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STRENGTHENING UPPER BOUND FOR THE NUMBER OF CRITICAL LEVELS OF NONLINEAR FUNCTIONALS

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1. Introduction. Let $r > 0$ and let f, g be two nonlinear functionals defined on a real Banach space X. Denote f' and g' the Fréchet derivatives of f and g respectively. Set $M_r(f) = \{x \in X; f(x) = r\}$. The point $u \in M_r(f)$ is said to be the critical point of the functional g with respect to the manifold $M_r(f)$ if there exists $\lambda \in E_1$ such that $\lambda f'(u) = g'(u)$. The value of the functional g at the critical point is called the critical level. Let Γ be the set of all critical levels, i.e. Γ is the set of all $\gamma \in E_1$ for which there exist $\lambda \in E_1$ and $u \in X$ such that the following equations hold:

(1) $\lambda f'(u) = g'(u)$,

(2) $f(u) = r$,

(3) $g(u) = \gamma$.

This paper deals with the investigation of the set Γ.

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The main result of this branch of nonlinear functional analysis is the Ljusternik-Schnirelmann theory. L.A. Ljusternik and L. Schnirelmann proved that the set Γ is under certain assumptions at least countable. The same result under more general assumptions was proved in many papers (for example, see [1, 2, 3]).

The main goal of our previous papers [4, 5] was to prove the converse of the Ljusternik-Schnirelmann theory, i.e. to prove that the set Γ is, under suitable assumptions, at most countable. Together with the Ljusternik-Schnirelmann theory we obtained that $\Gamma = \{\gamma_n\}$, $\gamma_n \to 0$ as in the linear case. One of the important assumptions is that

$$(*) \qquad g'(u) = 0 \Longrightarrow g(u) = 0 .$$

In this paper, the assumption $(*)$ is omitted and for the set Γ' of all cumulation points of the set Γ it is proved

$$\Gamma' \subset K = g(\{x \in X ; g'(x) = 0\})$$

(in the papers [4, 5] the main assertion was $\Gamma' = K = \{0\}$). If $K = \emptyset$ we obtain that Γ is at most a finite set and this is such a strong upper bound for the number of critical levels that one can see the importance of assumptions like $(*)$ in the Ljusternik-Schnirelmann theory.

In Section 2, we shall prove two abstract theorems and we give applications to the existence of solutions of the Dirichlet problem for ordinary and partial differential equations (in the case $K = \emptyset$) in Section 3.

Let T be a mapping defined on a real Banach space X with the values in a real Banach space Y. The Fréchet derivative of the mapping T at the point $x \in X$ is denoted by $T'(x)$ or $dT(x,\cdot)$. We recall that the mapping T is said to be

(a) completely continuous, if it maps bounded sets in X into compact sets in Y and is continuous,

(b) strongly continuous, if it maps weakly convergent sequences in X onto strongly convergent sequences in Y,

(c) bounded, if it maps bounded sets in X onto bounded sets in Y,

(d) weakly continuous, if it maps weakly convergent sequences in X onto weakly convergent sequences in Y,

(e) real-analytic on X (see [8]), if the following conditions are fulfilled:

(i) For each $x \in X$ there exist the Fréchet derivatives $d^n T(x, \dots)$ of arbitrary orders.

(ii) For each $x \in X$ there exists $\delta > 0$ such that for all $h \in X$, $\|h\| \leq \delta$ it is

$$T(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n T(x, h^n)$$

(the convergence of the series on the right hand side is locally uniform and absolute in the norm of the space Y), where h^n is the vector $[h, \dots, h]$ with n components.

We use the symbols " $\rightharpoonup$ " and " $\rightarrow$ " to denote the weak and the strong convergence, respectively.

The abstract theorems from Section 2 are based on Theorem A (for proof see [5, Theorem 3.1]). Let X_1, X_2, X_3 be three real Banach spaces such that $X_1 \subset X_3$. Suppose that $\langle \cdot , \cdot \rangle$ is a bilinear form on $X_1 \times X_2$ continuous on X_2 for fixed $h \in X_1$ and such that the following implication holds:

$$\langle h, x \rangle = 0 \quad \text{for each } h \in X_1 \Longrightarrow x = 0 .$$

Let $f, g : X_3 \longrightarrow E_1$ be two functionals such that

(f 1)(g 1) f, g are real-analytic on X_1.

