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Commentationes Mathematicae Universitatis Carolinae, Vol. 14 (1973), No. 1, 27--46

Persistent URL: http://dml.cz/dmlcz/105467

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14,1 (1973)

EXISTENCE THEOREMS FOR OPERATOR EQUATIONS AND NONLINEAR ELLIPTIC BOUNDARY-VALUE PROBLEMS

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Abstract:

Let V be a real reflexive Banach space with dual V^*. Suppose that T is - in one sense - the limit of bounded continuous mappings from V to V^* with domain $D(T) := \{u \in V : T(u) \in V^*\}$. Under suitable conditions the existence of a solution $u_0 \in D(T)$ of the nonlinear operator equation

$$T(u) = f$$

with $f \in V^*$ is proved. Application to a nonlinear elliptic boundary value problem is given.

Key words: Nonlinear operator equation, regularization method, elliptic differential equation, boundary condition.

AMS, Primary: 47H15, 47F05, 35J60   Ref. Ž. 7.956,
Secondary: 46B10, 46E35   7.978.5

Let V be a reflexive Banach space, and V^* its dual space. The theory of coercive, semi-monotone operators T from V to V^* and its applications to the study of nonlinear elliptic boundary-value problems have been treated extensively by Browder [3], Leray-Lions [8], Nečas [9, 10] and others.

In this paper we will consider operators T with domain of definition D contained in V and range in V^*. In Section 1, an existence theorem (Theorem 1) is proved for such operators T mapping D into V^*. This theorem generalizes the known existence theorem for mappings T from V to V^*. Its proof is based on regularization methods. Section 2 contains the application of Theorem 1 to nonlinear elliptic partial differential equations (Theorem 2). This theorem is a generalization of the existence theorems for elliptic equations, proved by Browder [2, 3], Leray-Lions [8], Višik [13 - 15], Nečas [9, 10], Bui An Ton [5] and others (see also [7]).

2. Let V, W be two real reflexive separable Banach spaces with $W \subset V$, where the natural injection mapping $\mathcal{J}_1$ of W into V is assumed to be continuous. Further suppose that W is dense in V.

Let V^*, W^* be the duals of V and W respectively. The pairing between V and V^* shall be denoted by $(\cdot,\cdot)$ and that of W and W^* by $((\cdot,\cdot))$.

By $\rightarrow$ and $\rightharpoonup$ we will denote the strong and weak convergence respectively.

In this section we use the following Theorem of Browder - Bui An Ton [4] (see also [5]).

Proposition 1. Let X be a real reflexive separable Banach space, S a denumerable subset of X. Then there exist a separable Hilbert space H and a linear compact mapping $\mathcal{J}$ of H into X such that $S \subset \mathcal{J}(H)$.

Applying Proposition 1 to $X := W$ we obtain the existence of a separable Hilbert space H (the inner product shall be denoted by $\langle\cdot,\cdot\rangle$) and a compact linear mapping $\mathcal{J}$ of H into W such that $\mathcal{J}(H)$ is dense in W.

We assume

Assumption 1. (a) Let $A : V \longrightarrow V^*$ be bounded (i.e. maps bounded sets into bounded sets) and demi-continuous (i.e. continuous from the strong to the weak topology).
(b) Any sequence $\{w_n\} \subset W$ satisfying $\mathcal{J}_1 w_n \rightharpoonup u_0$ in V, $A(\mathcal{J}_1 w_n) \rightharpoonup g$ in V^* with $\limsup_n (A(\mathcal{J}_1 w_n), \mathcal{J}_1 w_n) \leq (g, u_0)$ implies $A(u_0) = g$.

Remark 1. (a) A bounded demicontinuous operator A from V to V^* which is semi-monotone (see e.g. Bui An Ton [5]) satisfies Assumption 1 (s.[5]).
(b) A bounded continuous operator A from V to V^* satisfying Condition $(S+)$ (see e.g. Browder [3]) implies Assumption 1.
(c) Assumption 1 (b) is related to mappings of type (M) introduced by Brézis [1].

To prove an existence theorem for mappings from V to V^* we will use regularization methods. Therefore we introduce

Assumption 2. (a) Let there exist $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$, $u \in V$, $B(\varepsilon, u, \cdot) : V \rightarrow \mathbb{R}^1$ is linear and continuous and $B(\varepsilon, \cdot, u) : V \rightarrow \mathbb{R}^1$ is continuous. Further suppose that for all $\varepsilon \in]0, \varepsilon_0]$ and all $w \in W$, $B(\varepsilon, \mathcal{J}_1 w, \mathcal{J}_1 w) \geq 0$.
(b) Any sequences $\{\varepsilon_n\} \subset]0, \varepsilon_0]$ and $\{w_{\varepsilon_n}\} \subset W$ satisfying $\varepsilon_n \rightarrow 0$, $\mathcal{J}_1 w_{\varepsilon_n} \rightharpoonup u_0$ in V and $0 \leq B(\varepsilon_n, \mathcal{J}_1 w_{\varepsilon_n}, \mathcal{J}_1 w_{\varepsilon_n}) \leq \mathcal{C}$ with some constant $\mathcal{C} > 0$ imply the existence of $B(0, u_0, \mathcal{J}_1 w)$ for all $w \in W$. Furthermore there exists a subsequence $\{n'\}$

such that for all $w \in W$, $B(\varepsilon_{n'}, \mathcal{J}_1 w_{\varepsilon_{n'}}, \mathcal{J}_1 w) \to$ $\to B(0, u_0, \mathcal{J}_1 w)$. In addition suppose that the existence of $B(0, u_0, u_0)$ implies (perhaps by taking a subsequence)

$$B(0, u_0, u_0) \leq \lim_{n'} B(\varepsilon_{n'}, \mathcal{J}_1 w_{\varepsilon_{n'}}, \mathcal{J}_1 w_{\varepsilon_{n'}}) .$$

We introduce further the following coercivity condition.

