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A GENERALIZATION OF REFLEXIVE BANACH SPACES AND WEAKLY  
COMPACT OPERATORS

Joe HOWARD, Stillwater <sup>x)</sup>

Abstract

A Banach space  $X$  is almost reflexive if every bounded sequence in  $X$  contains a weak Cauchy subsequence. A continuous linear operator  $T: X \rightarrow Y$  is a weak Cauchy operator if it maps bounded sequences of  $X$  into sequences in  $Y$  which have a weak Cauchy subsequence. A comparison of this operator with other related operators is given along with certain properties of a Banach space involving the weak Cauchy operator. Conditions are given when the weak Cauchy operator is equivalent to other related operators.

1. Preliminaries. A Banach space  $X$  is said to be almost reflexive if every bounded sequence in  $X$  contains a weak Cauchy subsequence. A weakly complete space which is almost reflexive is reflexive. A reflexive space is always almost reflexive.

Let  $X$  and  $Y$  be Banach spaces and  $T: X \rightarrow Y$  a continuous linear operator.  $T$  is said to be a weak Cauchy operator if it maps bounded sequences of  $X$  into sequences

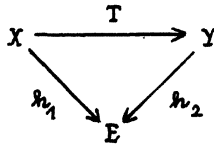
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AMS, Primary: 4610; Secondary 4601. Ref. Ž.

Key Phrases: Almost reflexive, weak Cauchy, complete continuous operator, weakly metrizable sets, Dunford - Pettis property, uc operator, weakly compactly generated.

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in  $Y$  which have a weak Cauchy subsequence.

If  $Y$  is also weakly complete, then  $T$  is weakly compact. All weakly compact operators are weak Cauchy.  $T$  is said to be a completely continuous operator if it maps weak Cauchy sequences in  $X$  into norm convergent sequences in  $Y$ .  $X$  is said to be an unconditionally converging (uc operator) if it sends every weakly unconditionally converging (wuc) series in  $X$  into an unconditionally converging (uc) series in  $Y$ .  $X$  is said to be an  $\ell_1$ -cosingular operator provided that for no Banach space  $E$  isomorphic to  $\ell_1$  does there exist epimorphisms  $h_1: X \rightarrow E$  and  $h_2: Y \rightarrow E$  such that the diagram



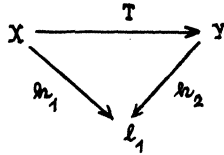
is commutative.  $T$  is  $\ell_1$ -cosingular if and only if  $T'$ , the conjugate of  $T$ , is a uc operator (see [3]).

## 2. Weak Cauchy, $\ell_1$ -cosingular, and uc operators

We now compare the operators weak Cauchy,  $\ell_1$ -cosingular, and uc.

Proposition 2.1. If  $T: X \rightarrow Y$  is weak Cauchy, then  $T$  is  $\ell_1$ -cosingular.

Proof: Assume that  $T$  is not an  $\ell_1$ -cosingular operator, i.e. that there exist epimorphisms  $h_1: X \rightarrow \ell_1$  and  $h_2: Y \rightarrow \ell_1$  such that the diagram

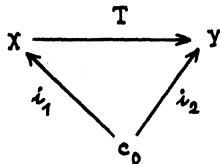


is commutative. Since  $T$  maps bounded sets into sets where every sequence has a weak Cauchy subsequence, then  $h_1 = h_2 T : X \rightarrow l_1$  must do the same. Let  $K$  denote the unit sphere of  $X$ . Since  $l_1$  is weakly complete, every sequence of  $h_1(K)$  contains a weakly convergent subsequence. Hence  $h_1$  is weakly compact, and since  $h_1$  is an epimorphism,  $l_1$  must be reflexive. This contradiction completes the proof.

Corollary 2.2. If  $T$  is weak Cauchy, then  $T'$  is a uc operator.

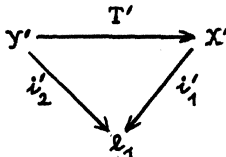
Proposition 2.3. If  $T' : Y' \rightarrow X'$  is weak Cauchy, then  $T$  is a uc operator.

Proof: Assume  $T$  is not uc. By Lemma 1 of [2], the diagram



is commutative where  $i_1$  and  $i_2$  are isomorphic embeddings.

Hence the diagram



is commutative where  $i'_1$  and  $i'_2$  are epimorphisms. Since weak Cauchy convergence implies norm convergence in  $\ell_1$ ,  $i'_1$  is completely continuous. Since  $T'$  is weak Cauchy,  $i'_2 = i'_1 T'$  is compact. Now  $i'_2$  is onto, so  $\ell_1$  is finite dimensional. This contradiction shows that  $T$  must be uc.

