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Tomáš Kepka

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THE TOTALLY SIMPLE QUASIGROUPS

Tomáš KEPKA, Praha

Basic definitions used in this paper can be found in [1] or [2].

Let Q be a quasigroup and ρ a relation on the set Q. We shall define the following conditions:

(1) $\forall a, b, c \in Q,\ a \rho b \Rightarrow ac \rho bc$.

(2) $\forall a, b, c \in Q,\ a \rho b \Rightarrow ca \rho cb$.

(3) $\forall a, b, c \in Q,\ ac \rho bc \Rightarrow a \rho b$.

(4) $\forall a, b, c \in Q,\ ca \rho cb \Rightarrow a \rho b$.

(5) $\forall a, b, c, d \in Q,\ (ac = bd \text{ et } c \rho d) \Rightarrow a \rho b$.

(6) $\forall a, b, c, d \in Q,\ (ac = bd \text{ et } a \rho b) \Rightarrow c \rho d$.

(7) $\forall a, b, c, d \in Q,\ (a \rho b \text{ et } c \rho d) \Rightarrow ac \rho bd$.

(8) $\forall a, b, c, d \in Q,\ (ac \rho bd \text{ et } c \rho d) \Rightarrow a \rho b$.

(9) $\forall a, b, c, d \in Q,\ (ac \rho bd \text{ et } a \rho b) \Rightarrow c \rho d$.

Let M be a set. By δ_M, we shall denote the set of all pairs (a, a), $a \in M$. Further denote $\pi_M = M \times M$ and $\sigma_M = \pi_M \setminus \delta_M$; that is, σ_M consists of all prdered pairs (a, b), where $a, b \in M$, $a \neq b$. Hence

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δ_M , π_M and σ_M are relations on the set M .

If Q is a quasigroup then it is evident that the relations δ_Q , π_Q , σ_Q satisfy the conditions (1),(2),(3),(4),(5),(6). Moreover, δ_Q and π_Q satisfy (7),(8),(9). Every equivalence relation $\wp$ on Q that satisfies (7),(8),(9) is called a normal congruence relation.

A quasigroup Q is called simple if every its normal congruence relation is equal to one of the relations δ_Q , π_Q . A quasigroup Q is called totally simple if every its relation $\wp$ that satisfies at least one of the conditions (1),(2),(3),(4),(5),(6) is equal to one of the relations δ_Q , π_Q , σ_Q . Evidently, every totally simple quasigroup is simple. In this paper we shall prove that every quasigroup can be imbedded in a totally simple quasigroup.

<u>Lemma 1</u>. Let $Q(*)$ be the right inverse quasigroup of a quasigroup Q . Let $\wp$ be a relation on the set Q . Then:

(i) $\wp$ satisfies (1) on Q if and only if $\wp$ satisfies (6) on $Q(*)$.

(ii) $\wp$ satisfies (2) on Q if and only if $\wp$ satisfies (4) on $Q(*)$.

(iii) $\wp$ satisfies (3) on Q if and only if $\wp$ satisfies (5) on $Q(*)$

<u>Proof</u>. (i) Let $\wp$ satisfy (1) on Q and let $a, b, c, d \in Q$ be such that $a * c = b * d$ and $a \wp b$. Put $x = a * c$. Hence $ax = c$, $bx = d$. Since $a \wp b$, $ax \wp bx$. Thus $\wp$ satisfies (6) on $Q(*)$. Again,

let ϱ satisfy (6) on $Q(*)$ and $a, b, c \in Q$ be such that $a \varrho b$. There are $x, y \in Q$ such that $a * x = b * y = c$. Since ϱ satisfies (6), $x \varrho y$. But $x = ac$, $y = bc$. The proof of (ii) and (iii) is similar to that of (i).

<u>Lemma 2</u>. Let $Q(o)$ be the left inverse quasigroup of a quasigroup Q. Let ϱ be a relation on the set Q. Then:
(i) ϱ satisfies (1) on Q if and only if ϱ satisfies (3) on $Q(o)$.
(ii) ϱ satisfies (2) on Q if and only if ϱ satisfies (5) on $Q(o)$.
(iii) ϱ satisfies (4) on Q if and only if ϱ satisfies (6) on $Q(o)$.

<u>Proof</u>. Similarly as for Lemma 1.

<u>Lemma 3</u>. For every quasigroup Q there is a quasigroup $\overline{Q}$ such that Q is a subquasigroup of $\overline{Q}$ and if $a, b, c, d \in Q$, $a \neq b$, $c \neq d$, then there is $x \in \overline{Q}$ such that $ax \cdot x = c$, $bx \cdot x = d$.

<u>Proof</u>. For every $a, b, c, d \in Q$, $a \neq b$, $c \neq d$, select (pair-wise different) symbols $x(a, b, c, d)$, $y(a, b, c, d)$, $u(a, b, c, d)$. Let H be the set consisting of all these symbols and of all elements of Q. We shall define a partial binary operation $*$ on H. Let $m, n \in H$. Then $m * n$ is defined only in the following cases:

(i) If $m, n \in Q$. Then $m * n = mn$.