Assumption 3. For all $\varepsilon \in]0, \varepsilon_0]$ and all $w \in W$ let

$$(A(\mathcal{J}_1 w), \mathcal{J}_1 w) + B(\varepsilon, \mathcal{J}_1 w, \mathcal{J}_1 w) \geq c(\| \mathcal{J}_1 w \|_V) \| \mathcal{J}_1 w \|_V ,$$

where $c(\varkappa)$ is a function on $\mathbb{R}^1_+$ satisfying: (i) $c(\varkappa) \to \infty$ as $\varkappa \to \infty$; (ii) $c(\varkappa) \geq -c_0$ on $\mathbb{R}^1_+$ with some $c_0 \geq 0$.

By Assumption 2(a) there exists $B(\varepsilon, u) \in V^*$ for all $\varepsilon \in]0, \varepsilon_0]$ and all $u \in V$ such that

$$(B(\varepsilon, u), v) = B(\varepsilon, u, v)$$

for all $v \in V$. Further for each $\varepsilon \in]0, \varepsilon_0]$, $B(\varepsilon, \cdot)$: $V \to V^*$ is demi-continuous.

We define $D(B) := \{u \in V : B(0, u, \cdot) : V \to \mathbb{R}^1$ is a linear continuous mapping$\}$. Hence for all $u \in D(B)$ there exists $B(u) \in V^*$ satisfying

$$(B(u), v) = B(0, u, v)$$

for all $v \in V$.

The problem, to be considered in this section, is to prove the existence of a solution $u_0 \in D(B)$ to

$$A(u) + B(u) = f \tag{2.1}$$

with $f \in V^*$.

We formulate our main theorem.

Theorem 1. Suppose that Assumptions 1, 2, 3 hold. Let $f \in V^*$. Then there exists at least one $u_0 \in D(B)$ satisfying (2.1).

Proof: The proof follows by several steps.

(a) We first remark that the dual $\mathcal{J}_1^*$ of $\mathcal{J}_1$ is a linear continuous mapping of V^* into W^* and we have $V^* \subset W^*$. Furthermore the dual $\mathcal{J}^*$ of $\mathcal{J}$ is a linear compact mapping of W^* into H. For $\varepsilon \in]0, \varepsilon_0]$ we now consider the problem

(2.2) $$\varepsilon u + \mathcal{J}^* \mathcal{J}_1^* A(\mathcal{J}_1 \mathcal{J} u) + \mathcal{J}^* \mathcal{J}_1^* B(\varepsilon, \mathcal{J}_1 \mathcal{J} u) = \mathcal{J}^* \mathcal{J}_1^* f$$

with $u \in H$. We set with $\varepsilon \in]0, \varepsilon_0]$, $u \in H$

$$T(\varepsilon, u) := \frac{1}{\varepsilon}(\mathcal{J}^*\mathcal{J}_1^* f - \mathcal{J}^*\mathcal{J}_1^* A(\mathcal{J}_1 \mathcal{J} u) - \mathcal{J}^* \mathcal{J}_1^* B(\varepsilon, \mathcal{J}_1 \mathcal{J} u)).$$

By Assumption 1.2 and the above remarks it follows that, for each $\varepsilon \in]0, \varepsilon_0]$, the mapping $T(\varepsilon, \cdot)$ is continuous and compact from H to H. Further it follows by Assumption 3

$$\langle u - T(\varepsilon, u), u \rangle = \|u\|_H^2 - \frac{1}{\varepsilon}\langle \mathcal{J}^* \mathcal{J}_1^* f, u \rangle +$$

$$+ \frac{1}{\varepsilon}\{\langle \mathcal{J}^* \mathcal{J}_1^* A(\mathcal{J}_1 \mathcal{J} u), u\rangle + \langle \mathcal{J}^* \mathcal{J}_1 B(\varepsilon, \mathcal{J}_1 \mathcal{J} u), u\rangle\}$$

$$= \|u\|_H^2 - \frac{1}{\varepsilon}(f, \mathcal{J}_1 \mathcal{J} u) + \frac{1}{\varepsilon}\{(A(\mathcal{J}_1 \mathcal{J} u), \mathcal{J}_1 \mathcal{J} u) +$$

$$+ (B(\varepsilon, \mathcal{J}_1 \mathcal{J} u), \mathcal{J}_1 \mathcal{J} u)\} \geq \|u\|_H^2 - \frac{1}{\varepsilon}\|f\|_{V*}\|\mathcal{J}_1 \mathcal{J} u\|_V +$$

