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SET FUNCTORS

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In the following paper we shall investigate set functors. We shall characterize the behaviour of a functor on all objects (sets) from its behaviour on its unattainable cardinals, where a cardinal α is an unattainable cardinal of a functor F if there exists X with $card\ X = \alpha$ and $x \in FX$ such that $x \notin Im\ Ff$ as soon as $card\,(domain\ f) < \alpha$. (A precise definition is given in the part three.) We shall give a necessary and sufficient condition for a functor to reflect monomorphisms, epimorphisms, isomorphisms.

In the first part we introduce some definitions and necessary conventions. In the second part we form some auxiliary propositions about sets. With their help we investigate the behaviour of a functor with respect to its unattainable cardinals in part three, where there is also the formulation of the main theorem on estimation of the behaviour of a functor. In the fourth part we show some constructions of functors with a given class of unattainable cardinals. Semiconstant functors, i.e. functors naturally equivalent with a constant functor up to a certain cardinality,

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are investigated in the part five. In the sixth part we discuss the relation between a functor and the preservation of monomorphisms, epimorphisms and isomorphisms.

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1.

<u>Convention</u>: Denote by $\mathcal{S}$ the category of all sets and their mappings. Let α be a cardinal. Then $\mathcal{S}^{\alpha}$ denotes the complete subcategory of $\mathcal{S}$ with $X \in (\mathcal{S}^{\alpha})^{\sigma} \Longleftrightarrow card\, X < \alpha$. In agreement with the set theory a cardinal α is a set and so $card\, X = \alpha$ means that there exists a bijection of X and α.

<u>Convention</u>: Writing $X \leqq Y$ we mean $card\, X \leqq card\, Y$ while $X \subset Y$ means X is a subset of Y. By $X \simeq Y$ we mean $card\, X = card\, Y$. An ordinal also means the naturally ordered set of all smaller cardinals. Denote by $\prec$ the natural ordering of the ordinals.

If A, B are sets (categories), f a mapping (functor) $f : A \longrightarrow B$ and C a subset of A (subcategory of A) then f/C denotes the restriction of f to the domain C.

<u>Definition</u>: A set functor F is regular if:

1) $F\vartheta_X$ is a monomorphism where $\vartheta_X : \emptyset \rightarrow X$.

2) Every monotransformation from $C_1/\mathcal{S}_0$ to $F/\mathcal{S}_0$ in $\mathcal{S}_0$ has an extension to a monotransformation from C_1 to F in $\mathcal{S}$ where $\mathcal{S}_0$ is the category of nonvoid sets and their mappings and C_1 is a constant functor to one-point set.

There is a difference between the notion of the regular functor, as defined above, from the one in [5].

Lemma 1.1: A functor F is regular if and only if it preserves prosections i.e.

$$\forall A, B \quad Fi_A[FA] \cap Fi_B[FB] = Fi_{A\cap B}[F(A\cap B)]$$

where i_A, i_B, $i_{A\cap B}$ are the inclusions from A, B, $A\cap B$ to $A \cup B$ respectively.

Proof: see [5] .

Lemma 2.1: For every set functor F there exists a regular set functor $F^{\varkappa}$ such that $F^{\varkappa}/S_0 = F/S_0$.

Proof: see [5] .

Convention: All functors throughout this paper will be covariant regular functors from S to S. The superposition $F \circ G$ of arbitrary functors F and G is written left-hand i.e.

$$(F \circ G)X = F(GX) .$$

Let us introduce some of the most commonly used functors:

I - denotes the identical functor,

C_M - a constant functor to M .

Convention: X^Y denotes the set of all mappings from Y to X where Y and X are sets. Let $A \subset B$. Then i_A^B denotes the inclusion from A to B.

We recall the definitions of a distinguished point and of a component of a functor.

Let F be a functor. A point $a \in F1$ will be called a distinguished point of F if there exists a transformation $\tau : C_1 \to F$ such that $\tau^1(0) = a$ where 1 is ordinal.

Subfunctor F_a of F, $a \in F1$ is a component of F

if

$$x \in F_a X \Longleftrightarrow Fh(x) = a , \quad h : X \longrightarrow 1 .$$

There is a difference between the notion of a distinguished point, as defined above, from the one in [5] .

Convention: Let X be a set, F a functor. F^X denotes the subfunctor of F where $F^X Z = \bigcup_{Y<X} \bigcup_{f \in Z^Y} Ff [FY]$.
Let α be a cardinal. Denote by α' the follower of α .

2.

Definition: Let X be a set, α a cardinal such that $\alpha \leqq X$, let $\mathcal{a}$ be a system of sets such that:

$$\mathcal{a} \subset \exp X ; \; Z \in \mathcal{a} \Rightarrow Z \geq \alpha ; \; Z_1 , Z_2 \in \mathcal{a} \Rightarrow (Z_1 \cap Z_2) < \alpha .$$

Then we call the system $\mathcal{a}$ a $\binom{X}{\alpha}$-system.

Lemma 1.2: Let $\alpha \leqq X \leqq \aleph_0$. Then there exists a $\binom{X}{\alpha}$-system Φ such that

$$\Phi \simeq \binom{\mathrm{card}\, X}{\alpha} , \quad \text{i.e. } \mathrm{card}\, \Phi = \binom{\mathrm{card}\, X}{\alpha} .$$

Proof is evident.

Lemma 2.2: Let $\alpha < \aleph_0 \leqq X$. Then there exists a $\binom{X}{\alpha}$-system Φ such that $\Phi \simeq X$.

Proof is evident.

Convention. Denote by $\binom{\overline{X}}{\alpha}$ the system of all subsets Z of a set X with $Z \simeq \alpha$, $\alpha < \aleph_0$.
Clearly $\binom{\overline{X}}{\alpha}$ is a $\binom{X}{\alpha}$-system.