Suppose that for each $x \in X_1$ there exists a couple $F(x)$, $G(x) \in X_2$ (if there exists at least one, then it is unique) such that

(f 2) $df(x, h) = \langle h, F(x) \rangle$,

(g 2) $dg(x, h) = \langle h, G(x) \rangle$

for each $h \in X_1$ and let

(f 3)(g 3) $F, G : X_1 \longrightarrow X_2$ be real-analytic on X_1 and

(g 4) $G : X_1 \longrightarrow X_2$ is completely continuous on X_1.

Let $\varkappa > 0$ and denote $M_\varkappa(f) = \{x \in X_3 ; f(x) = \varkappa\}$, $B_1 = \{x \in M_\varkappa(f) \cap X_1$; there exists $\lambda \in E_1$ such that $\lambda\, df(x, h) = dg(x, h)$ for each $h \in X_1\}$.

Let $x_0 \in B_1$ and let $\lambda_0 \in E_1$ be the corresponding eigenvalue from the definition of the set B_1. Suppose $\lambda_0 \neq 0$ and

(f 4) $$F'(x_0) = J + L ,$$

where J is an isomorphism from X_1 onto X_2 and $L : X_1 \longrightarrow X_2$ is a completely continuous linear operator.

Then there exists a neighborhood $\mathcal{U}(x_0)$ in X_1 of the point x_0 such that the set $g(B_1 \cap \mathcal{U}(x_0))$ is a one-point set.

<u>2. Abstract results</u>. <u>Theorem 1</u>. Let X_1, X_2, X_3 be three real Banach spaces, let X_3 be a reflexive Banach space and $X_1 \subset X_3$. Let the identity mapping from X_1 into X_3 be continuous. Suppose that $\langle \cdot , \cdot \rangle$ is a bilinear form on $X_1 \times X_2$ with the properties from Theorem A. Let $f, g : X_3 \longrightarrow E_1$ be two functionals with the Fréchet derivatives f' and g' on X_3. Denote $B = \{x \in M_r(f) ;$ there exists $\lambda \in E_1$ such that $\lambda f'(x) = g'(x)\}$. Further, assume (f 1),(f 2),(f 3),(f 4),(g 1),(g 2),(g 3),(g 4),

(g 5) for each $\sigma > 0$ the set $B(\sigma)$ is a compact subset of X_1, where $B(\sigma) = \{x \in M_r(f) ;$ there exists $\lambda, |\lambda| \geq \sigma$, such that $\lambda f'(x) = g'(x)\}$,

(g 6) $g' : X_3 \longrightarrow X_3^*$ is a strongly continuous mapping,

(f 5) $f' : X_3 \longrightarrow X_3^*$ is a continuous and bounded mapping,

(f 6) $f'(x) \neq 0$ for $x \in M_r(f)$,

(f 7) $\lim_{\|x\| \to \infty} f(x) = \infty$.

Denote $K = g(\{x \in X_3 ; g'(x) = 0\})$.

Then the set $g(B) - K$ is isolated and $\Gamma' \subset K$, where Γ' is the set of all accumulation points of the set $\Gamma = g(B)$.

__Proof__. Let $u_n \in B$ $(n = 1, 2, \dots)$ be such a sequence that $g(u_n) \to \gamma \in K$, then there exist λ_n $(n = 1, 2, \dots)$ such that $\lambda_n f'(u_n) = g'(u_n)$. The assumption (f 7) implies that the set $M_k(f)$ is bounded and since X_3 is reflexive, we can suppose that $u_n \rightharpoonup u_0$ in X_3.