$$+ \frac{1}{\varepsilon}\{(A(\mathcal{J}_1 \mathcal{J} u), \mathcal{J}_1 \mathcal{J} u) + B(\varepsilon, \mathcal{J}_1 \mathcal{J} u, \mathcal{J}_1 \mathcal{J} u)\}$$

$$\geq \|u\|_H \{\|u\|_H + \frac{1}{\varepsilon}(c(\|\mathcal{J}_1 \mathcal{J} u\|_V) - \|f\|_{V*})\frac{\|\mathcal{J}_1 \mathcal{J} u\|_V}{\|u\|_H}\} \geq 0$$

for all $u \in S_{R_\varepsilon} := \{ u \in H : \| u \|_H = R_\varepsilon \}$, where R_ε is a suitable positive constant. This follows by the assumption on c and the inequality $\| \mathcal{J}_1 \mathcal{J} u \|_V \le \gamma \| u \|_H$ with some constant $\gamma > 0$. Hence by a theorem of Krasnoselskii (see e.g.[6]), there exists for each $\varepsilon \in]0, \varepsilon_0]$ a fixed point $u_\varepsilon \in H$ of $T(\varepsilon, \cdot)$, i.e. u_ε is a solution to (2.2). Therefore by Assumption 3

$$
\begin{aligned}
0 &= \langle \varepsilon u_\varepsilon + \mathcal{J}^* \mathcal{J}_1^* A(\mathcal{J}_1 \mathcal{J} u_\varepsilon) + \mathcal{J}^* \mathcal{J}_1^* B(\varepsilon, \mathcal{J}_1 \mathcal{J} u_\varepsilon) - \mathcal{J}^* \mathcal{J}_1^* f, u_\varepsilon \rangle \\
&= \varepsilon \| u_\varepsilon \|_H^2 + (A(\mathcal{J}_1 \mathcal{J} u_\varepsilon), \mathcal{J}_1 u_\varepsilon) + (B(\varepsilon, \mathcal{J}_1 \mathcal{J} u_\varepsilon), \mathcal{J}_1 \mathcal{J} u_\varepsilon) - (f, \mathcal{J}_1 \mathcal{J} u_\varepsilon) \\
&\ge \varepsilon \| u_\varepsilon \|_H^2 + (c(\| \mathcal{J}_1 \mathcal{J} u_\varepsilon \|_V) - \| f \|_{V^*}) \| \mathcal{J}_1 \mathcal{J} u_\varepsilon \|_V .
\end{aligned}
$$

Hence there exist positive constants $\mathcal{C}_1$, $\mathcal{C}_2$ such that

$$
(2.3) \qquad \sqrt{\varepsilon} \| u_\varepsilon \|_H \le \mathcal{C}_1 , \quad \| \mathcal{J}_1 \mathcal{J} u_\varepsilon \|_V \le \mathcal{C}_2
$$

for all $\varepsilon \in]0, \varepsilon_0]$.

(b) By virtue of (2.3) and Assumption 1(a) there exists a sequence $\{ \varepsilon_n \} \subset]0, \varepsilon_0]$ such that $\varepsilon_n \to 0$, $\varepsilon_n u_{\varepsilon_n} \to 0$ in H, $\mathcal{J}_1 \mathcal{J} u_{\varepsilon_n} \rightharpoonup u_0$ in V and $A(\mathcal{J}_1 \mathcal{J} u_{\varepsilon_n}) \rightharpoonup g$ in V^* .

Further it follows by (2.3) Assumptions 1(a),2(a) and $\mathcal{J} u_\varepsilon \in W$

$$
\begin{aligned}
0 \le B(\varepsilon, \mathcal{J}_1 \mathcal{J} u_\varepsilon, \mathcal{J}_1 \mathcal{J} u_\varepsilon) &= (B(\varepsilon, \mathcal{J}_1 \mathcal{J} u_\varepsilon), \mathcal{J}_1 \mathcal{J} u_\varepsilon) \\
&= -\varepsilon \| u_\varepsilon \|_H^2 - (A(\mathcal{J}_1 \mathcal{J} u_\varepsilon), \mathcal{J}_1 \mathcal{J} u_\varepsilon) + (f, \mathcal{J}_1 \mathcal{J} u_\varepsilon) \\
&\le (\| A(\mathcal{J}_1 \mathcal{J} u_\varepsilon) \|_{V^*} + \| f \|_{V^*}) \| \mathcal{J}_1 \mathcal{J} u_\varepsilon \|_V \le \mathcal{C}_3
\end{aligned}
$$

with some constant $\mathcal{C}_3 > 0$. By Assumption 2(b) there exists a subsequence $\{\varepsilon_{n'}\}$ such that for all $w \in W$ $B(\varepsilon_{n'}, \mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}}, \mathcal{J}_1 w) \to B(0, u_0, \mathcal{J}_1 w)$. Because u_ε satisfies (2.2) we have for all $\Psi \in H$

$$0 = \langle \varepsilon_{n'}, u_{\varepsilon_{n'}}, \Psi \rangle + (A(\mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}}), \mathcal{J}_1 \mathcal{J} \Psi) +$$

$$+ (B(\varepsilon_{n'}, \mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}}), \mathcal{J}_1 \mathcal{J} \Psi) - (f, \mathcal{J}_1 \mathcal{J} \Psi) = \langle \varepsilon_{n'}, u_{\varepsilon_{n'}}, \Psi \rangle +$$

$$+ (A(\mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}}), \mathcal{J}_1 \mathcal{J} \Psi) + B(\varepsilon_{n'}, \mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}}, \mathcal{J}_1 \mathcal{J} \Psi) - (f, \mathcal{J}_1 \mathcal{J} \Psi)$$

from which by (2.3) as $n' \to \infty$

$$(g, \mathcal{J}_1 \mathcal{J} \Psi) + B(0, u_0, \mathcal{J}_1 \mathcal{J} \Psi) = (f, \mathcal{J}_1 \mathcal{J} \Psi) .$$