Lemma 3.2: Let $\aleph_0 \leqq \alpha \leqq X$. Then there exists a $\binom{X}{\alpha}$-system Φ such that $\Phi \simeq X$.

Proof is evident.

Let us introduce this known lemma:

Lemma 4.2: Let us assume the generalized continuum hypothesis. Let $\alpha \geq \aleph_0$ be a cardinal. Let X be a set such that $X \simeq \alpha$. Then there exists a $\binom{X}{\alpha}$-system Φ such that $\Phi \simeq 2^{\alpha}$.

Proof: Let ω_0 be an ordinal such that $\omega_0 \simeq \alpha$ and that $\omega' \lneq \omega_0 \Rightarrow \omega' < \alpha$. Let $\mathcal{S} = \bigcup_{\omega \lneq \omega_0} 2^{\omega}$ where 2 is ordinal. Clearly $\mathcal{S} \simeq \alpha$. Let f be a mapping from ω_0 to 2. Let $s_f = \{ g \mid g = f/\mathit{domain}\, g, g \in \mathcal{S} \}$. Clearly $g \simeq \alpha$ and $f_1 \neq f_2 \Rightarrow s_{f_1} \cap s_{f_2} < \alpha$ as there exists an ordinal $\omega_1 \lneq \omega_0$ and $f_1(\omega_1) \neq f_2(\omega_1)$. As $2^{\omega_0} \simeq 2^{\alpha}$, $\{ s_f \mid f \in 2^{\omega_0} \}$ is the system we were looking for. Q.E.D.

3.

Definition 1: A cardinal $\alpha > 1$ is said to be an unattainable cardinal of a functor F if $F\alpha \neq F^{\alpha}\alpha$, $\mathit{Card}(F\alpha - F^{\alpha}\alpha)$ is said to be the increase of the functor F on α.

Denote by $\mathcal{A}_F$ the class of all unattainable cardinals of the functor F.

Lemma 1.3: Let α be an unattainbale cardinal of F. Let $f : X \to Y$ be a monomorphism Then $Ff(FX - F^{\alpha}X) \subset \subset FY - F^{\alpha}Y$.

Proof: Suppose $x \in FX - F^{\alpha}X$ and $Ff(x) = y$, $y \in \in F^{\alpha}Y$. There exists $g : Y \to X$ such that $g \circ f = id$ and so $Fg(y) = x$. We have $Fg(F^{\alpha}Y) \subset F^{\alpha}X$, hence $x \in \in F^{\alpha}X$. That is a contradiction. Q.E.D.

Lemma 2.3: Let α be an unattainable cardinal of F. Let Z_1, Z_2 be sets such that $Z_1 \subset X$, $Z_2 \subset X$, $(Z_1 \cap Z_2) < \alpha$. Then

$$(Fi^X_{Z_1}[FZ_1] - F^{\alpha}X) \cap (Fi^X_{Z_2}[FZ_2] - F^{\alpha}X) = \emptyset .$$

Proof: There exists a morphism $g : X \to Z_1$ such that

$g \circ i_{Z_1}^X = id$ and $g(Z_2) < \alpha$. Suppose $x \in [(Fi_{Z_1}^X[FZ_1] - F^\alpha X) \cap (Fi_{Z_2}^X[FZ_2] - F^\alpha X)$.
As $g \circ i_{Z_1}^X = id$ there exists $z \in FZ_1 - F^\alpha Z_1$ such that $Fi_{Z_1}^X(z) = x$ and therefore $Fg(x) = z$, $g \circ i_{Z_2}^X = h_1 \circ h_2$ where $h_2 : Z_2 \to Y$, $h_1 : Y \to Z_1$ and $Y < \alpha$. Then $F^\alpha Y = FY$ and therefore

$Fg(Fi_{Z_2}^X[FZ_2] - F^\alpha X) \subset Fh_1[FY] \subset Fh_1[F^\alpha Y] \subset F^\alpha Z_1$

and $Fg(x) \in F^\alpha Z_1$. That is a contradiction. Q.E.D.

<u>Lemma 3.3</u>: Let α be an unattainable cardinal of F. Let Φ be a $\binom{X}{\alpha}$-system.
Then there exists a monomorphism $\tau : \Phi \to FX - F^\alpha X$.

<u>Proof</u>: Lemma 1.3 implies $Fi_Z^X[FZ] \cap (FX - F^\alpha X) \neq \emptyset$ for every $Z \in \Phi$. Lemma 2.3 implies $(Fi_{Z_1}^X[FZ_1] - F^\alpha X) \cap (Fi_{Z_2}^X[FZ_2] - F^\alpha X) = \emptyset$ for every $Z_1, Z_2 \in \Phi$. Choose $x_Z \in Fi_Z^X[FZ] - F^\alpha X$ for every $Z \in \Phi$. Put $\tau : \Phi \to FX - F^\alpha X$, $\tau(Z) = x_Z$ for every $Z \in \Phi$. τ is evidently a monomorphism. Q.E.D.

<u>Convention</u>: Denote
$max(X, Y) = max(card\, X, card\, Y)$, $min(X, Y) = min(card\, X, card\, Y)$,
where X and Y are sets.

<u>Lemma 4.3</u>: Let α be an unattainable cardinal of a functor F. Then $FX \geqq max(F\alpha, X)$ for every set X with $X \geqq max(\alpha, \aleph_0)$.

<u>Proof</u>: Lemmas 3.2 and 3.3 imply $FX \geqq X$. As every functor maps monomorphisms into monomorphisms it holds that $F\alpha \leqq FX$. Q.E.D.