(ii) If $m \in Q$ and if there are

$a, b, c, d \in Q$, $a \neq b$, $c \neq d$,

such that $m = a$, $n = x(a, b, c, d)$. Then

$m * n = y(a, b, c, d)$.

(iii) If $m \in Q$ and if there are

$a, b, c, d \in Q$, $a \neq b$, $c \neq d$,

such that $m = b$, $n = x(a, b, c, d)$. Then

$m * n = u(a, b, c, d)$.

(iv) If there are $a, b, c, d \in Q$, $a \neq b$, $c \neq d$, such that $m = y(a, b, c, d)$, $n = x(a, b, c, d)$. Then $m * n = c$.

(v) If there are $a, b, c, d \in Q$, $a \neq b$, $c \neq d$, such that $m = u(a, b, c, d)$, $n = x(a, b, c, d)$. Then $m * n = d$.

It is easy to show that $H(*)$ is a cancellation halfgroupoid. Hence $H(*)$ can be imbedded in a quasigroup $\overline{Q}(*)$. Evidently, Q is a subquasigroup in $\overline{Q}(*)$. Let $a, b, c, d \in Q$, $a \neq b$, $c \neq d$. Put $x = x(a, b, c, d)$. Then $(a * x) * x = c$, $(b * x) * x = d$.

__Theorem 1__. Let Q be a quasigroup. Then Q can be imbedded in a quasigroup P having the following property: If $a, b, c, d \in P$, $a \neq b$, $c \neq d$, then there is $x \in P$ such that $ax \cdot x = c$, $bx \cdot x = d$.

__Proof__. Put $Q_1 = Q$ and $Q_{i+1} = Q_i$ for $i = 1, 2, 3, \ldots$ ($\overline{Q}_i$ by Lemma 3). We have $Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \ldots$. Hence there is a quasigroup P such that $P = \bigcup_{i=1}^{\infty} Q_i$. Let $a, b, c, d \in P$, $a \neq b$, $c \neq d$. There is i such that $a, b, c, d \in Q_i$. By Lemma 3, there exists $x \in Q_{i+1}$ such that $ax \cdot x = c$, $bx \cdot x = d$.

Let $i = 1,2,3,4,5,6$. A quasigroup Q will be called an (i)-quasigroup if every its relation satisfying the condition (i) is equal to one of δ_Q, π_Q, σ_Q.

Lemma 4. Every quasigroup Q can be imbedded in an (i)-quasigroup $Q^{(i)}$ for every $i = 1,2,3,4,5,6$.

Proof. First for $i = 1$. By Theorem 1, there is a quasigroup P having the following property:
Q is a subquasigroup of P and for every $a, b, c, d \in P$, $a \neq b$, $c = d$, there is $x \in P$ such that $ax \cdot x = c$, $bx \cdot x = d$. Let ϱ be a relation on P and let ϱ satisfy (1). Let $\varrho \neq \delta_P$. Since ϱ satisfies (1) there are $a, b \in P$ such that $a \neq b$ and $a \varrho b$. Let $c, d \in P$, $c \neq d$, be arbitrary. There is $x \in P$ such that $ax \cdot x = c$, $bx \cdot x = d$. Since ϱ satisfies (1), $c \varrho d$. Hence $\sigma_P \subseteq \varrho$. Further, let $\varrho \neq \sigma_P$. Hence there is $a \in P$ such that $a \varrho a$. Let $b \in P$ be arbitrary and $c \in P$ be such that $ac = b$. Since $a \varrho a$, $ac \varrho ac$. Thus $b \varrho b$ and $\varrho = \pi_P$. Now it is sufficient to put $P = Q^{(1)}$.

Now for $i = 6$. Let $Q(*)$ be the right inverse quasigroup of Q. There is a (1)-quasigroup $P(*)$ such that $Q(*)$ is a subquasigroup of $P(*)$. Let P be the right inverse quasigroup of $P(*)$. Evidently, Q is a subquasigroup in P. Let ϱ be a relation on the set P and let ϱ satisfy (6) on P. By Lemma 1, ϱ satisfies (1) on $P(*)$. Hence ϱ is equal to one of δ_P, σ_P, π_P.

Thus P is a (6)-quasigroup.

For $i = 2,3,4,5$ similarly as for $i = 6$ by Lemmas 1,2.

<u>Theorem 2</u>. Every quasigroup can be imbedded in a totally simple quasigroup.

<u>Proof</u>. Let Q be the given quasigroup. Let $\alpha(i)$, for $i = 1, 2, 3, \dots$ be such a number that $1 \leq \alpha(i) \leq 6$ and $\alpha(i) \equiv i \bmod 6$. Put $Q_1 = Q$ and $Q_{i+1} = Q_i^{(\alpha(i))}$ for $i = 1, 2, 3, \dots$

($Q_i^{(\alpha(i))}$ by Lemma 4). Thus $Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \dots$.

Hence there is a quasigroup P such that $P = \bigcup_{i=1}^{\infty} Q_i$.

It is easy to show that P is totally simple.

References

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Matematicko-fyzikální fakulta

Karlova universita

Praha 8, Sokolovská 83

Československo

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