$\mathcal{J} H$ is dense in W and W is dense in V by assumption, hence $\mathcal{J}_1 \mathcal{J} H$ is dense in V. Further from the above relation it follows

$$|B(0, u_0, \mathcal{J}_1 \mathcal{J} \Psi)| = |(f - g, \mathcal{J}_1 \mathcal{J} \Psi)| \leq \| f - g \|_{V*} \| \mathcal{J}_1 \mathcal{J} \Psi \|_V ,$$

i.e. $B(0, u_0, \cdot) : \mathcal{J}_1 \mathcal{J} H \to \mathbb{R}^1$ is a linear continuous mapping from the strong topology in V. Hence $B(0, u_0, \cdot)$ can uniquely be continued to a linear continuous mapping from the Banach space V to $\mathbb{R}^1$ such that

(2.4) $$(g, v) + B(0, u_0, v) = (f, v)$$

holds for all $v \in V$. Further there exists $B(u_0) \in V^*$ such that for all $v \in V$

(2.5) $$(B(u_0), v) = B(0, u_0, v) .$$

By the last two relations we obtain

(2.6) $$g + B(u_0) = f .$$

From (2.5) it follows $(B(u_0), u_0) = B(0, u_0, u_0)$.

Further by u_ε satisfying (2.2) we obtain (perhaps by taking a subsequence)

$$\limsup_{n'} (A(\mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}}), \mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}}) = \limsup_{n'} \{-\varepsilon_{n'} \|u_{\varepsilon_{n'}}\|_H^2 + (f, \mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}}) - (B(\varepsilon_{n'}, \mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}}), \mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}})\}$$

$$\leq (f, u_0) - \lim_{n'} B(\varepsilon_{n'}, \mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}}, \mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}}) .$$

By Assumption 2(b) and (2.6) we have

$$\limsup_{n'} (A(\mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}}), \mathcal{J}_1 \mathcal{J} u_{\varepsilon_{n'}}) \leq (f, u_0) - B(0, u_0, u_0)$$

$$= (f, u_0) - (B(u_0), u_0) = (f - B(u_0), u_0) = (g, u_0) ,$$

from which by Assumption 1(b)

$$A(u_0) = g = f - B(u_0) ,$$

i.e. u_0 satisfies (2.1). By (2.5) it follows $u_0 \in D(B)$, proving Theorem 1.

Remark 2. (a) The method used to prove Theorem 1 is a combination of the elliptic super-regularization studied in [4] and another regularization applied in [12].

(b) In Theorem 1, the domain of Definition $D(B)$ of the operator $T := A + B$ is a subset of the Banach space V and the range of T is contained in V^*. This theorem generalizes most of the known existence theorems for mappings T from V to V^* (see e.g. [3,8,9]).

(c) Another method to obtain existence theorems for mappings T with domain $D(T)$ contained in some Banach space B_1 and range in some other Banach space B_2 is given in [11].

3. In this section we will apply Theorem 1 to elliptic differential equations. We use the notation of Browder in [3]. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open domain with sufficiently smooth boundary $\partial\Omega$ such that the Imbedding Theorems of Sobolev are applicable (see e.g. Browder [3]). It is our purpose, to study differential equations of the form

$$\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, \xi_m(u)(x)) + \\ + \sum_{|\beta|\leq m-1} (-1)^{|\beta|} D^{\beta} B_{\beta}(x, \xi_{m-1}(u)(x)) = f(x)$$

for $x \in \Omega$ with Dirichlet boundary conditions. Precisely, denote by $[f, g] := \int_{\Omega} f(x)\, g(x)\, dx$ and consider the Sobolev space $V := \overset{\circ}{W}_{m,p}$ with $1 < p < \infty$. Let f be an element of V^*. Then we ask for an element $u_0 \in V$ which satisfies the condition

$$(3.1) \qquad \sum_{|\alpha|\leq m} [A_{\alpha}(\cdot, \xi_m(u_0)), D^{\alpha} v] + \\ + \sum_{|\beta|\leq m-1} [B_{\beta}(\cdot, \xi_{m-1}(u_0)), D^{\beta} v] = [f, v]$$

for all $v \in V$.