<u>Lemma 5.3</u>: Let α_1, α_2 be cardinals such that there exists no unattainable cardinal α_3 of the functor F with $\alpha_1 < \alpha_3 < \alpha_2$. Let $\alpha_1 \geqq \aleph_0$. Then for every X with

$\alpha_1 \leqq X < \alpha_2$, $FX \leqq (\max F\alpha_1, X^{\alpha_1})$.

Proof: As there does not exist any unattainable cardinal α of F with $\alpha_1 < \alpha \leqq X$, we have $FX = \bigcup_{f \in X^{\alpha_1}} Ff[F\alpha_1]$. It implies $FX \leqq (\max F\alpha_1, X^{\alpha_1})$. Q.E.D.

Lemma 6.3: Let α_1 , α_2 be unattainable cardinals of F with $\alpha_1 < \alpha_2$, $\alpha_1 < \aleph_0$ and let there exist no unattainable cardinal α_3 with $\alpha_1 < \alpha_3 < \alpha_2$. Let $F\alpha_1$ be finite. Let a be the increase of F on α_1 . Let X be a set with $\alpha_1 \leqq X < \min(\alpha_2, \aleph_0)$. Then $FX \simeq F^{\alpha_1} X \vee a . \binom{\operatorname{card} X}{\alpha_1}$.

Proof: We prove $FX \geqq F^{\alpha_1} X \vee a . \binom{\operatorname{card} X}{\alpha_1}$. For every $\mathcal{Z} \subset X$, $\mathcal{Z} \simeq \alpha$ there exists a monomorphism $f_{\mathcal{Z}}$ from α_1 into $\mathcal{Z}$. Lemmas 1.2 and 2.3 imply $FX \geqq F^{\alpha_1} X \vee a . \binom{\operatorname{card} X}{\alpha_1}$. As for every monomorphism $g : \alpha_1 \to X$ there exists an isomorphism $h_g : \alpha_1 \to \alpha_1$ and $\mathcal{Z} \in \binom{X}{\alpha_1}$ such that $g : i^X_{\mathcal{Z}} \circ f_{\mathcal{Z}} \circ h_g$ we have $Fg[F_{\alpha_1}] = F(i^X_{\mathcal{Z}} \circ f_{\mathcal{Z}})[F\alpha_1]$. Evidently $F^{\alpha_1} X \cup (\bigcup_{\mathcal{Z} \in \binom{X}{\alpha_1}} F(i^X_{\mathcal{Z}} \circ f_{\mathcal{Z}})[F\alpha_1]) \simeq F^{\alpha_1} X \vee a . \binom{\operatorname{card} X}{\alpha_1}$. Also clearly $F^{\alpha_1} X \cup (\bigcup_{\mathcal{Z} \in \binom{X}{\alpha_1}} F(i^X_{\mathcal{Z}} \circ f_{\mathcal{Z}})[F\alpha_1]) = F^{\alpha_1} X \cup (\bigcup_{f \in X^{\alpha_1}} Ff[F\alpha_1])$. As there does not exist any unattainable cardinal α of F with $\alpha_1 < \alpha \leqq X$ it holds that $FX = F^{\alpha_1} X \cup (\bigcup_{f \in X^{\alpha_1}} Ff[F\alpha_1])$ and therefore $FX \simeq F^{\alpha_1} X \vee a . \binom{\operatorname{card} X}{\alpha_1}$. Q.E.D.

Lemma 7.3: Under the presumptions of Lemma 6.3. Let $\aleph_0 \leqq X < \alpha_2$. Then $FX \simeq X$.

Proof: Lemma 2.2 implies $FX \geqq X$. As there does not exist any unattainable cardinal α of F with $\alpha_1 < \alpha \leqq X$ we have $FX = \bigcup_{f \in X^{\alpha_1}} Ff[F\alpha_1] \simeq X$. Q.E.D.

Remark: Let α be a finite unattainable cardinal of F and let $F\alpha \geqq \aleph_0$. Let X be a set such that $\alpha = \sup \mathcal{A}_{FX}$. Then $FX \simeq \max(F\alpha, X)$.

Proof is evident.

Theorem 1.3: Let X be a set with $\sup \mathcal{A}_{FX} = \beta > 1$.
1) If X is finite then $FX \simeq F^{\beta}X \vee a . \binom{\mathrm{card}\, X}{\beta}$ where a is the increase of F on β.
2) If X is infinite then $\max (F\beta, X) \leqq FX \leqq$ $\leqq \max (F\beta, X^{\beta})$.

Proof: The theorem is a consequence of Lemmas 4.3, 5.3, 6,3 and 7.3.

Corollary: Under the presumptions of Theorem 1.3 and assuming the generalized continuum hypothesis it holds for every set $X \geqq \aleph_0$ with $\mathrm{conf}\, X > \beta$ that $FX \simeq \max(F\beta, X)$.

Proposition 2.3: Let us assume the generalized continuum hypothesis. Let $\alpha \geqq \aleph_0$, $\beta = 2^{\alpha}$. Let $F\beta > \max(F\alpha, \beta)$. Then β is an unattainable cardinal of F.

Proof: It follows from Lemma 5.3 that $F^{\beta}\beta \leqq \max(F\alpha, \beta)$; $F\beta > F^{\beta}\beta$ and therefore $F\beta - F^{\beta}\beta \neq \emptyset$, hence β is an unattainable cardinal of F.

Proposition 3.3: Let $\alpha \geqq \aleph_0$ be an unattainable cardinal of F. Then $\beta \geqq \alpha$ where β is the increase of F on α.

Proof: Lemmas 3.2 and 3.3 imply $\beta \simeq F\alpha - F^{\alpha}\alpha \geqq \alpha$.

