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A NOTE ON VOLTERRA INTEGRAL EQUATIONS WITH DEGENERATE KERNEL

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In the paper several relations between the linear vector - valued Volterra integral equation

$$(I) \qquad x(t) = a(t) + \int_0^t B(t,s)x(s)\,ds$$

and the initial - value problem

$$(D) \qquad \begin{aligned} \dot{x} - P(t)x &= q(t) \, , \\ x(0) &= x_0 \end{aligned}$$

are investigated. Particularly it is shown that under some weak assumptions the following three assertions are equivalent:

(i) the kernel B of the equation (I) is degenerate;

(ii) there exists a matrix $P(t)$ such that the function $B(\cdot, s)$ satisfies the equation in (D) with $q = 0$;

(iii) the solution of the equation (I) satisfies some special initial - value problem of the type (D).

Analogous results are obtained for the case of an

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initial - value problem for a differential equation of a higher order.

The results generalize those obtained by J. Nagy and E. Nováková in [2] for a special type of the kernel B .

1. Notation. Let for $m, n = 1, 2, \ldots$ $R^{m \times n}$ ($K^{m \times n}$) denote the space of all real (complex) matrices of the type $m \times n$. The m -dimensional vectors will be identified with the column matrices (of the type $m \times 1$) for $m = 1, 2, \ldots$, and R^m, K^m will stand for $R^{m \times 1}$, $K^{m \times 1}$ respectively. Analogously for vector valued functions. We shall denote the identity matrices by I and the zero matrices by 0 .

Let $G \subset R^n$ be a domain in R^n , let $\bar{G}$ be the closure of G . Then $C^{(k)}_{m \times n}(G)$ for $m, n = 1, 2, \ldots$; $k = 0, 1, 2, \ldots$ denotes the space of all $m \times n$ complex k-times continuously differentiable matrix-valued functions on $\bar{G}$. (The function is 0-times continuously differentiable if it is continuous; we define the 0-th derivative of a given function to be equal to the function itself.)

Let $m_i > 0$, $n_j > 0$ for $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$ be integers, $Y_{ij} \in K^{m_i \times n_j}$. We shall identify the matrix

$$\begin{bmatrix} Y_{11} & Y_{12} & \ldots & Y_{1q} \\ Y_{21} & Y_{22} & \ldots & Y_{2q} \\ \ldots & \ldots & \ldots & \ldots \\ Y_{p1} & Y_{p2} & \ldots & Y_{pq} \end{bmatrix}$$

with the corresponding element of $K^{M\times N}$, where $M = m_1 + m_2 + \dots + m_p$, $N = n_1 + n_2 + \dots + n_q$.

The partial derivatives of a function f with the domain in R^p will be denoted by

$$D^i f(u) = D^{i_1,\dots,i_p} f(u) = \frac{\partial^{i_1+\dots i_p} f(u_1,\dots,u_p)}{\partial u_1^{i_1}\dots\partial u_p^{i_p}}$$

where $i = (i_1,\dots,i_p)$ denotes some multiindex, $p = 1,2,\dots$. Further, the set $\{[t,s] \in R^2 : t \geq s \geq 0\}$ will be denoted by Δ and the interval $\langle 0, \infty)$ by R_+. Finally, in what follows, the symbols m, n will stand for integers, $m \geq 1$, $n \geq 1$, $k \geq 0$ and P, B will be elements of $C^{(0)}_{m\times n}(R_+)$, $C^{(0)}_{n\times n}(\Delta)$ respectively.

2. Problem. The main purpose of the paper is to find some assumptions on the kernel B and the forcing function a so that the solution of the Volterra integral equation

$$\text{(I)} \qquad x(t) = a(t) + \int_0^t B(t,s)\,x(s)\,ds\,, \quad t \geq 0$$

may satisfy some special initial value problem for an ordinary differential equation.

The following theorem is well known.

3. Theorem. Let $P \in C^{(0)}_{n\times n}(R_+)$, $q \in C^{(0)}_{n\times m}(R_+)$, $s \geq 0$, $x_s \in K^{n\times m}$. Then there exists a unique solution $x \in C^{(1)}_{n\times m}(\langle s, \infty))$ of the initial value problem

$$\text{(D)}\quad \begin{aligned} &(3.1) \quad \dot{x} - P(t)x = q(t), \quad t > s\,, \\ &(3.2) \quad x(s) = x_s\,. \end{aligned}$$

4. Remark. The following well known variation of constants formula:

$$(4.1)\quad x(t) = H(t)H(s)^{-1}x_s + \int_s^t H(t)H(u)^{-1}q(u)\,du, \quad t \geq s,$$

holds for the solution x of (D) where $H \in C^{(1)}_{m\times m}(R_+)$ is the solution of the square-matrix initial value problem (K is a regular square matrix)

$$(4.2)\qquad \dot{X} = P(t)X, \quad X(0) = K \ .$$

The following theorem holds for the equation (I). (See R.K. Miller [1].)