We assume (see Browder [3])

<u>Assumption 4</u>. (a) Each $A_{\alpha}(x, \xi_m)$ is measurable in x for fixed ξ_m in $\mathbb{R}^{s_m}$ and continuous in ξ_m on $\mathbb{R}^{s_m}$ for almost all x in Ω. Let b be the greatest integer less than $m - n/p$, and let ξ_b denote the vector $\{\xi_{\alpha} : |\alpha| \leq b\}$, from the vector space $\mathbb{R}^{s_b}$. There exist continuous functions c_{α} and c_1 from $\mathbb{R}^{s_b}$ to $L^{p_{\alpha}}$ and $\mathbb{R}^1$, respectively, such that the following inequalities hold:

$$|A_\alpha(x,\xi_m)| \le c_\alpha(\xi_\ell)(x) + c_1(\xi_\ell)\sum_{m-n/p\le|\beta|\le m}|\xi_\beta|^{p_{\alpha\beta}}$$

with the exponents p_α and $p_{\alpha\beta}$ satisfying

$$p_\alpha = p' \quad (p^{-1}+p'^{-1}=1) \qquad \text{for } |\alpha| = m ,$$

$$p_\alpha > s'_\alpha \quad \text{for } m-n/p \le |\alpha| < m, \ s_\alpha^{-1} = p^{-1} - n^{-1}(m-|\alpha|),$$

$$p_\alpha = 1 \quad \text{for } |\alpha| < m-n/p$$

and

$$p_{\alpha\beta} \le p-1 \quad \text{for } |\alpha| = |\beta| = m ,$$

$$p_{\alpha\beta} < s_\beta(s'_\alpha)^{-1} \quad \text{for } m-n/p \le |\alpha|, |\beta| \le m, |\alpha|+|\beta| < 2m,$$

$$p_{\alpha\beta} \le s_\beta \quad \text{for } |\alpha| < m-n/p, \ m-n/p \le |\beta| \le m .$$

(b) If $\xi_m = (\xi_{m-1}, \zeta_m)$ is the division of ξ_m into its m-th order components ζ_m and the corresponding $(m-1)$-st order jet ξ_{m-1}, then for each $x \in \Omega$ and each $\xi_{m-1} \in \mathbb{R}^{s_{m-1}}$

$$\sum_{|\alpha|=m}[A_\alpha(x,\xi_{m-1},\zeta_m) - A_\alpha(x,\xi_{m-1},\zeta'_m)][\zeta_\alpha - \zeta'_\alpha] > 0$$

for $\zeta_m \ne \zeta'_m$.

(c) There exist two continuous functions c_0 and c from $\mathbb{R}^{s_\ell}$ to $\mathbb{R}^1$ with $c_0(\xi_\ell) \ge \tilde{c}_0 > 0$ for all ξ_ℓ such that for all $x \in \Omega$, all ζ_m and all ξ_{m-1} we have

$$\sum_{|\alpha|=m} A_\alpha(x,\xi_{m-1},\zeta_m)\zeta_\alpha \ge c_0(\xi_\ell)|\zeta_m|^p -$$
$$- c(\xi_\ell)\sum_{m-n/p\le|\beta|\le m-1}|\xi_\beta|^{t_\beta}$$

where $t_\beta < s_\beta$.

<u>Proposition 2</u> (see [3]). Let Assumption 4 be satisfied. Then there exists a bounded continuous mapping A of V into V^* such that for all $u, v \in V$

$$\sum_{|\alpha| \le m} [A_\alpha(\cdot, \xi_m(u)), D^\alpha v] = (A(u), v) .$$

The mapping A is coercive and satisfies Condition $(S+)$.

<u>Assumption 5</u>.(a) $B_\beta(x, \xi_{m-1})\ (|\beta| \le m-1)$ is a continuous function from $\Omega \times \mathbb{R}^{s_{m-1}}$ to $\mathbb{R}^1$ such that for all $\varepsilon \in [0,1]$ all ξ_{m-1} in $\mathbb{R}^{s_{m-1}}$ and almost all x in Ω

$$\sum_{|\beta| \le m-1} \frac{B_\beta(x, \xi_{m-1})\, \xi_\beta}{1 + \varepsilon\, |B_\beta(x, \xi_{m-1})|} \ge 0$$

(b) Suppose that there exist a constant $c_2 \ge 0$ and a function $F: \Omega \times \mathbb{R}^{s_{m-1}} \times \mathbb{R}^{s_{m-1}} \longrightarrow \mathbb{R}^1$ such that for all ε in $[0,1]$ all ξ_{m-1}, ξ'_{m-1} in $\mathbb{R}^{s_{m-1}}$ and almost all x in Ω

$$\left| \sum_{|\beta| \le m-1} \frac{B_\beta(x, \xi_{m-1})\, \xi'_\beta}{1 + \varepsilon |B_\beta(x, \xi_{m-1})|} \right| \le$$

$$\le c_2 \sum_{|\beta| \le m-1} \frac{B_\beta(x, \xi_{m-1})\, \xi_\beta}{1 + \varepsilon\, |B_\beta(x, \xi_{m-1})|}$$

$$+ F(x, \xi_{m-1}, \xi'_{m-1}) .$$

Further suppose that for all $w \in W_{m^*, p}$ with $m^* > m + n/p$, the mapping $F(\xi_{m-1}(\cdot), \xi_{m-1}(w))$, defined by $F(\xi_{m-1}(u), \xi_{m-1}(w))(x) := F(x, \xi_{m-1}(u)(x),$ $\xi_{m-1}(w)(x))$, is bounded and continuous from $W_{m-1,p}$ to L^1 .