Proposition 4.3: Let us assume the generalized continuum hypothesis. Let $\alpha \geqq \aleph_0$ be an unattainable cardinal of F. Then $\beta \geqq 2^{\alpha}$ where β is the increase of F on α.

Proof: Lemmas 4.2 and 3.3 imply $\beta \simeq F\alpha - F^{\alpha}\alpha \geqq 2^{\alpha}$.

Corollary: Let us assume the generalized continuum hypothesis. Let $\alpha \geqq \aleph_0$ be an unattainable cardinal of F. Then $F\alpha \geqq 2^{\alpha}$

Convention: Let α, β be cardinals. Define a functor ${}^{\alpha}R_{\beta}$

$${}^{\alpha}R_{\beta}X = \{(A, \psi, \alpha) \mid A \simeq \beta, A \subset X, \psi \in \alpha\} \cup \{0\}, \ f: X' \to X'',$$

$${}^{\alpha}R_{\beta}f(A, \psi, \alpha) = 0 \iff f(A) < \beta, \ {}^{\alpha}R_{\beta}f(0) = 0,$$

$${}^{\alpha}R_{\beta}f(A, \psi, \alpha) = (f(A), \psi, \alpha) \iff f(A) \simeq \beta .$$

Proposition 1.4: Let $\mathcal{A}$ be a class of cardinals with $\alpha \in \mathcal{A} \implies \alpha > 1$. Let f be a mapping from $\mathcal{A}$ to the class of all cardinals with $f(\alpha) \geqq 2^{\alpha}$. Then there exists a functor F such that $\mathcal{A} = \mathcal{A}_F$ and $f(\alpha)$ is the increase of F on α.

Proof: Define a functor F

$$FX = \bigcup_{\alpha \in \mathcal{A}} {}^{f(\alpha)}R_{\alpha}X \ ; \ \varphi: X' \to X'', \ F\varphi \mid {}^{f(\alpha)}R_{\alpha}X' = {}^{f(\alpha)}R_{\alpha}\varphi \ \forall \alpha \in \mathcal{A}.$$

Clearly F is correctly defined and satisfies the conditions of the proposition. Q.E.D.

Corollary: Let us assume the generalized continuum hypothesis. Let $\mathcal{A}$ be a class of cardinals with $\alpha \in \mathcal{A} \implies \alpha \geqq \aleph_0$. Let f be a mapping from $\mathcal{A}$ to the class of all cardinals. Then there exists a functor F such that $\mathcal{A} = \mathcal{A}_F$ and $f(\alpha)$ is the cardinal of increase of F on α if and only if $f(\alpha) \geqq 2^{\alpha}$.

Proposition 2.4: Let $\mathcal{A}$ be a class of cardinals with $\alpha \in \mathcal{A} \implies \alpha \geqq \aleph_0$. Let f be a mapping from $\mathcal{A}$ to the class of all cardinals with $f(\alpha) \geqq 2^{\alpha}$ and $\alpha, \beta \in \mathcal{A}$ $\alpha < \beta \implies f(\alpha) \leqq f(\beta)$. Then there exists a functor F such that $\mathcal{A} = \mathcal{A}_F$ and $F\alpha \simeq f(\alpha)$ for every $\alpha \in \mathcal{A}$.

Proof: Define a functor F

$$FX = \bigcup_{\alpha \in \mathcal{A}} {}^{f(\alpha)}R_\alpha X ; \quad g : X' \to X'', \ Fg / {}^{f(\alpha)}R_\alpha X' =$$

$$= {}^{f(\alpha)}R_\alpha g \quad \forall \alpha \in \mathcal{A} .$$

Clearly F is correctly defined and satisfies the conditions of the proposition. Q.E.D.

Corollary: Let us assume the generalized continuum hypothesis. Let $\mathcal{A}$ be a class of cardinals with $\alpha \in \mathcal{A} \Longrightarrow \alpha \geqq \aleph_0$. Let f be a mapping from $\mathcal{A}$ to the class of all cardinals. Then there exists a functor F such that $\mathcal{A} = \mathcal{A}_F$ and $F\alpha \simeq f(\alpha)$ $\forall \alpha \in \mathcal{A}$ if and only if $f(\alpha) \geq 2^\alpha$ and $\alpha, \beta \in \mathcal{A}, \alpha < \beta \Longrightarrow f(\alpha) \leqq f(\beta)$.

We recall the definition of a small functor.

Convention: Denote by Q_α a functor from the category $\mathbb{K}$ into $\mathcal{S}$ defined by

$Q_\alpha b = \{g \mid g : \alpha \to b\}$ for b an object from $\mathbb{K}$,

$Q_\alpha f(g) = f \circ g$ for a morphism $f : b \to c$ and $g \in Q_\alpha b$,

Q_α is called covariant homfunctor.

A functor $F\,\mathbb{K} \to \mathcal{S}$ is small iff it is a colimit of a diagram the objects of which are covariant homfunctors.

Lemma 1.4: A functor is small iff it is a factorfunctor of a disjoint union of a set of covariant homfunctors.

Proof: see [2].

Lemma 2.4: If F is a factorfunctor of G, then $\mathcal{A}_F \subset \mathcal{A}_G$.

Proof is evident.

Lemma 3.4: $\mathcal{A}_{Q_M} = \{\alpha \mid \alpha$ is a cardinal, $M \geq \alpha > 1\}$.