5. Theorem. Let $a \in C^{(k)}_{m\times m}(R_+)$, $B \in C^{(k)}_{m\times m}(\Delta)$. Then there exists a unique solution $x \in C^{(k)}_{m\times m}(R_+)$ of the equation (I), which is given by

$$(5.1)\qquad x(t) = a(t) + \int_0^t R(t,s)a(s)\,ds, \quad t \geq 0,$$

where R is the resolvent kernel of the kernel B . This kernel R is the unique solution of the resolvent equation

$$(5.2)\quad R(t,s) = B(t,s) + \int_s^t B(t,u)R(u,s)\,du, \quad 0 \leq s \leq t \ .$$

6. Remark. In what follows we shall be interested especially in the case of degenerate kernels, i.e. kernels B of the form

$$(6.1)\qquad B(t,s) = [\, b_{ij}(t,s)\,]_{i,j=1}^{m}$$

with $b_{ij}(t,s) = u_{ij}(t)v_{ij}(s)$; $t \geq s \geq 0$; $i,j = 1,2,\dots,m$; u_{ij}, v_{ij} being some sufficiently many times continuously differentiable matrix functions of the type $1 \times k_{ij}$, $k_{ij} \times 1$ respectively, defined on R_+.

7. Lemma. Let $B \in C^{(k)}_{n \times n}(\Delta)$ be the degenerate kernel (6,1). Then there exist an integer $m \geq 1$ and $U \in C^{(k)}_{n \times m}(R_+)$, $V \in C^{(k)}_{m \times n}(R_+)$ so that

$$B(t,s) = U(t)V(s)\,, \quad t \geq s \geq 0 \,. \tag{7.1}$$

It is possible to choose $m \geq n$ and the matrix U in the form

$$U = [I, U_1] \,.$$

Proof. Obviously, we can choose U in the form

$$U = \left[I \;\; \begin{matrix} u_{11} & u_{12} \dots u_{1n} & 0 & 0 \;\dots 0 & \dots 0 & 0 \;\dots 0 \\ 0 & 0 \;\dots 0 & u_{21} & u_{22} \dots u_{2n} & \dots 0 & 0 \;\dots 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 \;\dots 0 & 0 & 0 \;\dots 0 & \dots u_{n1} & u_{n2} \dots u_{nn} \end{matrix} \right]$$

and the transposed matrix V^T of the matrix V in the form

$$V^T = \left[0 \;\; \begin{matrix} v^T_{11} & 0 \;\dots 0 & v^T_{21} & 0 \;\dots 0 & \dots v^T_{n1} & 0 \;\dots 0 \\ 0 & v^T_{12} \dots 0 & 0 & v^T_{22} \dots 0 & \dots 0 & v^T_{n2} \dots 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 \;\dots v^T_{1n} & 0 & 0 \;\dots v^T_{2n} & \dots 0 & 0 \;\dots v^T_{nn} \end{matrix} \right].$$

So we obtain

$$m = n + \sum_{i,j=1}^{n} k_{ij} \,.$$

8. Theorem. Let $\mathfrak{B} \in C^{(k)}_{m \times m}(\Delta)$ be the degenerate kernel (6.1), $a \in C^{(0)}_{m}(R_{+})$. Then there exist an integer p, $p > m$ and $\tilde{\mathfrak{U}}, \tilde{\mathcal{V}} \in C^{(k)}_{p \times p}(R_{+})$ such that $\mathfrak{U}(t)$ is a regular square-matrix for all $t \geq 0$ and the following assertion holds:

Let us define $\tilde{\mathfrak{B}} \in C^{(k)}_{p \times p}(\Delta)$, $\tilde{a} \in C^{(0)}_{p}(R_{+})$ by means of

$$\begin{aligned} (8.1) \qquad \tilde{\mathfrak{B}}(t,s) &= \tilde{\mathfrak{U}}(t)\tilde{\mathcal{V}}(s), \quad t \geq s \geq 0, \\ \tilde{a} &= \begin{bmatrix} a \\ 0 \end{bmatrix}. \end{aligned}$$

Let $\tilde{x} \in C^{(0)}_{p}(R_{+})$, $x \in C^{(0)}_{m}(R_{+})$, $y \in C^{(0)}_{p-m}(R_{+})$ and let it hold

$$\tilde{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then

(i) if $\tilde{x}$ is a solution of the equation

$$(8.2) \quad \tilde{x}(t) = \tilde{a}(t) + \int_0^t \tilde{\mathfrak{B}}(t,s)\tilde{x}(s)\,ds, \quad t \geq 0,$$

then x is a solution of (I);

(ii) if x is a solution of (I) then there exists such a $y \in C^{(0)}_{p-m}(R_{+})$ that $\tilde{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ is a solution of (8.2).