Set

$\mathcal{D}(B) := \{ u \in V = \overset{\circ}{W}_{m,p}$: such that for all $v \in V$ there exists a constant $\mathcal{C}_u \geq 0$ (not depending on v) satisfying

$$\Big| \sum_{|\beta| \leq m-1} [B_\beta(\cdot, \xi_{m-1}(u)), D^\beta v] \Big| \leq \mathcal{C}_u \, \|v\|_V \} ,$$

then we state

<u>Theorem 2</u>. Suppose that Assumptions 4, 5 hold. Then there exists at least one solution $u_0 \in \mathcal{D}(B)$ of (3.1) for all $v \in V$

Proof: (a) We apply Theorem 2 by setting

$V := \overset{\circ}{W}_{m,p}$, $W := W_{m*,p} \cap \overset{\circ}{W}_{m,p}$ with norm of $W_{m*,p}$

$A := A$ and

$$B(\varepsilon, u, v) := \sum_{|\beta| \leq m-1} \Big[\frac{B_\beta(\cdot, \xi_{m-1}(u))}{1 + \varepsilon \, |B_\beta(\cdot, \xi_{m-1}(u))|}, D^\beta v \Big]$$

with $\varepsilon \in]0,1]$.

We remark that by the Imbedding Theorem of Sobolev it follows:

(a) $W_{m*,p} \subset C^{(m)}(\overline{\Omega})$; (b) $W_{m*,p} \subset W_{m,p}$

with continuous injection mapping. Hence W and V are two real reflexive separable Banach spaces with $W \subset V$, where the injection mapping $\mathcal{J}_1$ of W into V is continuous. Furthermore W is dense in V .

(b) Assumption 1 and Assumption 3 follow directly by Assumptions 4, 5(a), Proposition 2 and Remark 1(b), while Assumption 2(a) is a simple consequence of Assumption 5(a) and the definition of $B(\varepsilon, u, v)$.

(c) It rests to prove Assumption 2(b). Suppose that $\{\varepsilon_m\} \subset$ $\subset]0,1]$ and $\{w_m\} \subset W$ satisfy $\varepsilon_m \to 0$, $u_m := J_1 w_{\varepsilon_m} \longrightarrow u_0$ in $V := \overset{\circ}{W}_{m,p}$ and $0 \le B(\varepsilon_n, u_m, u_m) \le \mathcal{C}$ with some $\mathcal{C} > 0$.

By the Imbedding Theorem of Sobolev we have $W_{m,p} \subset$ $\subset W_{m-1,p}$ and the injection mapping is continuous and compact. Hence there exists a subsequence $\{m'\}$ such that

$$(3.2) \qquad u_{m'} \longrightarrow u_0$$

in $W_{m-1,p}$, from which the existence of a subsequence follows (also denoted by $\{m'\}$), which satisfies

$$(3.3) \qquad D^\alpha u_{m'}(x) \longrightarrow D^\alpha u_0(x)$$

a.e. on Ω for all $|\alpha| \le m-1$.

For any $w \in W \subset W_{m^*,p}$ we define the measurable functions

$$f_{1,m}(x) := \sum_{|\beta| \le m-1} \frac{B_\beta(x, \xi_{m-1}(u_m)(x))\, D^\beta u_m(x)}{1 + \varepsilon_m |B_\beta(x, \xi_{m-1}(u_m)(x))|} ,$$

$$f_{2,m}(x) := \sum_{|\beta| \le m-1} \frac{B_\beta(x, \xi_{m-1}(u_m)(x))\, D^\beta w(x)}{1 + \varepsilon_m |B_\beta(x, \xi_{m-1}(u_m)(x))|} ,$$

$$f_1(x) := \sum_{|\beta| \le m-1} B_\beta(x, \xi_{m-1}(u_0)(x))\, D^\beta u_0(x) ,$$

$$f_2(x) := \sum_{|\beta| \le m-1} B_\beta(x, \xi_{m-1}(u_0)(x))\, D^\beta w(x) .$$

We first remark that the assumption $0 \le B(\varepsilon_{m'}, u_{m'}, u_{m'}) \le \mathcal{C}$ may be written in the form

$$(3.4) \qquad 0 \le \int_\Omega f_{1,m'}(x)\,dx \le \mathcal{C} .$$

Therefore there exists a constant $\mathscr{C}_0 \leq \mathscr{C}$ such that (perhaps by taking a subsequence)

$$\lim_{n'} \int_\Omega f_{1,n'}(x)\,dx = \mathscr{C}_0 \ .$$

By (3.3) and Assumption 5(a) follows as $n' \to \infty$

$$f_{1,n'}(x) = |f_{1,n'}(x)| \to f_1(x) = |f_1(x)|$$

a.e. on Ω. Hence it follows by the Theorem of Fatou

$$(3.5)\quad B(0, u_0, u_0) = \int_\Omega f_1(x)\,dx \leq \lim_{n'} \int_\Omega f_{1,n'}(x)\,dx$$

$$= \lim_{n'} B(\varepsilon_{n'}, u_{n'}, u_{n'})$$

which proves one part of Assumption 2(b).

