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SOME EXISTENCE THEOREMS FOR NONLINEAR EQUATIONS OF HAMMERSTEIN TYPE

(Preliminary communication)

Josef DANEŠ, Praha

We will use the notations and definitions of [1] and [2]. Let X be a Banach space, Y a closed linear subspace of X^*, a mnc on Y satisfying the conditions (1) and (4) of [2]. Let $K : X \longrightarrow Y$, $F : Y \longrightarrow X$ be some mappings and consider the equation:

(H.e.) $$y + KFy = 0 .$$

<u>Theorem 1</u>. Suppose that for some $\rho > 0$, we have:

(i) $-KF \in \mathcal{D}(\mu, \bar{B}_Y(0, \rho))$,

(ii) $y \in \partial B_Y(0, \rho)$ implies $\langle y, Fy \rangle > 0$,

(iii) $\langle Kx, x \rangle \geq 0$ for all x in $R(F) \subset X$.

Then (H.e.) has at least one solution in $\bar{B}_Y(0, \rho)$. Moreover, if

(ii′) y in $Y \setminus B_Y(0, \rho)$ implies $\langle y, Fy \quad 0$, then each solution of (H.e.) lies in $B_Y(0, \rho)$

<u>Theorem 2</u>. Suppose that K is injective, p-homogenous

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$(p > 0)$, odd. Let $\rho > 0$ be such that

(i) $\quad -KF \in \mathcal{D}(\mu, \bar{B}_Y(0,\rho))$,

(ii) $\quad y \in \partial B_Y(0,\rho)$ implies $\langle y, (K^{-1}+F)y \rangle > 0$,

(iii) $\quad \langle Kx, x \rangle \geq 0$ for all x in X .

Then (H.e.) has at least one solution in $B_Y(0,\rho)$. Moreover, if

(ii') y in $R(K) \setminus B_Y(0,\rho)$ implies $\langle y, (K^{-1}+F)y \rangle > 0$, then all solutions of (H.e.) lie in $B_Y(0,\rho) \cap R(K)$.

Proposition 1. Let $R(K)$ be symmetric and starshaped wrt. 0 . Suppose that for some $\rho > 0$:

(i) $\quad -KF \in \mathcal{D}(\mu, \bar{B}_Y(0,\rho))$,

(ii) $\quad x$ in X, $\|Kx\| = \rho$ implies $\langle (K+KFK)x, x \rangle > 0$,

(iii) $\quad \langle Kx, x \rangle \geq 0$ for all x in X .

Then (H.e.) has at least one solution in $\bar{B}_Y(0,\rho) \cap R(K)$. Moreover, if

(ii') x in X, $\|Kx\| \geq \rho$ implies $\langle (K+KFK)x, x \rangle > 0$, then each solution of (H.e.) lies in $\bar{B}_Y(0,\rho) \cap R(K)$.

Proposition 2. Let K be symmetric linear. Suppose that for some $c > 0$ and a function $\varphi : \mathbb{R}_+ [0,\infty) \longrightarrow \mathbb{R}_+$ we have:

(i) $-KF \in \mathcal{D}(\mu)$ on bounded sets,

(ii) $\langle y, Fy \rangle \geq -\varphi(\|y\|)$ for all y in $R(K)$,

(iii) $\langle Kx, x \rangle \geq c\|Kx\|^2$ for all x in X,

(iv) $\lim_{s \to +\infty} s^{-2}\varphi(s) < c$.

Then (H.e.) has at least one solution in Y .

Proposition 3. Let X_0 be a normed linear space, K_0: : $X_0 \longrightarrow X_0$ a bounded angle-bounded mapping. Then there is a constant $c > 0$ such that for all x in X_0 :

$\langle K_0(x), x \rangle \geq c\|K_0(x)\|^2$.

Theorem 3. Let K be angle-bounded and $\varphi: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a function such that

(i) $-KF \in \mathcal{D}(\mu)$ on bounded sets,

(ii) $\langle y, Fy \rangle \geq -\varphi(\|y\|)$ for all y in $R(K)$,

(iv) $\lim_{s \to +\infty} s^{-2}\varphi(s) = 0$.

Then (H.e.) has at least one solution in Y .

Proposition 4. Suppose that $c > 0$ and $\varphi: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are such that

(i) $-KF \in \mathcal{D}(\mu)$ on bounded sets,

(ii) $\langle y, Fy \rangle \geq -\varphi(\|y\|)$ for all y in $R(K)$,

(iii) $\langle Kx, x \rangle \geq c\|Kx\|^2$ for all x in X ; K linear;

(iv) $\lim_{\|y\| \to +\infty} \dfrac{\varphi(\|y\|)}{\|y\| \, \|Fy\|} = 0$.

Then (H.e.) has at least one solution in Y .

<u>Theorem 4</u>. Let (X, H, X^*) be a triple in normal position, K quasi-accretive (with the constant μ_K), $\tilde{K} = K|_H : H \longrightarrow H$, $F_\lambda = F + \lambda$. It is known that $K_\lambda = (I - \lambda\tilde{K})^{-1} K$ is well-defined and maps X continuously into X^* for $\lambda < \mu_K$. Suppose that for some $\varphi > 0$ and $\lambda_0 < \lambda < \mu_K$ we have:

(i) $-K_\lambda F_\lambda \in \mathcal{D}(\mu, \bar{B}_Y(0, \varphi))$,

(ii) y in $\partial B_Y(0, \varphi)$ implies $\langle y, Fy\rangle + \lambda_0 \|y\|_H^2 \geq 0$.

Then (H.e.) has a solution in $\bar{B}_Y(0, \varphi)$.

<u>Theorem 5</u>. Let (X, H, X^*) be as in Theorem 4, K quasi-angle-bounded, $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that for some $\lambda < \mu_K$ we have:

(i) $-K_\lambda F_\lambda \in \mathcal{D}(\mu)$ on bounded sets,

(ii) y in Y implies

$$\langle y, Fy\rangle + \lambda \|y\|_H^2 \geq -\varphi(\|y\|_{X^*}) ,$$

(iv) $\lim_{s \to +\infty} s^{-2} \varphi(s) = 0$.

Then (H.e.) has at least one solution in Y .

<u>Remark</u>. Theorem 1 remains true if (ii) of Theorem 1 is replaced by :

(ii'') y in $\partial B_Y(0, \varphi)$ implies $\langle y, Fy\rangle \geq 0$,

K is linear, the mnc μ satisfies some additional conditions, $X/N(K)$ is separable, and (i) is replaced by:

(i') $\mu(KFM) \le k\,\mu(M)$ for any $M \subset \bar{B}_Y(0,\rho)$ $(k < 1)$.

<u>Remark</u>. Detailed proofs will be given in Math.Nachrichten, under the same title.

References

[1] AMANN H.: Existence theorems for equations of Hammerstein type, Applicable Analysis, to appear .

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