Proof. We can put

$$\tilde{\mathfrak{U}} = \begin{bmatrix} \mathfrak{U} & I \\ I & 0 \end{bmatrix} \in C^{(k)}_{(m+n)\times(m+n)}(R_{+}),$$

$$\tilde{\mathcal{V}} = \begin{bmatrix} \mathcal{V} & 0 \\ 0 & 0 \end{bmatrix} \in C^{(k)}_{(m+n)\times(m+n)}(R_{+}),$$

where $\mathfrak{U}, \mathcal{V}$ are matrices of the types $n \times m$, $m \times n$ respectively described in Lemma 7. Then $p = n + m$,

$$\tilde{B}(t,s) = \begin{bmatrix} B(t,s) & 0 \\ V(s) & 0 \end{bmatrix}$$

and (i) holds. Let x be the solution of (I). Set

$$y(t) = \int_0^t V(s)x(s)ds, \quad t \geq 0 .$$

Then $\tilde{x}$ satisfies (8.2) and (ii) holds as well.

9. <u>Remark</u>. For some special kernels B the conclusion of Theorem 8 (or, more precisely, its easy modification) holds with $p = m$.

Lemma 7 asserts that each degenerate kernel may be expressed in the form (7.1). So we shall pay attention only to the degenerate kernels of this type. From Theorem 8 it follows that each equation (8.2) with a degenerate kernel may be complemented so that the equation (8.2) with the kernel (8.1) will be obtained. Therefore it is sufficient to consider only such equations (I) with a degenerate kernel where the kernel B is of the form (7.1) with a regular square matrix U .

10. <u>Theorem</u>. Let $U \in C^{(0)}_{m\times m}(R_+)$, $V \in C^{(0)}_{m\times m}(R_+)$, $B(t,s) = U(t)V(s)$, $t \geq s \geq 0$ and let $E \in C^{(1)}_{m\times m}(R_+)$ be the solution of the matrix initial value problem

$$\dot{E} = V(t)U(t)E , \quad E(0) = I . \tag{10.1}$$

Then the function

$$R(t,s) = U(t)E(t)E(s)^{-1}V(s), \quad t \geq s \geq 0 \tag{10.2}$$

is the resolvent kernel of the kernel B .

Proof. Clearly

$$\int_s^t B(t,u)R(u,s)\,du = \int_s^t U(t)V(u)U(u)E(u)E(s)^{-1}V(s)\,du =$$

$$= U(t)\int_s^t V(u)U(u)E(u)\,du\, E(s)^{-1}V(s) = U(t)\int_s^t \dot{E}(u)\,du\, E(s)^{-1}V(s) =$$

$$= U(t)[E(t) - E(s)]\,E(s)^{-1}V(s) = R(t,s) - B(t,s) \ ,$$

$$t \geq s \geq 0 \ ,$$

so that R satisfies the resolvent equation (5.2).

11. Theorem. Let $B \in C^{(1)}_{m\times m}(\Delta)$.

Then the following three assertions are equivalent:

(i) there exist $U \in C^{(1)}_{m\times m}(R_+)$ regular on R_+ and $V \in C^{(1)}_{m\times m}(R_+)$ so that

(11.1) $\qquad B(t,s) = U(t)V(s) \ , \quad t \geq s \geq 0 \ ;$

(ii) there exists $P \in C^{(0)}_{m\times m}(R_+)$ (which is uniquely determined by B) so that

(11.2) $\qquad D^{1,0}B(t,s) - P(t)B(t,s) = 0, \quad t \geq s \geq 0 \ ;$

(iii) there exists $P \in C^{(0)}_{m\times m}(R_+)$ (which is uniquely determined by B) so that for all $a \in C^{(1)}_m(R_+)$ the solution x of the equation (I) satisfies the initial value problem

(11.3) $\qquad \dot{x} - [P(t) + B(t,t)]\,x = \dot{a}(t) - P(t)a(t) \ ,$

(11.4) $\qquad x(0) = a(0)$

(This P is the same as that in (ii).)

