}(x)) \setminus (P_f(e) \cap Q_f)| > 1$ there exist x_1, x_2, x_3 with the property (2) of Construction 2. In both cases Construction 2 gives us an algebraic monoid different of $\mathcal{M}$ from Construction 1.

Using Construction 4 in the case when (vi) is not fulfilled, we verify the necessity of the condition (vi) and the proof of necessity of Conditions (i) - (vi) has been finished.

Next let us show the sufficiency of the conditions stated. Our aim is to show that every algebraic monoid $\mathcal{M}'$ with $f \in L(\mathcal{M}')$ has the form specified by Construction 1.

Let us suppose we have some $\mathcal{M}'$ with $f \in L(\mathcal{M}')$; $L(\mathcal{M}') = \{f'_x \mid x \in X\}$ and $R(\mathcal{M}') = \{g'_y \mid y \in X\}$; it is evident that only e can be a source of both $L(\mathcal{M}')$ and $R(\mathcal{M}')$. Proceeding in the same way as in the proof of Theorem 1, we get $g'_y(t) = f^{d(t)}(y)$ for all $t \in K$. Because of the condition (ii) we can designate $x \in Q_f \cap T_{p,q}$ by $x_{p,q}$, because of $|Q_f \cap T_{p,q}| = 1$.

The following step is to show that for all $g \in R(\mathcal{M}')$

$$g(x_{p,q}) \notin P_f(e) . \tag{26}$$

Let first $g(x_{0,1}) = x_{0,0}$, then $g(x_{0,k}) = x_{0,k-1}$ for all $k \geq 1$ and $g(x_{i,0}) = x_{i+1,0}$ for all $i \geq 0$ (use commutativity of g and f). It is easy to show that for $\bar{f} \in L(\mathcal{M}')$ and $\bar{g} \in R(\mathcal{M}')$ with $\bar{f}(e) = \bar{g}(e) = x_{0,\mu(e)+1}$ it is $\bar{f}(x_{0,\mu(e)+1}) = \bar{g}(x_{0,\mu(e)+1}) = x_{0,\mu(e)+2}$. Using commutativity of $\bar{f}$ with $\bar{g}$ we get $\bar{f}(x_{0,0}) = x_{0,1}$ and

$\bar{f}(x_{1,0}) = x_{0,0}$, thus $f(\bar{f}(x_{0,1})) = \bar{f}(f(x_{0,0})) = x_{0,0}$. For every $t \in Q_f$ there exists a translation $\tilde{g}$ in $R(M')$ with $\tilde{g}(x_{0,0}) = t$, so $f(\bar{f}(t)) = \bar{f}(f(t)) = t$ for all $t \in Q_f$, i.e. $f \mid Q_f$ is injective and it is a contradiction.

It is not possible for g'_{y_0} $(y_0 = f(e))$ to find $m \geq 0$ with $g'_{y_0}(x_{m,1}) = x_{m,0}$ and $g'_{y_0}(x_{k,1}) = x_{k+1,1}$ for all $k = 0, 1, \dots, m-1$, because of $\bar{g}^{m}(g'_{y_0})^{m+1} \in R(M')$. Similarly if for some y , $g'_y(x_{p,q}) = x_{0,a}$, then $\bar{g}^{a} g'_y (g'_{y_0})^{p} (\bar{g})^{q-1}(x_{0,1}) = x_{0,0}$ for $\bar{g}^{a} g'_y (g'_{y_0})^{p} (\bar{g})^{q-1} \in R(M')$. It leads to a contradiction. So for all $g \in R(M')$ $g(x_{p,q}) \notin P_f(e)$.

We know already that $g'_y(t) = f^{d(t)}(y)$ for $t \in K$. Let $t = x_{p,q}$, then $g'_y(x_{p,q+1}) \in f^{-1}(h^{q} f^{u(e)+p}(y)) \cap (Q_f \setminus P_f(e))$, i.e. $g'_y(x_{p,q+1}) = h^{q+1} f^{u(e)+p}(y)$. (Use induction on q).