Remark: From [3] we know that if  $T': Y' \rightarrow X'$  is an  $\ell_1$ -cosingular operator, then  $T: X \rightarrow Y$  is a uc operator. The following example shows that the converse is not true. This example was communicated to me by A. Pelczynski.

Example 2.4. If  $T: X \rightarrow Y$  is a uc operator, then  $T'$  is not necessarily an  $\ell_1$ -cosingular operator.

Proof: Let  $X$  be a Banach space with a boundedly complete basis. Then by Theorem 1 of [4] there exists a separable space  $E$  such that  $E'' = JE + F$  where  $JE$  is the natural image of  $E$  into  $E''$  and where  $F$  is isomorphic to  $X$ .

Now put  $X = \ell_1$  and  $Y = E'$ . Since  $E''$  is separable,  $Y = E'$  is separable. Hence  $Y$  does not contain a subspace isomorphic to  $c_0$  because if a conjugate Banach space contains a subspace isomorphic to  $c_0$ , it contains a subspace isomorphic to  $m$  by Theorem 4 of [1] and hence  $Y$  could not be separable. Thus the identity operator  $I: Y \rightarrow Y$  is a uc operator but its conjugate  $I'$  is clearly not an  $\ell_1$ -cosingular operator.

Remark: The identity operator  $I: c_0 \rightarrow c_0$  is weak Cauchy and  $\ell_1$ -cosingular but not uc.  $I': \ell_1 \rightarrow \ell_1$  is uc but not weak Cauchy and not  $\ell_1$ -cosingular.  $I'': m \rightarrow m$  is

not weak Cauchy and not uc but is  $\ell_1$ -cosingular. Hence the converses of Propositions 2.1 and 2.3 are not true. Also if  $T$  is weak Cauchy, then  $T'$  is not necessarily weak Cauchy.

### 3. Weak Cauchy $V$ and weak Cauchy $V'$ properties.

We now consider spaces  $X$  which are such that the converses to Propositions 2.1 and 2.3 hold.

Definition 3.1. Let  $X$  be a Banach space.  $X$  has the weak Cauchy  $V$  property if it satisfies one of the following equivalent conditions:

- (a) For every  $B$ -space  $Y$ , every uc operator  $T: X \rightarrow Y$  is such that  $T': Y' \rightarrow X'$  is weak Cauchy.
- (b) Every subset  $K'$  of  $X'$  satisfying the condition
- (+)  $\lim_n \sup_{x' \in K'} x' x_n = 0$  for every wuc series  $\sum_n x_n$  in  $X$  has a weak Cauchy sequence.

Remark: The proof that (a) and (b) are equivalent is similar to the proof for Proposition 1 of [6].  $X$  is said to have property  $V$  if for every  $B$ -space  $Y$ , every uc operator  $T: X \rightarrow Y$  is weakly compact.  $X$  has weak Cauchy  $V$  property and  $X'$  is weakly complete if and only if  $X$  has property  $V$  (see Corollary 5 of [6]).

Proposition 3.2. Let  $X$  be weakly complete. Then  $X$  has weak Cauchy  $V$  if and only if  $X'$  is almost reflexive.

Proof: Since  $X$  is weakly complete, by Orlicz's theorem every wuc series in  $X$  is uc. Thus every bounded set in  $X'$  satisfies the condition (+). Since  $X$  has weak Cauchy  $V$ , every bounded set in  $X'$  has a weak Cauchy

sequence. So  $X'$  is almost reflexive. The converse is clear.

Definition 3.3. Let  $Y$  be a Banach space.  $Y$  has the weak Cauchy  $V'$  property if it satisfies one of the following equivalent conditions:

(c) For every  $B$ -space  $X$ , every  $\ell_1$ -cosingular operator  $T: X \rightarrow Y$  is weak Cauchy.

(d) Every subset  $K$  of  $Y$  satisfying the condition

(+ +)  $\lim_n \sup_{y \in K} \psi'_n y = 0$  for every wuc series  $\sum_n \psi'_n$  in  $Y'$  has a weak Cauchy sequence.

Remark: The proof that (c) and (d) are equivalent is similar to the proof for (a) and (b) in Definition 3.1 using the fact that  $T'$  is uc.  $Y$  is said to have property  $V'$  if for every  $B$ -space  $X$ , every  $\ell_1$ -cosingular operator  $T: X \rightarrow Y$  is weakly compact.  $Y$  has weak Cauchy  $V'$  and  $Y$  is weakly complete if and only if  $Y$  has property  $V'$  (see Proposition 6 of [6]).

Proposition 3.4. Let  $Y'$  be weakly complete. Then  $Y$  has weak Cauchy  $V'$  if and only if  $Y$  is almost reflexive.

Proof: The proof is similar to the proof of Proposition 3.2.

