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DUAL PROPERTIES FOR UNCONDITIONALLY CONVERGING OPERATORS

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Abstract: An operator $T: \mathcal{X} \to \mathcal{Y}$ ($\mathcal{X}, \mathcal{Y}$ are Banach spaces) is unconditionally converging (uc) if it maps weakly unconditionally converging series into unconditionally converging series. It is known that T' (the dual of T) is a uc operator if and only if T is ℓ_1-cosingular. The ℓ_1-cosingular operator is classified and then used to characterize Banach spaces with property V' (studied by Pelczynski).

Key words: Unconditionally converging operator, weakly compact operator, dual space.

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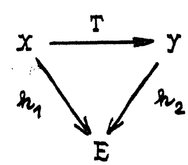
It is shown in [9] that an operator $T: \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces is an ℓ_1-cosingular operator if and only if its conjugate T' is an unconditionally converging (uc) operator. This paper is a study of ℓ_1-cosingular operators and other dual properties for uc operators.

We intend to preserve the notation and terminology of [2]. All operators are to be continuous and all spaces are to be Banach spaces. A series $\sum_n x_n$ of elements of a Banach space $\mathcal{X}$ is weakly unconditionally converging (wuc) [respectively unconditionally converging (uc)] if for every real sequence $\{t_n\}$ with $\lim_n t_n = 0$ [respectively for every bounded real sequence $\{t_n\}$] the series $\sum_n t_n x_n$ is convergent.

<u>Definition 0.1</u>. Let X and Y be Banach spaces. A linear operator $T: X \to Y$ is unconditionally converging (uc operator) if it sends every wuc series in X into uc series in Y.

<u>Definition 0.2.</u> A linear operator $T: X \to Y$ is ℓ_1-cosingular provided that for no Banach space E isomorphic to ℓ_1 does there exist epimorphisms $h_1 : X \to E$ and $h_2 : Y \to E$ such that the diagram



is commutative.

From [3] we know that T is a uc operator if and only if T has no bounded inverse on a subspace E of X isomorphic to c_0 .

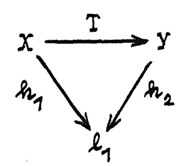
1. <u>ℓ_1 -cosingular operators</u>

From definition 0.2 it is clear that if $T: X \to Y$ and if in either X or Y a subspace isomorphic to ℓ_1 cannot be complemented, then T is an ℓ_1-cosingular operator (see also [9], p.38). Some Banach spaces which satisfy this condition are ℓ_∞, $C(S)$, c_0, c, and reflexive spaces.

It is shown in [4] that every weakly compact operator is ℓ_1-cosingular. The following proposition gives a weaker condition for an operator to be ℓ_1-cosingular.

Proposition 1.1. If $T: X \longrightarrow Y$ takes bounded sets of X into sets of Y such that every sequence contains a weak Cauchy subsequence, then T is an ℓ_1-cosingular operator.

Proof: Assume that T is not an ℓ_1-cosingular operator, i.e. that there exist epimorphisms $h_1 : X \longrightarrow \ell_1$ and $h_2 : Y \longrightarrow \ell_1$ such that the diagram



is commutative. Since T maps bounded sets into sets such that every sequence contains a weak Cauchy subsequence, then $h_1 = h_2\ T : X \longrightarrow \ell_1$ must do the same. Let K denote the unit sphere of X. Then since ℓ_1 is weakly complete, every sequence of $h_1(K)$ contains a weakly convergent subsequence. Hence h_1 is weakly compact, and since h_1 is an epimorphism, ℓ_1 must be reflexive. This contradiction completes the proof.

From [4] we know that if $T' : Y' \longrightarrow X'$ is an ℓ_1-cosingular operator, then $T: X \longrightarrow Y$ is a uc operator. The following example shows that the converse is not true. This example was communicated to me by A. Pelczynski.

Example 1.2. If $T: X \longrightarrow Y$ is a uc operator, then T' is not necessarily an ℓ_1-cosingular operator.

Proof: Let X be a Banach space with a boundedly complete basis. Then by theorem 1 of [5] there exists a separable space E such that $E'' = JE + F$ where JE is the natural image of E in E'' and where F is isomorphic to $\mathcal{X}$.

Now put $X = \ell_1$ and $Y = E'$. Since E'' is separable, $Y = E'$ is separable. Hence Y does not contain a subspace isomorphic to c_0 because if a conjugate Banach space contains a subspace isomorphic to c_0, it contains a subspace isomorphic to ℓ_∞ by theorem 4 of [1] and hence Y could not be separable.

Thus the identity operator $I : Y \longrightarrow Y$ is a uc operator but its conjugate I' is clearly not an ℓ_1-cosingular operator.

2. Property V'

We now consider property V' defined by A. Pelczynski in [7].

Definition 2.1. A Banach space X is said to have the property V' if every set K in X satisfying the condition (+ +) $\lim_n \sup_{x \in K} x'_n x = 0$ for every wuc series $\sum_n x'_n$ in X' is weakly sequentially compact.

The following proposition gives a connection between property V' and ℓ_1-cosingular operators.

Proposition 2.2. The following conditions are equivalent.

(a) Y has property V'

(b) For every Banach space X, every ℓ_1-cosingular operator $T: X \to Y$ is weakly compact.