Proof. (i) $\Longrightarrow$ (ii) From (11.1) it follows

$$D^{1,0}B(t,s) = \dot{U}(t)V(s) = \dot{U}(t)U(t)^{-1}U(t)V(s) = P(t)B(t,s)$$

where $P(t) = \dot{U}(t)U(t)^{-1}$, $t \geq 0$.

(ii) $\Longrightarrow$ (i) Let H be the fundamental matrix of the system $\dot{x} = P(t)x$. Then (11.2) and Theorem 4 imply (11.1), where $U(s) = H(s)$, $V(s) = H(s)^{-1}B(s,s)$, $s \geq 0$.

(ii) $\Longrightarrow$ (iii) Let x be the solution of (I); a , B continuously differentiable. Then

$$\dot{x}(t) = \dot{a}(t) + B(t,t) + \int_0^t D^{1,0}B(t,s)\,x(s)\,ds , \quad t \geq 0 .$$

Simple calculation gives

$$(11.5) \quad \dot{x}(t) - [P(t) + B(t,t)]x(t) = \int_0^t [D^{1,0}B(t,s) - P(t)B(t,s)]x(s)\,ds + \dot{a}(t) - P(t)a(t), \quad t \geq 0 .$$

Now (11.3) follows from (11.2) and (11.5).

(iii) $\Longrightarrow$ (ii) Let the solution x of (I) satisfy the initial value problem (11.3-4). Then (11.5) holds. Hence and from (11.3) we obtain

$$(11.6) \quad \int_0^t [D^{1,0}B(t,s) - P(t)B(t,s)]x(s)\,ds = 0 , \quad t \geq 0 .$$

From the equation (I) it follows that for each $x \in C_m^{(1)}(R_+)$ there exists $a \in C_m^{(1)}(R_+)$ so that x is the solution of (I). So (11.6) holds for all $x \in C_m^{(1)}(R_+)$ and (11.2) is satisfied.

12. Remark. Theorems 8 and 11 imply the following assertion for a degenerate kernel $B \in C^{(1)}_{m\times m}(\Delta)$ and $a \in C_m^{(1)}(R_+)$. It is always possible to complement the matrix B and the forcing function a in (I) so that the new kernel satisfies the equation of the form

(11.2) and the solution of the complemented equation satisfies the initial value problem of the form (11.3-4).

It also follows from Theorems 7 and 10 that the resolvent kernel $\mathbf{R}$ of a smooth degenerate kernel $\mathbf{B}$ is given by (10.2).

13. Remark. Theorems 10 and 11 imply immediately: if a kernel $\mathbf{B}$ fulfils the equation (11.2) then

(i) $\mathbf{B}$ is degenerate;

(ii) a solution of (I) (with smooth a) is also a solution of (11.3-4);

(iii) the resolvent kernel $\mathbf{R}$ may be written in the form (10.2).

The investigations described above may be modified and generalized in many ways. One of such modifications will be described now.

14. Theorem. Let $A_0, A_1, \dots, A_k \in C^{(0)}_{m\times m}(R_+)$, $A_k = I$, $B \in C^{(k)}_{m\times m}(\Delta)$, $a \in C^{(k)}_m(R_+)$.

Let for all $s \geq 0$ the function $B(\cdot, s)$ satisfy the equation

$$(14.0) \qquad \sum_{i=0}^{k} A_i(t) D^i x(t) = 0$$

on $\langle s, \infty)$.

Then

(i) the kernel B is degenerate;

(ii) the function $x \in C^{(k)}_m(R_+)$ is a solution of (I) if and only if it is a solution of the initial value problem

$$(14.1) \qquad \sum_{\ell=0}^{k} F_\ell(t) D^\ell x(t) = q(t), \quad t > 0 ,$$

$$(14.2) \quad D^i x(0) - \sum_{\ell=0}^{i-1} G_{i\ell}(0) D^\ell x(0) = D^i a(0); \quad i = 0,1,\dots,k-1 ;$$

where

$$(14.3) \quad F_\ell(t) = A_\ell(t) - \sum_{r=0}^{k-\ell-1} \binom{r+\ell}{\ell} \sum_{j=0}^{k-\ell-r-1} A_{\ell+r+j+1}(t) \times \times D^r D^{j,0} B(t,t) ; \quad \ell = 0, 1, \dots, k ;$$

$$(14.4) \qquad q(t) = \sum_{i=0}^{k} A_i(t) D^i a(t); \quad t \geq 0 ,$$

$$(14.5) \quad G_{i\ell}(t) = \sum_{j=0}^{i-\ell-1} \binom{i-j-1}{\ell} D^{i-j-\ell-1} D^{j,0} B(t,t), \quad t \geq 0 ; \quad \ell = 0, 1, \dots, i-1 ; \quad i = 0, 1, \dots, k ;$$

and where we set $\sum_{i=0}^{p} \dots = 0$ whenever $p < 0$.