Remark: By following [6], we have the following:

(A) Let  $X$  have weak Cauchy  $V'$  property. Then  $X$  is almost reflexive if and only if no complemented subspace of  $X$  is isomorphic to  $\ell_1$ . (B) Let  $X$  have weak Cauchy  $V$  property. Then  $X'$  is almost reflexive if and only if no subspace of  $X$  is isomorphic to  $c_0$ .

Proposition 3.5. If  $X$  has weak Cauchy  $V$  then  $X'$

has weak Cauchy  $V'$ ; if  $X'$  has weak Cauchy  $V$  then  $X$  has weak Cauchy  $V'$ .

Proof: The proof follows from Definitions 3.1 and 3.3.

Remark: We show that the converses of Proposition 3.5 are not true. For properties  $V$  and  $V'$  this is not known (see [6]).

Example 3.6. If  $X'$  has weak Cauchy  $V'$ , then  $X$  does not necessarily have weak Cauchy  $V$ .

Proof: Consider the space  $X = E'$  given in Example 2.4. Since  $I : E' \rightarrow E'$  is uc but  $I' : E'' \rightarrow E''$  is not weak Cauchy,  $X = E'$  does not have weak Cauchy  $V$ . But  $X' = E'' = JE + F$  where  $F$  is isomorphic to  $\ell_1$  and both  $E$  and  $\ell_1$  have weak Cauchy  $V'$  property. Therefore  $X'$  has weak Cauchy  $V'$ .

Example 3.7. If  $X$  has weak Cauchy  $V'$ , then  $X'$  does not necessarily have weak Cauchy  $V$ .

Proof: Consider the space  $X = E'$  as given in Example 3.6. Since  $E''$  is separable,  $E'$  is almost reflexive; therefore  $X = E'$  has weak Cauchy  $V'$  property. Since  $I : E'' \rightarrow E''$  is uc but  $I'$  is not weak Cauchy,  $X' = E''$  does not have the weak Cauchy  $V$  property.

Remark: The B-space  $E$  is an example which has weak Cauchy  $V$  but not property  $V$ . Also  $E$  has weak Cauchy  $V'$  but not property  $V'$ .

#### 4. Dunford - Pettis property

A Banach space  $X$  is said to have the Dunford-Pettis (D.P.) property provided that for every Banach space  $Y$ ,



every weakly compact linear operator  $T: X \rightarrow Y$  is completely continuous.

Theorem 4.1. Let  $X$  be a Banach space.  $X$  has the D.P. property if and only if for every  $B$ -space  $Y$ , every weak Cauchy operator  $T': Y' \rightarrow X'$  is such that  $T$  is completely continuous.

Proof: ( $\Leftarrow$ ) This follows since if  $T$  is weakly compact,  $T'$  is weakly compact and hence  $T'$  is weak Cauchy.

( $\Rightarrow$ ) It suffices to show for every  $B$ -space  $Y$ , if  $T'$  is weak Cauchy, then  $\lim_n \|Tx_n\| = 0$  for every weakly convergent to  $0$  sequence  $\{x_n\}$ . Let  $\overline{\lim}_n \|Tx_n\| = \sigma \geq 0$ . Let  $\psi'_n$  with  $\|\psi'_n\| = 1$  be such that  $\psi'_n(Tx_n) = \|Tx_n\|$  for all  $n$ . Put  $x'_n = T'\psi'_n$ . Thus w.l.o.g. we assume  $\{x'_n\}$  is a weak Cauchy sequence. We have

$$\overline{\lim}_n x'_n x_n = \overline{\lim}_n (T'\psi'_n) x_n = \overline{\lim}_n \psi'_n(Tx_n) = \overline{\lim}_n \|Tx_n\| = \sigma.$$

We now show  $\sigma = 0$  where  $\lim |x'_n x_n| = \sigma$ . Let  $\{m\}$  be a subsequence of  $\{n\}$  such that  $|x'_m x_m| \leq \sigma/2$ . Since  $\{x_m\}$  weakly converges to  $0$  such a subsequence  $\{m\}$  exists.

We have

$$x'_m x_m = (x'_m - x'_n) x_m + x'_n x_m.$$

Since  $\{x'_m - x'_n\}$  weakly converges to  $0$ , we obtain

$$\sigma = \lim_m |x'_m x_m| \leq \overline{\lim}_m |(x'_m - x'_n) x_m| + \overline{\lim}_m |x'_n x_m| \leq \sigma/2.$$

Thus  $\sigma = 0$ .

Corollary 4.2. Suppose  $X$  or  $X'$  has D.P. property and  $X'$  is almost reflexive. Then a sequence in  $X$  is weak Cauchy if and only if it is norm Cauchy.