Suppose that $\inf_n |\lambda_n| = 0$. Then there exists a subsequence $\lambda_{n_k} \to 0$ and thus $g'(u_{n_k}) \to 0 = g'(u_0)$ in X_3^*. From (g 6) it follows that g is weakly continuous (see [7, Chap.I, § 4]) and thus $g(u_{n_k}) \to g(u_0) = \gamma \notin K$ and this is a contradiction with the definition of the set K. Hence there exists $\delta > 0$ such that $|\lambda_n| \geq \delta$ for each positive integer n and thus $u_n \in B(\delta)$. Owing to Assumption (g 5), we can suppose that $u_n \to u_0 \in M_k(f)$ in X_1 and thus in X_3, too. With respect to Assumption (f 6) it is $f'(u_0) \neq 0$ and we have

$$|\lambda_n| = \frac{\|g'(u_n)\|}{\|f'(u_n)\|} \longrightarrow \frac{\|g'(u_0)\|}{\|f'(u_0)\|} \quad (\geq \delta) .$$

Thus we can suppose $\lambda_n \to \lambda_0$, $|\lambda_0| \geq \delta$. If n tends to infinity, we obtain $\lambda_0 \, df(u_0, \cdot) = dg(u_0, \cdot)$ and according to Theorem A there exists a neighborhood $U(u_0)$ in X_1 of the point u_0 such that the set $g(B \cap U(u_0))$ is a one-point set and thus there exists an index n_0 such that $g(u_n) = \gamma$ for $n \geq n_0$ and Theorem 1 is proved.

__Corollary 1__. Let the assumptions of Theorem 1 be fulfilled. Suppose that K is an empty set (i.e. $g'(x) \neq 0$

for each $x \in X_3$) .

Then the set Γ is at most a finite set.

<u>Theorem 2</u>. Let X be a real Hilbert space with the inner product $(\cdot,\cdot)$. Let us suppose:

(F 1) f is a real-analytic functional on X ,

(F 2) $f(0) = 0$, $f(u) > 0$ for all $u \neq 0$,

(F 3) there exists a continuous and nondecreasing function $c_1(t) > 0$ for $t > 0$ such that for all $u, h \in X$

$$d^2 f(u, h, h) \geq c_1(f(u)) \|h\|^2 ,$$

(F 4) the set $M_r(f) = \{x \in X ; f(x) = r\}$ is bounded,

(F 5) the operator f' is a bounded mapping,

(F 6) $\inf_{x \in M_r(f)} (f'(x), x) \geq c_2 > 0$,

(G 1) g is a real-analytic functional on X ,

(G 2) the Fréchet derivative g' is strongly continuous.

Denote $K = g(\{x \in X ; g'(x) = 0\})$ and $B = \{x \in M_r(f) ;$ there exists $\lambda \in E_1$ such that $\lambda f'(x) = g'(x)\}$.

Then the set $g(B) - K$ is isolated and $\Gamma' \subset K$.

<u>Proof</u>. Set in Theorem A: $X_1 = X_2 = X_3 = X$, $\langle \cdot,\cdot \rangle = (\cdot,\cdot)$, $F = f'$, $G = g'$. The assumptions of Theorem 2 obviously imply the assumptions (f 1),(f 2),(f 3),(g 1),(g 2),(g 3), (g 4). One can see that (F 3) implies the crucial assumption

(f 4) with $L = 0$. Thus the assumptions of Theorem A are fulfilled.

Suppose that $u_m \in B$ $(m = 1, 2, \dots)$, $g(u_m) = \gamma_m \to \gamma \notin X$. We can suppose $u_m \rightharpoonup u_0$, for $M_\kappa(f)$ is a bounded set. From (F 3) it follows that f is a convex functional, hence $R = \{x \in X;\ f(x) \le \kappa\}$ is convex and closed, hence weakly closed. We have $u_0 \in R$ and from (G 2) it follows that g is weakly continuous (see once more [7, Chap.I, § 4] as in the proof of Theorem 1). Hence $\gamma_m = g(u_m) \to$ $\to g(u_0) = \gamma \notin X$. There exist $\lambda_m \in E_1$ such that $\lambda_m f'(u_m) = g'(u_m)$. By the same way as in the proof of Theorem 1 we can prove that the sequence $\{\frac{1}{\lambda_m}\}$ is bounded. Hence we can suppose that $\frac{1}{\lambda_m} \to \alpha \in E_1$ and then there exists $w \in X$ such that $f'(u_m) \to w$. Now, Assumption (F 3) implies