Proof: A) $\alpha \leqq M$. Let f be an epimorphism with

$f : M \longrightarrow \alpha$. If $Q_M \alpha = (Q_M)^{\alpha} \alpha$ holds then there exist $g : M \longrightarrow \beta$, $h : \beta \longrightarrow \alpha$, $\beta < \alpha$ such that $f = g \circ h$. $Im\, g \leq \beta$ and therefore $Im\, g < \alpha$. That is a contradiction and therefore $Q_M \alpha \neq (Q_M)^{\alpha} \alpha$ and $\alpha \in \mathcal{A}_{Q_M}$. B | $\alpha > M$. Let $\varepsilon \in Q_M \alpha$. Then $\varepsilon = Q_M\, \varepsilon\, (id_M)$ and therefore $Q_M \alpha = (Q_M)^{\alpha} \alpha$ and $\alpha \notin \mathcal{A}_{Q_M}$. Q.E.D.

Theorem 3.4: A functor F is a small functor if and only if $\mathcal{A}_F$ is a set.

Proof: The theorem is a consequence of Lemmas 1.4, 2.4 and 3.4.

Definition 2: A functor F is said to be a semiconstant functor up to α

if F^{α} is a constant functor on $\mathcal{S}$.

F is said to be a semiconstant functor if there exists α such that F is a semiconstant functor up to α .

Definition 3: A functor is said to be a big functor if it is not a small functor.

Remark: F is a big functor if and only if $\mathcal{A}_F$ is a proper class.

Lemma 4.4: Let F, G be functors. Define a mapping h_G from $\mathcal{A}_G$ into the class of all cardinals:

$h_G(\alpha) = \min\limits_{F\delta \geq \alpha} \delta$ if the minimum exists; if contrary, put $h_G(\alpha) = 1$. If G is not a semiconstant functor then $(\mathcal{A}_F \cup h_G(\mathcal{A}_G)) - 1 \subset \mathcal{A}_{G \circ F}$. If G is a semiconstant functor then $[(\mathcal{A}_F \cup h_G(\mathcal{A}_G)) - (1 \cup \mathcal{A}_{F\beta})] \subset \mathcal{A}_{G \circ F})$, where $\beta = \min\limits_{F\delta \geq \min \mathcal{A}_G} \delta$.

Proof: We have $(F\alpha - F^{\alpha}\alpha) \geq \alpha$ where $\alpha \in \mathcal{A}_F$

(Proposition 4.3). If G is not a semiconstant functor or $F\alpha \geq \gamma$ where $\gamma = \min \mathcal{A}_G$ and $\alpha \in \mathcal{A}_G$, then $G(F\alpha - F^{\alpha}\alpha) \cap G^{\alpha} F_{\alpha} \subset G\emptyset$ and $G(F\alpha - F^{\alpha}\alpha) \neq G\emptyset$. Therefore α is an unattainable cardinal of $G \circ F$. $\delta \in h_G(\mathcal{A}_G)$ is evidently an unattainable cardinal of $G \circ F$. Q.E.D.

Theorem 4.4: Let F be a big functor, let G be a non-constant functor. Then $F \circ G$ and $G \circ F$ are big functors.

Proof is evident.

5.

Theorem 1.5: Let F be a semiconstant functor. Let α be the smallest cardinal such that $\{Ff \mid f \in \alpha^{\alpha}\} > 1$. Then $\alpha = \min \mathcal{A}_F$.

Proof: Every point of the set $F1$ is a distinguished point of the functor F and therefore for every $a \in F1$, $\tau^1(0) = a$ defines a transformation $\tau : C_1 \rightarrow F$. It implies that the functor F^{α} is a constant functor and therefore α is an unattainable cardinal of F. Q.E.D.

Theorem 2.5: Let F be a functor, X a set with $FX < X$. Then F is a semiconstant functor up to $(\mathrm{card}\, X - 1)'$.

Proof: We shall prove that every component has a distinguished point. For every component F_a of F where $a \in F1$, $F_a X < X$ and therefore there exist $f_0, f_1 : 1 \rightarrow X$ with $Ff_0 = Ff_1$ and $v_0, v_1 : 1 \rightarrow 2$ and a morphism $v : 2 \rightarrow X$ such that $v \circ v_0 = f_0$, $v \circ v_1 = f_1$. As $F_a(v)$ is a monomorphism it holds that $F_a(v_1) = F_a(v_0)$

and therefore a is a distinguished point. If $X \simeq 1$, then $FX = \emptyset$ and therefore $F = C_{\emptyset}$. If $X \simeq 2$, then $FX \leq 1$ and therefore the cardinal 2 is not an unattainable cardinal of F. If $X > 2$ and there exists an unattainable cardinal α of F such that $X - 1 \geq \alpha$ then $FX \geq X$ (Lemmas 4.3 and 6.3). That is a contradiction. Therefore there does not exist any unattainable cardinal of F smaller or equal to $\mathit{card}\, X - 1$ and hence F is a semiconstant functor up to $(\mathit{card}\, X - 1)'$. Q.E.D.

Corollary: Let F be a functor and let $\alpha = \min \mathcal{A}_F$. Then there exist A, B such that $(I \times C_A) \vee C_B /_{S\alpha}$ is naturally equivalent $F/_{S\alpha}$.

6.

Lemma 1.6: Let X be a set with $X > 1$. Let $\{Ff \mid f \in X^X\} \simeq 1$. Then the functor F is a semiconstant functor up to $(\mathit{card}\, X)'$.

Proof: Let Y be a set with $Y \leq X$. Let $f: Y \to X$ be a monomorphism. Then there exists an epimorphism $g: X \to Y$ such that $g \circ f = id$. It implies $Fg \circ Ff = F\, id$. It follows from the assumptions that $F(f \circ g) = id$. It implies that Ff and Fg are isomorphisms. Suppose there exist $h_1, h_2 : Y \to Y$, $Fh_1 \neq Fh_2$. Then $F(f \circ h_1 \circ g) \neq F(f \circ h_2 \circ g)$ which is a contradiction. Therefore for every $h: Y \to Y$, $Fh = id$. Hence for every $k: Y \to X$ it holds $k = k_1 \circ f \circ k_2$, where $k_1 : X \to X$, $k_2 : Y \to Y$ and $Fk = F(k_1 \circ f \circ k_2) = Ff$. The lemma is proved. Q.E.D.