As in the proof of Theorem 1, we have established the form of g'_{y_0}, $y_0 = f(e)$. Now we must show the form of g'_{y_0} in $t \in X \setminus (Q_f \cup P_f(e))$. From the conditions (iii) and (iv) it follows $g'_{y_0}(t) = f^{d(t)}(y_0) = f(t)$ for all $t \in K$. Using the condition (vi) and induction on $u(t)$ we get $g'_{y_0}(t) = h^{q} f^{u(e)+p}(y_0)$ for $t \in T_{p,q} \setminus Q_f$. We shall show it only for t with $u(t) = 1$, the rest is obvious. $g'_{y_0}(f(t)) = f(g'_{y_0}(t)) = x_{p+1,q-1}$, because of $f(t) = x_{p,q-1}$ and $q-1 \geq 1$; i.e. $g'_{y_0}(t) \in f^{-1}(h^{q-1} f^{u(e)+p+1}(e))$. By the condition (vi) if $z \in f^{-1}(h^{q-1} f^{u(e)+p}(y_0))$, $z \notin Q_f$, then $st(z) < st(t)$ and there is no transformation φ , $\varphi : L_t \longrightarrow L_z$ with (9), by Lemma 5. So the choice $g'_{y_0}(t) = z$ is not possible.

Let us turn to the form of $f'_x \in L(M')$. Using commutativity with g'_y, we get $f'_{x_{p,q}}(y) = h^q f^{u(e)+p}(y)$; further take $y \notin A$, i.e. there exists $y' \in X$ with $f(y') = y$, then $f'_x(y) = f'_x(g'_{y'}(f(e))) = g'_{y'}(f'_x(g'_{y_0}(e))) = g'_{y'}(g'_{y_0}(x))$. For $x \in K$ it is $f^{d(x)}(y)$ and for $x \in T_{p,q}$ it is $h^q f^{u(e)+p+1}(y') = h^q f^{u(e)+p}(y)$.

To show the rest, the form of g'_y for $y \in A$, we use the conditions (iv), (v), (vi) and Lemma 5 and proceed as for the translation g'_{y_0}. So we have that every $L(M')$ containing f have the form (10) - (13) of Construction 1. Thus the proof has been finished.

The following theorem deals with disconnected translations. It holds also for the periodical ones.

To our regret, the analogous condition in paper [1] has not been stated correctly and needs to be given the above form.

Theorem 4. Let $f : X \longrightarrow X$ be a disconnected translation. Then f is determining translation if and only if the following conditions hold:

(i) $f \mid E_f(e)$ is a determining translation;

(ii) Y has at most one element or $f|Y$ is a disconnected permutation with $Y \subset Z_f$

$n(x)$ does not divide $n(y)$ for any $x, y \in Y$, $x \notin Z(y)$;

(iii) if q, $q \neq 1$, is a common divisor of all $n(x)$, $x \in Y$ then there exists $x_0 \in Y$ such that for all p, where p and $\dfrac{n(x_0)}{q}$ are relatively prime,

$\frac{r(x_0)p-q}{q^2}$ is not an integer.

Proof: Necessity of the condition (i) is evident (see Construction 1 and Theorems 1, 2, 3).

By Lemma 6 $f|Y$ is an injective transformation. From Lemma 7 and Construction 5 it is obvious that the condition (ii) is necessary if there is no x in Y with $Q_f(x)=0$. Let there exist $x \in Y$, $Q_f(x)=0$. Injectivity of $f|Y$ gives another top element e' of f.

The necessity of the condition (iii) follows from Construction 6.

To finish the proof we must show that the conditions stated are sufficient; it will be done if we show that every $L(M')$ containing f must be equal to $L(M)$ from Construction 1.

Let there exist M' with $f \in L(M')$, $L(M')=\{f'_x \mid x \in X\}$, $R(M')=\{g'_y \mid y \in X\}$. The source of both systems must be the only top element which satisfies the condition (i) from Theorems 1 and 3. We shall designate it by e. $f'_x|E_f(e)$ and $g'_y|E_f(e)$, for $x, y \in E_f(e)$, have the form as specified in Construction 1 (see Theorems 1, 3).

Let $t \in K$, $y \in Y$, the $g'_y(f(t)) = f^{d(t)+1}(g'_y(e)) = f^{d(t)+1}(y)$, hence $g'_y(t) \in f^{-1}(f^{d(t)+1}(y))$, but $f|Y$ is a permutation, so $g'_y(t)=f^{d(t)}(y)$ for all $y \in X$, $t \in K$.