$$(f'(u_m) - f'(u_0),\ u_m - u_0) = \int_0^1 d^2 f(u_0 + t(u_m - u_0),\ u_m - $$
$$- u_0,\ u_m - u_0)\, dt \ \ge\ P_m \| u_m - u_0 \|^2 \quad ,$$

where $P_m = \int_0^1 c_1(f(u_0 + t(u_m - u_0)))\, dt > 0$. Suppose that $\liminf_{m \to \infty} P_m = 0$. Then there exists a subsequence u_{m_j} such that $P_{m_j} \to 0$ and (f being convex - see Assumption (F 3) - and continuous, thus being weakly lower semicontinuous);

$$0 = \lim_{j \to \infty} P_{m_j} \ge \int_0^1 c_1 \,(\liminf_{j \to \infty} f(u_0 + t(u_{m_j} - u_0)))\, dt \ge$$
$$\ge c_1(f(u_0)) \ge 0 \ .$$

Thus $f(u_0) = 0$ (see (F 3)) and $u_0 = 0$ (see (F 2)).

On the other hand, $(f'(u_n) - f'(u_0), u_n - u_0) = (f'(u_n), u_n) \to 0$ and this is a contradiction with (F 6). Thus $\liminf_{n \to \infty} P_n > 0$ and $u_n \longrightarrow u_0$, hence $u_0 \in B$.

With respect to Theorem A there exists an index n_0 such that $\gamma_n = \gamma$ for $n \geq n_0$ and Theorem 2 is proved.

Corollary 2. Let the assumptions of Theorem 2 be fulfilled. Suppose that K is an empty set (i.e. $g'(x) \neq 0$ for each $x \in X$).

Then the set Γ is at most a finite set.

Remark. The crucial assumptions in Theorem 1 are (f 4) and (g 5). (Assumption (g 5) says something about regularity of solution.) Theorem 2 is formulated without this assumption, but Assumption (F 3) is in the Banach spaces very difficult to verify and we think that it is very probable that Assumption (F 3) implies that a Banach space has the inner product.

Example. Let $X = \ell_2$ and for $x = \{x_n\} \in \ell_2$ set

$$f(x) = \frac{1}{2} \sum_{i=1}^{\infty} x_i^2, \quad g(x) = \left(\frac{1}{3} x_1^3 - \frac{1}{2} x_1^2 + \frac{1}{2} \sum_{i=2}^{\infty} 2^{-n} x_i^2 \right).$$

Let $r = 2$. Then the assumptions of Theorem 2 are fulfilled and $K = g(\{x \in X ; g'(x) = 0\}) = g(\{[0, \dots],$ $[1, 0, \dots]\}) = \{0, -\frac{1}{6}\}$. The equation $\lambda f'(x) = g'(x)$ has a solution $x \in M_2(f)$ iff

$$x_n^1 = [0, \dots, 0, \underset{n}{2}, 0, \dots], \quad \lambda = 2^{-n}$$

and

$$x_n^2 = [1 + 2^{-n}, 0, \dots, 0, \underbrace{(3 - 2^{-n+1} - 4^{-n})^{1/2}}_{n}, 0, \dots], \quad \lambda = 2^{-n}.$$

Thus

$$g(x_n^1) = 2^{-n+1} \to 0, \quad g(x_n^2) = \frac{1}{3}(1 + 2^{-n})^3 - \frac{1}{2}(1 + 2^{-n})^2 +$$
$$+ 2^{-n+1}(3 - 2^{-n+1} - 4^{-n}) \to -\frac{1}{6} \quad .$$

This example shows that $card\, K > 1$ and each point of the set K is an accumulation point of the set Γ. Thus the assertion about the set Γ' in Theorem 2 cannot be improved.

3. Applications. Let Ω be a fixed bounded domain in the Euclidean N-space E_N with the sufficiently smooth boundary $\partial\Omega$ in the case $N \geq 2$. Denote by $\mathring{W}_p^m(\Omega)$ the well-known Sobolev space (for definition and properties see [6]).