By Assumption 5(b) follows with any $\lambda > 0$

$$\left| \sum_{|\beta| \leq m-1} \frac{B_\beta(x, \xi_{m-1})\,\xi'_\beta}{1 + \varepsilon_n |B_\beta(x, \xi_{m-1})|} \right| \leq$$

$$\leq \lambda c_2 \sum_{|\beta| \leq m-1} \frac{B_\beta(x, \xi_{m-1})\,\xi_\beta}{1 + \varepsilon_n |B_\beta(x, \xi_{m-1})|} + \lambda F\left(x, \xi_{m-1}, \frac{\xi'_{m-1}}{\lambda}\right) .$$

Let $\varepsilon_1 > 0$ be arbitrary and set $\lambda := \dfrac{\varepsilon_1}{c_2 \mathscr{C}}$ $(c_2 \neq 0)$ and $\lambda = 1$ $(c_2 = 0)$ then we obtain for any $w \in W \subset$ $\subset W_{m^*,p}$ and any measurable set $\sigma \subset \Omega$

$$\int_\sigma |f_{2,n'}(x)|\,dx \leq \mathfrak{I}_{1,n'}(\sigma) + \mathfrak{I}_{2,n'}(\sigma)$$

with

$$\mathfrak{I}_{1,n'}(\sigma) := \frac{\varepsilon_1}{\mathscr{C}} \int_\sigma f_{1,n'}(x)\,dx \ ,$$

$$\mathfrak{I}_{2,n'}(\sigma) := \lambda \int_\sigma F\left(x, \xi_{m-1}(u_{n'})(x), \frac{\xi_{m-1}(w)(x)}{\lambda}\right) dx \ .$$

We have by (3.4), (3.2) and Assumption 5(b)

$$\mathcal{J}_{1,n'}(\sigma) \le \frac{\varepsilon_1}{\mathcal{C}} \int_\Omega f_{1,n'}(x)\,dx \le \varepsilon_1 \quad ,$$

$$\lim_{n'\to\infty} \mathcal{J}_{2,n'}(\sigma) = \lambda \int_\sigma F(x, \xi_{m-1}(u_0)(x), \quad \frac{\xi_{m-1}(u^*)(x)}{\lambda})\,dx \ .$$

Therefore it follows by the arbitrariness of ε_1

(3.6) $$\lim_{|\sigma|\to 0} \ \lim_{n'} \sup \int_\sigma |f_{2,n'}(x)|\,dx = 0$$

where $|\sigma|$ denotes the measure of σ .

Further we obtain by Assumption 5(b) and (3.5)

$$\int_\Omega |f_2(x)|\,dx \le c_2 \int_\Omega f_1(x)\,dx + \int_\Omega F(x, \xi_{m-1}(u_0)(x) \ ,$$
$$\xi_{m-1}(w)(x))\,dx \le \mathcal{C}_1$$

with some $\mathcal{C}_1 > 0$, i.e. $f_2 \in L^1$.

Let $\sigma > 0$ be arbitrary, then by (3.3) there exists a subset σ of Ω with $|\sigma| = \sigma$ and

(3.7) $$D^\alpha u_{n'}(x) \longrightarrow D^\alpha u_0(x)$$

uniformly on $\Omega - \sigma$ for all $|\alpha| \le m-1$. Now let $\{\sigma_k\}$ be such a sequence of subsets of Ω with $\sigma_{k+1} \subset \sigma_k$ and $|\sigma_k| \to 0$. Then we obtain by (3.7) and Assumption 5(a)

$$f_{2,n'}(x) \longrightarrow f_2(x)$$

uniformly on $\Omega - \sigma_k$. Therefore by virtue of (3.6) and $f_2 \in L^1$ it follows

$$\lim_{n'} \int_\Omega |f_{2,n'}(x) - f_2(x)|\,dx \le \lim_{n'} \int_{\Omega-\sigma_k} |f_{2,n'}(x) - f_2(x)|\,dx \ +$$

$$+ \limsup_{n'} \int_{\delta_k} |f_{2,n'}(x)|\,dx + \int_{\delta_k} |f_2(x)|\,dx$$

$$\leq \limsup_{n'} \int_{\delta_k} |f_{2,n'}(x)|\,dx + \int_{\delta_k} |f_2(x)|\,dx \to 0$$

as $k \to \infty$, i.e. $f_{2,n'} \longrightarrow f_2$ in L^1 . Hence it follows for all $w \in W$

$$B(\varepsilon_{n'}, u_{n'}, \mathcal{J}_1 w) \longrightarrow B(0, u_0, \mathcal{J}_1 w) ,$$

proving the rest of Assumption 2(b). Therefore Theorem 2 follows from Theorem 1.

We shall now formulate the conditions on $B_\beta(x, \xi_{m-1})$ which are more useful in applications.

<u>Proposition 3</u>. Suppose that for each $|\beta| \leq m-1$, B_β is a continuous function from $\Omega \times \mathbb{R}^{s_{m-1}}$ to $\mathbb{R}^1$. Set for all $|\beta| \leq m-1$

$$g_\beta(x, \xi'_{m-1}, \xi_{m-1}) := B_\beta(x, \xi_{m-1})\Big|_{\xi_\beta = \xi'_\beta} .$$