Lemma 2.6: Let X, Y be sets with $Y > 1$, $X > \emptyset$

and $\{Ff \mid f \in Y^X\} \simeq 1$. Then

1) Every point of the set $F1$ is a distinguished point of F.

2) If $X > 1$ then the functor is a semiconstant functor up to $[\min(\mathrm{card}\, X, \mathrm{card}\, Y)]'$.

Proof: The proposition 2) implies the proposition 1) with the exception $X \simeq 1$ in which case the proposition 1) is evident. We shall prove the proposition 2). Let $X \leq Y$. Then for every $f: X \to Y$, Ff is a monomorphism and therefore for every $g: X \to X$, $Fg = F\mathrm{id}_X$ and the rest follows from Lemma 1.6. Let $X \geq Y$. Then for every $f: X \to Y$, Ff is an epimorphism and therefore for every $g: Y \to Y$, $Fg = F\mathrm{id}_Y$ and the rest follows from Lemma 1.6. Q.E.D.

Lemma 3.6: Let $f: X \to Y$ be not a monomorphism and let Ff be a monomorphism. Let there exist $\max_{y \in Y}(\mathrm{card}\, f_{-1}(y))$. Then F is a semiconstant functor up to $[\max_{y \in Y}(\mathrm{card}\, f_{-1}(y))]'$.

Proof: We shall prove that $1 \simeq (Ff \mid f \in \beta^\beta)$ where $\beta = \max_{y \in Y}(\mathrm{card}\, f_{-1}(y))$ and the proof then follows from Lemma 1.6. There exists $y \in Y$ with $f_{-1}(y) \simeq \beta$. Therefore there exists a monomorphism $g: \beta \to X$ such that $f \circ g(\beta) \simeq 1$; clearly $F(f \circ g)$ is a monomorphism. For every $h: \beta \to \beta$, $f \circ g \circ h = f \circ g$. It implies $Fh = F\mathrm{id}_\beta$ for every $h: \beta \to \beta$. Q.E.D.

Lemma 4.6: Let $f: X \to Y$ be not a monomorphism and let Ff be a monomorphism. Let $\sup_{y \in Y} \mathrm{card}\, f_{-1}(y)$ be a singular cardinal. Then F is a semiconstant functor up to $(\sup_{y \in Y} \mathrm{card}\, f_{-1}(y))'$

Proof: If $(\sup_{y\in Y} card\ f_{-1}(y)) = (\max_{y\in Y} card\ f_{-1}(y))$, the proposition of Lemma 4.6 is a consequence of Lemma 3.6. Let there not exist $\max_{y\in Y} card\ f_{-1}(y)$. Let $\alpha = conf(\sup_{y\in Y} card\ f_{-1}(y))$. Then there exist $g: X \to Y$, $h: X \to X$ such that $f = g \circ h$ and $\sup_{y\in Y} card\ f_{-1}(y) = \sup_{y\in X} card\ f_{-1}(y)$. Clearly Fh is a monomorphism. There exists $Z \subset X$ such that $Z \simeq \sup_{y\in X} card\ h_{-1}(y)$ and $h(Z) \simeq \alpha$. Therefore there exists a monomorphism $k: X \to X$ such that $h \circ k \circ h(Z) \simeq 1$ and Lemma 3.6 implies the proposition. Q.E.D.

Lemma 5.6: Let $f: X \to Y$ be not a monomorphism and let Ff be a monomorphism. Then F is a semiconstant functor up to $\sup_{y\in Y} card\ f_{-1}(y)$.

Proof is evident.

Definition: Put $\mathbf{F}X = \{\mathcal{F} \mid \mathcal{F}$ is a filter on $X\} \cup \cup \{exp\ X\}$. $f: X \to Y$, $Z \in \mathbf{F}f(\mathcal{H}) \Longleftrightarrow \exists Z_1 \in \mathcal{H}$ with $f(Z_1) \subset Z$. Clearly $\mathbf{F}$ is a functor. Define a mapping $\mathcal{F}_{F,X}$ from FX into $\mathbf{F}X$, $Z \in \mathcal{F}_{F,X}(x) \Longleftrightarrow x \in \in Fi_Z^X [FZ]$.

There is a difference between the notion of mapping $\mathcal{F}_{F,X}$, as defined above, from the one in [6]. In [6] the mapping $\mathcal{F}_{F,X}$ is not defined in case $f(x)$ where x is a distinguished point and $f: 1 \to X$.

Definition: Let $\mathcal{H}, \mathcal{G} \in \mathbf{F}X$. Define $\mathcal{H} \subset \mathcal{G} \Longleftrightarrow \Longleftrightarrow (Z \in \mathcal{H} \Longrightarrow Z \in \mathcal{G})$.

Lemma 6.6: The relation $\subset$ is an ordering.

Proof is evident.

We recall the definition of essential cardinality.

For every $\mathcal{H} \in \mathbf{F}X$ put $\min_{Z\in\mathcal{H}} card\ Z = \|\mathcal{H}\|$. The number

$\| \mathcal{H} \|$ will be called essential cardinality of $\mathcal{H}$.

The definition of essential cardinality is the same as in [3] in case $\mathcal{H}$ is a filter.

Lemma 7.6: Let F be a functor, α an unattainable cardinal of F. Let $X \geq \alpha$. Then there exists $\mathcal{H} \in \mathcal{F}_{F,X}(FX)$ with $\| \mathcal{H} \| = \alpha$.