Proof. We prove the assertion (i). Let us introduce the matrix functions

$$\tilde{x} = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(k-1)} \end{bmatrix} , \qquad \tilde{A} = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -A_0 & -A_1 & -A_2 & \dots & -A_{k-1} \end{bmatrix} .$$

Clearly $\tilde{A} \in C^{(0)}_{kn \times kn}(R_+)$ and $x \in C^{(k)}_n(R_+)$ satisfies the equation (14.1) if and only if $\tilde{x} \in C^{(1)}_{kn}(R_+)$ and $\tilde{x}$ is the solution

$$\dot{\tilde{x}} = \tilde{A}(t) \tilde{x} .$$

If $\widetilde{H}$ is the fundamental matrix of the last equation, then its solution $\widetilde{x}$ may be written in the form

$$\widetilde{x}(t) = \widetilde{H}(t)\widetilde{x}(0) \ , \ t \geq 0 \ .$$

Setting

$$\widetilde{H} = \begin{bmatrix} H_0 \\ H_1 \end{bmatrix}$$

where the sub-matrix $H_0 \in C^{(1)}_{m \times km}(R_+)$, we get for the solution x of the equation (14.0)

$$x(t) = H_0(t)\widetilde{x}(0), \quad t \geq 0 \ .$$

Using this and the assumptions of the theorem we obtain

$$B(t,s) = H_0(t)\widetilde{H}(s)^{-1} \begin{bmatrix} B(s,s) \\ D^{1,0}B(s,s) \\ \vdots \\ D^{k-1,0}B(s,s) \end{bmatrix} , \ t \geq s \geq 0$$

so that B is degenerate and the assertion (i) holds.

Now we shall consider (ii). From (14.3) it follows $F_k = A_k = 1$. The values $x(0), Dx(0), \dots, D^{k-1}x(0)$ are uniquely defined by (14.2). It is possible to transform the equation (14.1) into the first order differential equation as we transformed the equation (14.0) above. Hence and from Theorem 3 it follows that the initial value problem (14.1-2) is uniquely solvable in $C^{(k)}_m(R_+)$.

Let x be the solution of the equation (I). Let us express the derivatives $D^i x$ in the form

(14.6) $D^i x(t) = D^i a(t) + \sigma^i x(t) + \int_0^t D^{i,0}B(t,s)x(s)ds;$

$i = 0, 1, \dots, k$. Differentiating the boths sides of the equation (I) and using (14.5-6) we obtain

$$(14.7)\quad \delta^{i} x(t) = \sum_{j=0}^{i-1} D^{i-j-1}[(D^{j,0}B(t,t))x(t)] = $$
$$= \sum_{\ell=0}^{i-1} G_{i\ell}(t) D^{\ell} x(t) \; ; \quad t \geq 0 \; ; \quad i = 0, 1, \dots, k \; ;$$

$$(14.8)\quad \sum_{i=0}^{k} A_i(t)\delta^{i} x(t) = \sum_{i=0}^{k} A_i(t) \sum_{\ell=0}^{i-1} G_{i\ell}(t) D^{\ell} x(t) =$$

$$= \sum_{\ell=0}^{k} [A_\ell(t) - F_\ell(t)] D^{\ell} x(t) \, , \quad x \geq 0 \; .$$

One has

$$\sum_{i=0}^{k} A_i(t) D^{i} x(t) = \sum_{i=0}^{k} A_i(t) D^{i} a(t) + \sum_{i=0}^{k} A_i(t) \delta^{i} x(t) +$$
$$+ \int_0^t \sum_{i=0}^{k} A_i(t) D^{i,0} B(t,s) x(s) ds \, , \quad t \geq 0 \; .$$

Since $B(\cdot, s)$ is the solution of (14.1), the last term equals zero. Hence using (14.8), we obtain (14.1), where F_ℓ, q are defined by means of (14.3-4) respectively. Putting $t = 0$ in (14.6) and using (14.7) we obtain the initial conditions (14.2).

Conversely, since the solution of the initial value problem (14.1-2) is unique and the solution of (I) exists, it follows from the above argument that the solution of the initial value problem (14.1-2) solves (I).

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R e f e r e n c e s

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