Suppose that for all $w \in W_{m^*,p}$ the mapping $g_\beta(\xi_{m-1}(w), \xi_{m-1}(\cdot))$, defined by

$$g_\beta(\xi_{m-1}(w), \xi_{m-1}(u))(x) := g_\beta(x, \xi_{m-1}(w)(x), \xi_{m-1}(u)(x)) ,$$

is bounded and continuous from $W_{m-1,p}$ to L^1 . Further suppose that there exists a function $G(x, \xi_{m-1})$ from $\Omega \times \mathbb{R}^{s_{m-1}}$ to $\mathbb{R}^1$ such that for all $\varepsilon \in [0, 1]$, almost all x in Ω and all ξ_{m-1} in $\mathbb{R}^{s_{m-1}}$

$$\sum_{|\beta| \leq m-1} \frac{|B_\beta(x, \xi_{m-1})\,\xi_\beta|}{1 + \varepsilon\,|B_\beta(x, \xi_{m-1})|} \leq$$

$$\leq c_3 \sum_{|\beta| \leq m-1} \frac{B_\beta(x, \xi_{m-1})\,\xi_\beta}{1 + \varepsilon\,|B_\beta(x, \xi_{m-1})|} + G(x, \xi_{m-1})$$

with some constant $c_3 \geq 0$. In addition suppose that the mapping $G(\xi_{m-1}(\cdot))$, generated by $G(x, \xi_{m-1}(u)(x))$, is bounded and continuous from $W_{m-1,p}$ to L^1 . Then Assumption 5(b) holds.

Proof. We define

$$\tilde{g}_\beta(x, \xi'_{m-1}, \xi_{m-1}) := \sup_{\substack{\xi_\beta \\ |\xi_\beta| \leq |\xi'_\beta|}} |B_\beta(x, \xi_{m-1})| \ .$$

Then $\tilde{g}_\beta$ also satisfies the assumptions of g_β . Let β be fixed then either (i) $|\xi'_\beta| \leq |\xi_\beta|$ or (ii) $|\xi'_\beta| > |\xi_\beta|$. Therefore it follows

$$|B_\beta(x, \xi_{m-1})\xi'_\beta| \leq \begin{cases} |B_\beta(x, \xi_{m-1})\xi_\beta| & \text{- case (i),} \\ \tilde{g}(x, \xi'_{m-1}, \xi_{m-1})|\xi'_\beta| & \text{- case (ii).} \end{cases}$$

Hence we have

$$|B_\beta(x, \xi_{m-1})\xi'_\beta| \leq |B_\beta(x, \xi_{m-1})\xi_\beta| + \tilde{g}(x, \xi'_{m-1}, \xi_{m-1})|\xi'_\beta| \ ,$$

from which it follows by Assumption

$$\left|\sum_{|\beta| \leq m-1} \frac{B_\beta(x, \xi_{m-1})\xi'_\beta}{1+\varepsilon|B_\beta(x, \xi_{m-1})|}\right| \leq \sum_{|\beta| \leq m-1} \frac{|B_\beta(x, \xi_{m-1})\xi'_\beta|}{1+\varepsilon|B_\beta(x, \xi_{m-1})|} \leq$$

$$\leq \sum_{|\beta| \leq m-1} \frac{|B_\beta(x, \xi_{m-1})\xi_\beta|}{1+\varepsilon|B_\beta(x, \xi_{m-1})|} +$$

$$+ \sum_{|\beta| \leq m-1} \tilde{g}_\beta(x, \xi'_{m-1}, \xi_{m-1})|\xi'_\beta| \leq$$

$$\leq c_3 \sum_{|\beta| \leq m-1} \frac{B_\beta(x, \xi_{m-1})\xi_\beta}{1+\varepsilon|B_\beta(x, \xi_{m-1})|} + F(x, \xi_{m-1}, \xi'_{m-1}) \ ,$$

where

$$F(x, \xi_{m-1}, \xi'_{m-1}) := G(x, \xi_{m-1}) + \sum_{|\beta| \leq m-1} \tilde{g}_\beta(x, \xi'_{m-1}, \xi_{m-1})|\xi'_\beta| \ .$$

By Assumption of the Proposition and $W_{m*,p} \subset C^{(m-1)}(\overline{\Omega})$

Assumption 5(b) is satisfied.

Remark 3. (a) Suppose that for each $|\beta| \leq m-1$ the function $B_\beta(x, \xi_{m-1})$ is continuous from $\Omega \times \mathbb{R}^{s_{m-1}}$ to $\mathbb{R}^1$. Let there exist continuous nondecreasing functions $h_\beta(v)$ from $\mathbb{R}^1_+$ to $\mathbb{R}^1_+$ such that for all $x \in \Omega$ and all $\xi_{m-1} \in \mathbb{R}^{s_{m-1}}$

$$|B_\beta(x, \xi_{m-1})| \leq h_\beta(|\xi_\beta|), \; B_\beta(x, \xi_{m-1})\xi_\beta \geq 0 .$$

Then Assumption 5 is satisfied.

(b) Theorem 2 generalized most of the known results on weak solutions for nonlinear elliptic differential equations. The special case of Remark 3(a) shows that there are less restrictive growth conditions on $B_\beta(x, \xi_{m-1})$ with respect to ξ_β.

(c) The inequalities of Assumption 5(β) and Proposition 3 are related to the conditions used by Zabreiko [16], studying systems of integral equations of Hammerstein type.

Proof. Remark 3(a) follows easily by virtue of Proposition 3 and $W_{m^*,p} \subset C^{(m)}(\bar{\Omega})$.

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(Oblatum 5.9.1972)