Proof: α is an unattainable cardinal of F and therefore for every $X \geq \alpha$, $F^{\alpha'}X - F^{\alpha}X \neq \emptyset$. Put $x \in F^{\alpha'}X - F^{\alpha}X$. Definition 1) and definition $\mathcal{F}_{F,X}$ imply

$x \in Fi_Z^X [FZ] \Rightarrow Z \geq \alpha, \exists Z_1 \simeq \alpha, x \in Fi_{Z_1}^X [FZ_1]$.

Therefore $\| \mathcal{F}_{F,X} \| = \alpha$. Q.E.D.

Lemma 8.6: Let F be a functor. Then for every $x \in FX$ and every $f : X \to Y$ it holds $Ff(\mathcal{F}_{F,X}(x)) \subset \mathcal{F}_{F,Y}(Ff(x))$.

Proof: $Z \in Ff(\mathcal{F}_{F,X}(x)) \Leftrightarrow \exists Z_1 \in \mathcal{F}_{F,X}(x)$ with

$f(Z_1) \subset Z \Rightarrow x \in Fi_{Z_1}^X [FZ_1]$,

$Ff(x) \in F(f \circ i_{Z_1}^X)[FZ_1] \Rightarrow Ff(x) \in Fi^Y_{f(Z_1)}[Ff(Z_1)] \subset$

$\subset Fi_Z^Y [FZ] \Rightarrow Z \in \mathcal{F}_{F,Y}(Ff(x))$.

Q.E.D.

Lemma 9.6: Let F be a functor, $\mathcal{H} \in \mathcal{F}_{F,X}(FX)$. Let f be a mapping from X into Y such that f/Z is a monomorphism for some $Z \in \mathcal{H}$. Then $Ff(\mathcal{F}_{F,X}(x)) = \mathcal{F}_{F,Y}(Ff(x))$ where $\mathcal{F}_{F,X}(x) = \mathcal{H}$.

Proof: There exists $g : Y \to X$ such that $g \circ f/Z = id/Z$.

$\mathcal{H} = \mathcal{F}_{F,X}(x) = Fg \circ f(\mathcal{F}_{F,X}(x)) \subset Fg(\mathcal{F}_{F,Y}(Ff(x)) \subset$

$\subset \mathcal{F}_{F,X}(F(g \circ f)(x)) = \mathcal{F}_{F,X}(x)$.

$Fg(\mathcal{F}_{F,Y}(Ff(x)) = \mathcal{F}_{F,X}(x) \Rightarrow Ff(\mathcal{F}_{F,X}(x)) = \mathcal{F}_{F,Y}(Ff(x))$. Q.E.D.

Lemma 10.6: Let $f: X \longrightarrow Y$ be not a monomorphism. Let Ff be a monomorphism. Let $\alpha = \sup_{y \in Y} \mathrm{card}\, f_{-1}(y)$. Then F is a semiconstant functor up to α'.

Proof: If α is a singular cardinal or $\alpha = \max_{y \in Y} \mathrm{card}\, f_{-1}(y)$ then the proposition follows from the lemmas 3.6 and 4.6. Now let α be a regular cardinal with no predecessor. Lemma 5.6 implies that F is a semiconstant functor up to α. Presume α is an unattainable cardinal of F. There exists $Z \subset Y$ such that $Z \simeq \alpha$ and $y \in Z \Rightarrow f_{-1}(y) > 1$. For every $y \in Z$ choose $x_y^i \in f_{-1}(y)$, $i = 1, 2$; $x_y^1 \neq x_y^2$ and put $X_i = \bigcup_{y \in Z} x_y^i$, $i = 1, 2$. Clearly $X_1 \simeq X_2 \simeq \alpha$ and f / X_1, f / X_2 are monomorphisms. Let $\mathcal{H}$ be a filter such that $\| \mathcal{H} \| = \alpha$ and $\mathcal{H} \in \mathcal{F}_{F,X}(FX)$. Let $Z_1 \in \mathcal{H}$ with $Z_1 \simeq \alpha$, let $h: X \to X$ such that h / Z_1 is a monomorphism and $h(X) \subset X_1$. Define $k: X \to X$ as follows: $k(x) = x_y^2 \Longleftrightarrow h(x) = x_y^1$. Lemma 9.6 implies $Fh(\mathcal{F}_{F,X}(x)) = \mathcal{F}_{F,X}(Fh(x))$, $Fk(\mathcal{F}_{F,X}(x)) = \mathcal{F}_{F,X}(Fk(x))$ as soon as $\mathcal{F}_{F,X}(x) = \mathcal{H}$. Further, $Ff \circ Fh(x) = F(f \circ h)(x) = F(f \circ k)(x) = Ff \circ Fk(x)$. But $Fh(x) \neq Fk(x)$ and therefore Ff is not a monomorphism. That is a contradiction. Q.E.D.

Theorem 1.6: Let $f: X \longrightarrow Y$ be not a monomorphism and let Ff be a monomorphism. Then F is a semiconstant functor up to $\max(\min(\mathrm{card}\, X + 1, \aleph_0), (\sup_{y \in Y} \mathrm{card}\, f_{-1}(y))')$.

Proof: A) $X \leq Y$. Then there exist a monomorphism $g: X \to Y$ and a morphism $h: X \to X$ such that $g \circ h = f$. h is not a monomorphism and Fh is a monomorphism. Let $X < \aleph_0$. Then there exist isomorphisms $g_1, g_2, \ldots, g_n$ such that $h \circ g_1 \circ h \circ g_2 \circ \ldots \circ h \circ g_n \circ h(X) \simeq 1$.