Proof: (a) implies (b): Let X be an arbitrary Banach space and let $T: X \to Y$ be such that T is ℓ_1-cosingular. Then T' is uc. We show T is a weakly compact operator. Let $\{x_n\}$ be an arbitrary bounded sequence in X and let $\sum_n y_n'$ be an arbitrary wuc series in Y'. Since T' is a uc operator, $\sum_n T' y_n'$ is a uc series in X'. Therefore by condition (H) of [6]

$$\lim_n \sup_m Jx_m(T'y_n') = \lim_n \sup_m T'y_n'(x_m) = \lim_n \sup_m y_n'(Tx_m) = 0$$

where J is the canonical map of X into X''. From definition 2.1 $\{Tx_m\}$ contains a weakly convergent subsequence. Therefore T is a weakly compact operator. (b) implies (a): Let $K \subseteq Y$ be such that $\lim_n \sup_{y \in K} y_n'(y) = 0$ for all wuc series $\sum_n y_n'$ in Y'. Then K is bounded by the Uniform Boundedness Principle. Denote by $B(K)$ the space of all bounded real valued functions on K with the norm $\|f\| = \sup_{y \in K} f(y)$ and consider the linear transformation $S: Y' \to B(K)$ where $Sy'(y) = Jy(y')$ for all $y' \in Y'$ and $y \in K$. By (+ +) of definition 2.1, S is a uc operator.

We show $S = T'$ where $T: \ell_1(K) \to Y$ is defined by $Tf = \sum_{k \in K} k f(k)$. We have

$$\begin{aligned}\langle T'y', f \rangle &= \langle y', Tf \rangle \\ &= \langle y', \sum_K k f(k) \rangle \\ &= \sum_K \langle y', k f(k) \rangle \\ &= \sum_K f(k) \langle y', k \rangle\end{aligned}$$

and

$$\langle Sy', f \rangle = \sum_K f(k)\, Sy'(k)$$
$$= \sum_K f(k) \langle y', k \rangle \quad .$$

Therefore $T'y' = Sy'$ for every $y' \in Y'$. So $S = T'$ is a uc operator, i.e., T is ℓ_1-cosingular. By assumption, T is weakly compact and hence T'' is weakly compact.

Let $\{y_n\}$ be an arbitrary sequence in Y. Set $F_n' f = f(y_n)$ for $f \in B(K)$ and for $n = 1, 2, \dots$. Then $\|F_n'\| = 1$ for every n, and $\{F_n'\}$ is a bounded sequence in $[B(K)]'$. Now $T'' F_n'(y') = F_n'(T'y') = T'y'(y_n) = Jy_n(y')$ for all $y' \in Y'$. Therefore $T'' F_n' = Jy_n$ for all n. Since T'' is weakly compact, one may choose from the sequence $\{Jy_n\}$ a weakly convergent subsequence. Hence $\{y_n\}$ has a subsequence which weakly converges. Therefore K is a weakly sequentially compact set in Y .

3. Applications

A space X is said to have the property D.P. (Dunford-Pettis) if for every Banach space Y every weakly compact operator $T: X \longrightarrow Y$ maps weak Cauchy sequence in X into Cauchy sequences in the norm topology of Y. We now consider a Banach space with both properties D.P. and V' .

Theorem 3.1. Let Y have properties DP and V' and let $T: X \longrightarrow Y$. Then the following are equivalent:

(a) T is strictly cosingular [8]

(b) T is ℓ_1-cosingular

(c) T is weakly compact

(d) T takes bounded sets of X into sets of Y such that every sequence contains a weak Cauchy subsequence.

Proof: (a) implies (b): This is clear from the definition of strictly cosingular given in [8]. (b) implies (c): Y has property V'. (c) implies (a): Y has property DP, hence if T is weakly compact, then T is strictly co-

lar by proposition 4(b) of [8].

Hence (a), (b), and (c) are all equivalent. The proof will be complete if (d) implies (b) and (c) implies (d). (d) implies (b): This follows from proposition 1.1. (c) implies (d): This is clear from the definition of a weakly compact operator.

Remark: Examples of spaces that have properties DP and V' are L_1, ℓ_1 and every abstract L-space.

Suppose Y has property V'. What additional properties on Y would imply Y reflexive? Two such conditions are given in [7]. We give a different proof to one of these and also prove Y reflexive for the following condition.

Definition 3.2. A Banach space X is almost reflexive if every bounded sequence in X contains a weak Cauchy subsequence.

Proposition 3.3. Let Y have property V'. Then if either

(1) no subspace isomorphic to ℓ_1 is complemented in Y or

(2) Y is almost reflexive

then Y is reflexive.

Proof: Consider the identity operator $I : Y \to Y$. If (1) is true, then clearly I is $\mathcal{L}_1$-cosingular. If (2) is true, I is $\mathcal{L}_1$-cosingular by proposition 1.1. So in either case I is $\mathcal{L}_1$-cosingular and hence weakly compact by proposition 2.2. So Y is reflexive.

R e f e r e n c e s

[1] C. BESSAGA and A. PELCZYNSKI: On bases and unconditional convergence of series in Banach spaces, Studia Math.18(1958),151-164.

[2] N. DUNFORD and J.T. SCHWARTZ: Linear Operators, Part I, Interscience Publishers,Inc.,New York,1958.

[3] J. HOWARD: The comparison of an unconditionally converging operator, Studia Math.23(1969),295-298.

[4] J. HOWARD: F-singular and G-cosingular operators, Colloquium Math.22(1970),85-89.

[5] R.C. JAMES: Separable conjugate spaces, Pacific J.Math. 10(1960),563-571.

[6] Ch.W. McARTHUR: A note on subseries convergence, Proc. Amer.Math.Soc.12(1961),540-545.

[7] A. PELCZYNSKI: Banach spaces in which every unconditionally converging operator is weakly compact, Bull.Acad.Polon.Sci.10(1962),641-648.

[8] A. PELCZYNSKI: On strictly singular and strictly cosingular operators, I, Bull.Acad.Polon.Sci.13(1965), 31-36.

[9] A. PELCZYNSKI: On strictly singular and strictly cosingular operators, II, Bull.Acad.Polon.Sci.13 (1965),37-41.

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