A. Consider the weak solution of the equation

$$\text{(3.1)} \quad \begin{cases} \lambda(-1)^m \Delta^m u + h(u) = 0 , \\ D^\alpha u = 0 \quad \text{on } \partial\Omega \quad \text{for } |\alpha| \leq m - 1 , \end{cases}$$

i.e. we seek a function $u \in \mathring{W}_2^m(\Omega)$ such that for each $v \in \mathring{W}_2^m(\Omega)$ the relation

$$\text{(3.2)} \quad \lambda \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x)\, dx = \int_\Omega h(u(x)) v(x)\, dx$$

holds.

Suppose that h is a polynomial of the degree $k < \frac{N+2m}{N-2m}$ in the case $2m < N$ and h is a real analytic function on real line in the case $2m > N$. In each case we suppose $h(0) \neq 0$. Define

$$f(u) = \frac{1}{2} \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u(x)|^2 dx$$

and

$$g(u) = \int_0^1 \int_\Omega h(tu(x))\, u(x)\, dt\, dx$$

for $u \in \overset{\circ}{W}{}_2^m(\Omega)$.

The functionals f and g satisfy the assumptions of Theorem 2 (for validity of Assumption (G 1) see [5, Section 6]). Suppose there exists $u \in \overset{\circ}{W}{}_2^m(\Omega)$ such that

$\int_\Omega h(u(x))\, v(x)\, dx = 0$ for each $v \in \overset{\circ}{W}{}_2^m(\Omega)$,

then in each case we obtain that $h(u(x)) = 0$ almost everywhere on Ω. Denote $\alpha = \inf\{|t|;\, h(t) = 0\}$. Thus $|u(x)| \geq \alpha > 0$ almost everywhere on Ω and the trace of u cannot be zero. This is a contradiction with $u \in \overset{\circ}{W}{}_2^m(\Omega)$. Hence K is an empty set and Corollary 2 may be applied, i.e. the set Γ is at most finite.

Remark. One can see that we obtain the same result if we replace the operator Δ^m by a general linear elliptic operator.

Example. The problem

$$\lambda u'' + h(u) = 0, \quad u(0) = u(1) = 0, \tag{3.3}$$

$$\int_0^1 (u'(t))^2\, dt = r, \tag{3.4}$$

$$\int_0^1 \int_0^1 h(\tau u(t))\, u(t)\, dt\, d\tau = \gamma \tag{3.5}$$

has for a fixed r a solution $u \in \mathring{W}_2^1(0,1)$ only for the finite number of γ's.

B. Let m be an even number. Denote $X_3 = \mathring{W}_m^1(\Omega)$, $X_1 = \mathring{W}_m^1(\Omega) \cap W_p^2(\Omega)$, $X_2 = L_p(\Omega)$, where $p > N$, and let $\langle \cdot , \cdot \rangle$ be the L_2-duality between X_1 and X_2. Let $\alpha \in \langle 0, \frac{Nm}{N-m} - 1)$ and let h be a real-analytic function on the real line satisfying

$$|h(u)| \leq c(1 + |u|^{\alpha})$$

for suitable $c > 0$ and all $u \in E_1$.
Assume that $h(0) \neq 0$ and define

$$f(u) = \frac{1}{m}\left[\int_{\Omega} \left(1 + \sum_{i=1}^{N} \left(\frac{\partial u(x)}{\partial x_i}\right)^2\right)^{m/2} dx - \operatorname{meas} \Omega\right]$$

and

$$g(u) = \int_0^1 \int_{\Omega} h(t\, u(x))\, u(x)\, dx\, dt$$

for $u \in \overset{\circ}{W}{}^{1}_{m}(\Omega)$.

By the same way as in Part A of this section we can prove that $K = \emptyset$ and that all assumptions of Theorem 1 except Assumption (g 5) are valid. The validity of Assumption (g 5) follows from [5, Section 5]. Hence Corollary 1 may be applied. All conditions on the function h are fulfilled for example in the case $h(u) = \frac{1}{1+u^2}$ or $h(u) = (1+u^2)^{\frac{m}{2}-1} u - 1$.

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