As $F(h \circ g_1 \circ h \circ \ldots \circ g_m \circ h)$ is a monomorphism, Lemma 10.6 implies the proposition. Let $X \geq \aleph_0$. Then for every finite cardinal γ there exist isomorphisms $g_1, g_2, \ldots, g_m$ such that $h \circ g_1 \circ h \circ g_2 \circ \ldots \circ g_m \circ h = \bar{h}$. $F\bar{h}$ is a monomorphism and $\gamma < \sup_{y \in X} \operatorname{card} \bar{h}_{-1}(y)$. Lemma 10.6 implies the proposition.

B) $X > Y$. Then there exists a monomorphism $g: Y \to X$ such that $g \circ f$ is not a monomorphism and $F(g \circ f)$ is a monomorphism. Then we proceed as in the case discussed above. Q.E.D.

<u>Lemma 11.6</u>: Let $f: X \to Y$ be not an epimorphism. Let Ff be an epimorphism. Then F is a semiconstant functor up to $(\operatorname{card}(Y - f(X)) + 1)'$.

<u>Proof</u>: Let Z be a set such that $Z \simeq (Y - f(X)) + 1$. Then there exists an epimorphism $g: Y \to Z$ such that $g \circ f(X) \simeq 1$. $F(g \circ f)$ is an epimorphism and therefore for every morphism $h: Z \to Z$ for which $h \circ g \circ f = g \circ f$ we have $Fh = id$. Let $\bar{h} \to Z \to Z$ be a constant morphism with $\bar{h} \circ g \circ f = g \circ f$. Then $F\bar{h}$ is a monomorphism and $\sup_{y \in Z} \operatorname{card} \bar{h}_{-1}(y) \simeq Z$. Lemma 11.6 is proved due to Theorem 1.6. Q.E.D.

<u>Theorem 2.6</u>: Let $f: X \to Y$ be not an epimorphism. Let Ff be an epimorphism. Then F is a semiconstant functor up to $\max[\min(Y + 1, \aleph_0), (\operatorname{card}[Y - f(X)])']$.

<u>Proof</u>: A) $X \geq Y$. Then there exist an epimorphism $g: X \to Y$ and a morphism $h: Y \to Y$ such that $h/f(X)$ is a monomorphism and $h \circ g = f$. h is not an epimorphism and Fh is an epimorphism. Let $Y < \aleph_0$. Then there exist isomorphisms $g_1, g_2, \ldots, g_m$ such

that $h \circ g_1 \circ h \circ g_2 \circ \dots \circ h \circ g_n \circ h(Y) \simeq 1$ and $F(h \circ g_1 \circ h \circ \dots \circ g_n \circ h)$ is an epimorphism. Lemma 11.6 proves the proposition. Let $Y \geq \aleph_0$. Then for finite cardinal γ there exist isomorphisms $g_1, g_2, \dots, g_n$ such that $h \circ g_1 \circ h \circ g_2 \circ \dots \circ h \circ g_n \circ h = \bar{h}$. $F\bar{h}$ is an epimorphism and $\gamma < (Y - \bar{h}(Y)) + 1$. Lemma 11.6 proves the proposition.

B) $X < Y$. If $Y \geq \aleph_0$, the proposition is evident. Let $Y < \aleph_0$. Then there exists an epimorphism $g: Y \longrightarrow X$ such that $f \circ g$ is not an epimorphism and $F(f \circ g)$ is an epimorphism. Then we proceed as in the case discussed above. Q.E.D.

Corollary: Let X, Y be sets such that $X \neq Y$. Let $f: X \longrightarrow Y$ be a morphism such that Ff is an isomorphism. Then F is a semiconstant functor up to $[max(X,Y)]'$.

In [2] P. Freyd considers the reflecting of retractions, co-retractions and isomorphisms. Much stronger results are obtained when we work with set functors only.

Theorem 3.6: The following conditions are equivalent:

1) F reflects isomorphisms.

2) F reflects epimorphisms.

3) F reflects monomorphisms.

4) F is not a semiconstant functor.

Proof: Implications 1) $\Longleftarrow$ 4), 2) $\Longleftarrow$ 4), 3 $\Longleftarrow$ 4) are consequences of Theorems 1.6 and 2.6. Let F be a semiconstant functor. Let $v: 1 \to 2$ be a morphism. Then Fv is an isomorphism and so an epimorphism. Let $f: 2 \longrightarrow 1$ be a morphism. Then Ff is an isomorphism and so a monomorphism. Implications 1) $\Longrightarrow$ 4), 2) $\Longrightarrow$ 4), 3) $\Longrightarrow$ 4) are proved. Q.E.D.

Proposition 4.6: The estimate of the smallest unattainable cardinal of the functor in Theorems 1.6 and 2.6 is the best possible.

Proof : Let $\alpha < \aleph_0$. Then the functor ${}^1R_\alpha$ proves the proposition. Let $\alpha \geqq \aleph_0$. Let $\cong_X$ be an equivalence on ${}^1R_\alpha X$ defined as follows: $Y, Z \in {}^1R_\alpha X$, $Y \cong_X X \Longleftrightarrow (Y - Z) \cup (Z - Y) < \alpha$. This equivalence defines a factorfunctor B^+_α of the functor ${}^1R_\alpha$. Let β be a cardinal with $\beta < \alpha$. Let f_β be a morphism defined like this: $f_\beta : X \to X$; $X \geqq \alpha$, $\exists Z \subset X$, $Z \cong \beta$, $f_\beta / _{X - Z} = id / _{X - Z}$, $f_\beta (Z) \simeq 1$. Evidently f_β is neither an epimorphism nor a monomorphism. Clearly $B^+_\alpha f_\beta =$ $= B^+_\alpha id_X$. Q.E.D.

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