Let $t \in T_{p,q}$, $y \in Y$, then similarly $g'_y(t) = h^q f^{u(e)+p}(y)$ and

(27) $\quad f'_t(y) = h^q f^{u(e)+p}(y) \qquad$ for $y \in X$, $t \in T_{p,q}$.

If Y has exactly one element x, $g'_x(t) = x$ for all t and $f'_x(t) = x$, i.e. M' is equal to M.

Let Y have more than one element, then $f|Y$ is a disconnected permutation. If we show that $g'_y|Y = 1_Y$ for all $y \in X$, the proof will be finished because of

(28) $$f_x(y) = x$$

for all $y \in X$ and $x \in Y$.

With regard to (26) and (27), M' is equal to M.

Now we show that $g'_t|Z(x) = (f|Z(x))^k$ for some $k \geq 0$. If there exist x_0, $z_0 \in Y$ with $x_0 \notin Z(z_0)$, $g'_t(x_0) = z_0$, then $f(g'_t(f^{\varkappa(x_0)-1}(x_0))) = g'_t(x_0) = z_0 = f^{\varkappa(x_0)}(z_0)$, i.e. $\varkappa(x_0) = m.\varkappa(z_0)$ and this is a contradiction with the condition (ii).

Hence $g'_t(Z(x)) \subset Z(x)$. From the commutativity of g'_t and f, we get $g'_t|Z(x) = f^k|Z(x)$.

Providing that there exists g'_t such that $g'_t|Z(x) = f^k|Z(x)$, $0 < k < \varkappa(x)$ for some $x \in Y$, we obtain that $g'_t|Y$ is a disconnected permutation and has cycles of the same order. Now we shall show that also $g'_{y_0}|Y$ is not an identity transformation ($y_0 = f(e)$). Let $x \in Y$ be an element with $g'_{y_0}(x) = x$; then for all $y \in Y$ it is $g'_{y_0}(y) = y$. (The last assertion follows from the same reasons as that all cycles of g'_t have the same order.) Because of $g'_{y_0}(E_f(e)) \subset E_f(e)$ we get $f'_x(E_f(e)) = \{x\}$, $x \in Y$. Therefore $f'_x \circ f = f'_x$ and hence $|f'_x(Z(x))| = 1$; but $g'_t|Z(x) = f^k|Z(x)$, thus g'_t does not commute with f'_x.

Hence $g'_{y_0}|Y$ is a disconnected permutation having all cycles of the same order; we shall designate this order by q. It is evident that q is a common divisor of all $r(x)$, $x \in Y$.

Let there exist $x \in Y$ such that for all p, where p and $\frac{r(x)}{q}$ are relatively prime,

(28) $$\frac{p \cdot r(x_2) - q}{q} \text{ is not an integer.}$$

We may suppose that $g'_x | Y = g'_{y_0} | Y$; in case this supposition does not apply we take $x' = g'_{y_0} \cdot (g'_x)^m (e)$, where $(g'_x)^m | Y = 1_Y$. It is obvious that x' has the property (28). Let us take $x \in Y$ with $g'_x | Y = g'_{y_0} | Y$. Then for $t = f^{k(x)(q-1)}(e)$, where $g'_x | Z(x) = f^{k(x)} | Z(x)$, it is $f'_x(g'_x(t)) = f'_x(f^{k(x)(q-1)}(x)) = f'_x((g'_x)^{q-1}(x)) = (g'_x)^{q-1}(g'_x(x)) = (g'_x)^q(x)$ and $g'_x(f'_x(t)) = g'_x(f'_x((g'_{y_0})^{k(x)(q-1)}(e))) = g'_x((g'_{y_0})^{k(x)(q-1)}(x)) = (g'_x)^{k(x)(q-1)+1}(x)$.

Therefore $k(x)(q-1)+1-q = mq$, where m is an integer, i.e. $k(x)-1 = aq$, where a is an integer.

Simultaneously $k(x)$ is such a number that $f^{k(x)} | Z(x)$ has cycles of the order q; thus $k(x) = \frac{p \cdot r(x)}{q}$, where p and $\frac{r(x)}{q}$ are relatively prime; in other words there is a number p with (28). This is a contradiction with the condition (iii).

Thus the proof of Theorem 4 has been accomplished.

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