Proof: If  $X'$  has D.P. property then so does  $X$  (see [7]); so it suffices to take  $X$  with the D.P. property. Since  $X'$  is almost reflexive,  $I': X' \rightarrow X'$  is weak Cauchy. By Theorem 4.1,  $I: X \rightarrow X$  is completely continuous and the result follows.

Corollary 4.3. Let  $X$  have weak Cauchy  $Y$  and D.P. properties, and let  $T: X \rightarrow Y$ . Then the following are equivalent.

- (a)  $T$  is uc,
- (b)  $T'$  is weak Cauchy,
- (c)  $T$  is completely continuous,
- (d)  $T'$  is  $\ell_1$ -cosingular.

Proof: (a)  $\implies$  (b)  $\implies$  (c) is clear. (c)  $\implies$  (a) follows from Proposition 1.9 of [2]. To complete the proof it suffices to show (b)  $\implies$  (d)  $\implies$  (a). Now (b)  $\implies$  (d) follows from Proposition 2.1 and (d)  $\implies$  (a) is found in [3].

We now consider somewhat a dual notion for the D.P. property.

Theorem 4.4. Let  $Y$  be a Banach space.  $Y$  has the D.P. property if and only if for every  $B$ -space  $X$ , every weak Cauchy operator  $T: X \rightarrow Y$  is such that  $T'$  is completely continuous.

Proof: ( $\Leftarrow$ ) By [7] it suffices to show for every weakly convergent to 0 sequence  $\{y_n\}$  in  $Y$  and for every weakly convergent to 0 sequence  $\{y'_n\}$  in  $Y'$ ,  $\lim_n y'_n y_n = 0$ . Let  $\{y_n\}$  be an arbitrary weakly convergent to 0 sequence in  $Y$ . Consider the linear operator  $T: c_0 \rightarrow Y$  with  $Te_n = y_n$  where  $e_n$  denotes the  $n$ -th unit vector in

$c_0$ . Then  $T': Y' \rightarrow \ell_1$  is completely continuous. By the properties of  $T'$ ,  $T'y'(e_n) = y'(Te_n) = y'(y_n)$  for every  $y'$  in  $Y'$ . Now let  $\{y'_m\}$  be an arbitrary sequence in  $Y'$  weakly convergent to 0. Then  $0 = \lim_n \|T'y'_m\| = \lim_n \sup_n |y'_m(y_n)|$ . Hence  $\lim_n y'_m y_n = 0$  and so  $Y$  has the D.P. property.

( $\implies$ ) Suppose  $T: X \rightarrow Y$  is weak Cauchy and  $Y$  has D.P. property. It suffices to show  $\lim_n \|T'y'_m\| = 0$  for every weakly convergent to 0 sequence  $\{y'_m\}$ . Let  $x_n$  with  $\|x_n\| = 1$  be such that  $T'y'_m(x_n) = \|T'y'_m\|$  for all  $n$ . Put  $y_n = Tx_n$ . The rest of the proof is analogous to that given in the proof of Theorem 4.1.

Corollary 4.5. If  $X$  is almost reflexive and  $X$  or  $X'$  has D.P. property, then a sequence in  $X'$  is weak Cauchy if and only if it is norm Cauchy.

Remark: The proof of Corollary 4.5 is similar to that of Corollary 4.2. Using Corollaries 4.2 and 4.5 we have that if  $X'$  has D.P. and is almost reflexive, then weak Cauchy sequences correspond to norm Cauchy sequences in both  $X$  and  $X''$ .

Corollary 4.6. Let  $Y$  have weak Cauchy  $Y'$  and D.P. properties, and let  $T: X \rightarrow Y$ . Then the following are equivalent.

- (a)  $T'$  is uc.
- (b)  $T$  is  $\ell_1$ -cosingular.
- (c)  $T$  is weak Cauchy.
- (d)  $T'$  is completely continuous.

Proof: (a)  $\implies$  (b)  $\implies$  (c)  $\implies$  (d) is clear. (d)  $\implies$  (a) follows from Proposition 1.9 of [2].

Corollary 4.7. Let  $X$  have weak Cauchy  $V$ ,  $X'$  have D.P. property,  $T: X \rightarrow Y$  and  $T': Y' \rightarrow X'$ . The following are equivalent.

- (a)  $T$  is uc.
- (b)  $T'$  is weak Cauchy.
- (c)  $T$  is completely continuous.
- (d)  $T'$  is  $\mathcal{L}_1$ -cosingular.
- (e)  $T''$  is uc.
- (f)  $T''$  is completely continuous.

Proof: The proof follows easily using Corollaries 4.3 and 4.6, Proposition 3.5, and the fact that  $X'$  has D.P. implies  $X$  has D.P.

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