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INVARIANT MANIFOLDS I

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Introduction. Let M be a submanifold of a manifold N, U a neighbourhood of M, $f: U \longrightarrow N$ a $C^{(1)}$ map such that $f|_M : M \longrightarrow M$ is a diffeomorphism onto M. There are found conditions which guarantee that for any $g: U \longrightarrow N$ sufficiently $C^{(1)}$ - close to f there exists a submanifold M_g of N such that $g|_{M_g} : M_g \longrightarrow M_g$ is a diffeomorphism onto M_g. It is assumed that U is diffeomorphic to a subset of E and the theory is developed on E. E is a bundle which differs from a vector bundle in that that bundle transformations preserve fibres but need not be linear on fibres.

This is motivated by an application to delayed differential equations, which will be published separately. In this application, E is the set of continuous maps $u: \langle -1, 0 \rangle \longrightarrow M$, M being a manifold. For $x \in M$, E_x - the fibre over x - is the set of such $u \in E$ that $u(0) = x$. There is no natural vector structure on E_x and therefore it seems preferable to restrain from it from the beginning.

Main Theorem is proved for $C^{(1)}$ -maps f in case that M is attractive exponentially. It can be extended formally to $C^{(k)}$ -maps and to the case of a hyperbolic structure of f near M. In this problem, uniformity properties (such as uniform boundedness and uniform continuity of differentials of certain maps) are of special importance.

Notations, Assumptions and Main Theorem are formulated in § 1, § 2 contains the proof of Main Theorem and in § 3 the theory is extended for the flows with a continuous parameter.

§ 1. <u>Notations, Assumptions, Main Theorem</u>

$\omega : \langle 0, \infty) \longrightarrow \langle 0, \infty)$ will be called a modulus of continuity, if it is continuous, nondecreasing and $\omega(0) = 0$. If Y is a Banach space, $y \in Y, \rho > 0$, then $\mathcal{B}(y, \rho) = \mathcal{B}(Y, y, \rho) = \{x \in Y \mid \|x - y\| < \rho\}$.

Let M be a Hausdorff space, $\tilde{X}$ - a Banach space, I - a set of indices, $\{\mathring{U}_i \mid i \in I\}$ an open covering of M, $\phi_i : \mathring{U}_i \longrightarrow \tilde{X}$ homeomorphisms.

<u>Definition 1.1</u>. $\{(\mathring{U}_i, \phi_i)\} \mid i \in I\}$ is called a uniform $C^{(k)}$ -atlas on M, $k = 1, 2, \ldots$ provided that there exist $K_1 \geq 1$, $R_1 > 0$ and a modulus of continuity ω_1 such that

(1.1) $\phi_i(\mathring{U}_i)$ is convex, $i \in I$;

(1.2) for any $x \in M$ there exists an $i \in I$ such that $x \in \mathring{U}_i$ and $\mathcal{B}(\phi_i(x), 2R_1) \subset \phi_i(\mathring{U}_i)$;

(1.3) maps $\overset{\circ}{\varphi}_{j,i} = \overset{\circ}{\varphi}_j \circ \overset{\circ}{\varphi}_i^{-1}$ are continuously differentiable up to order k and

$$\|D\overset{\circ}{\varphi}_{j,i}\|, \|D^2\overset{\circ}{\varphi}_{j,i}\|, \ldots \|D^k\overset{\circ}{\varphi}_{j,i}\| \leqq K_1$$

at any $v \in \overset{\circ}{\varphi}_i(\mathring{U}_i \cap \mathring{U}_j)$ and $D^k\varphi_{j,i}$ admits ω_1 as a modulus of continuity (i.e.

$$\|D^k\overset{\circ}{\varphi}_{j,i}(v_1) - D^k\overset{\circ}{\varphi}_{j,i}(v_2)\| \leqq \omega_1(\|v_1 - v_2\|)).$$

Two uniform $C^{(k)}$-atlases are called equivalent, if their union is a uniform $C^{(k)}$-atlas; the equivalence class of such atlases defines a uniform $C^{(k)}$-structure on M. M together with this structure is called a uniform $C^{(k)}$-manifold.

Assume that M is a uniform $C^{(1)}$-manifold and $\{(\mathring{U}_i, \overset{\circ}{\varphi}_i) \mid i \in I\}$ a uniform $C^{(1)}$-atlas on M.

Let $\hat{X}$ be a Banach space and put $X = \tilde{X} \times \hat{X}$. Let the norms in $X, \tilde{X}, \hat{X}$ be given in such a way that

(1.4) $\max(\|\tilde{x}\|, \|\hat{x}\|) \leqq \|x\| \leqq \|\tilde{x}\| + \|\hat{x}\|$,

if $x = (\tilde{x}, \hat{x}) \in X$, $\tilde{x} \in \tilde{X}$, $\hat{x} \in \hat{X}$, by $\tilde{P}, \hat{P}$ denote projectors, $\tilde{P}x = \tilde{x}$, $\hat{P}x = \hat{x}$.

Let there be given a Hausdorff space E, $\pi: E \to M$, $\pi(E) = M$, open sets $U_i \subset E$ and homeomorphisms $\varphi_i: U_i \to X$ for $i \in I$, and $R_2 > 0$ such that

(1.5) $U_i \subset \pi^{-1}(\mathring{U}_i)$ and $\overset{\circ}{\varphi}_i \circ \pi(u) = \tilde{P} \circ \varphi_i(u)$

for $u \in U_i$, $i \in I$;

(1.6) $\overset{\circ}{\varphi}_i(\mathring{U}_i) \times \mathcal{B}(\hat{X}, 0, R_2) \subset \varphi_i(U_i)$, $i \in I$;

(1.7) if $u \in U_i$, $\pi(u) \in \mathring{U}_j$, $i, j \in I$, $\hat{P} \circ \varphi_i(u) = 0$;

then $u \in U_j$ and $\hat{P} \circ \varphi_j(u) = 0$.

(1.8) $\varphi_{j,i} = \varphi_j \circ \varphi_i^{-1}$ are uniform $C^{(1)}$-maps, in detail, $\| D\varphi_{j,i}(v) \| \leqq K_1$ at any $v \in \varphi_i(U_i \cap U_j)$ and $D\varphi_{j,i}$ admits ω_1 as a modulus of continuity $i, j \in I$.

Put $\tilde{\varphi}_i = \tilde{P} \circ \varphi_i$, $\hat{\varphi}_i = \hat{P} \circ \varphi_i$, $\tilde{\varphi}_{j,i} = \tilde{P} \circ \varphi_{j,i}$, $\hat{\varphi}_{j,i} = \hat{P} \circ \varphi_{j,i}$, $j, i \in I$.

By (1.5) and (1.6) it follows that

(1.9) $\pi(U_i) = \mathring{U}_i$, $\tilde{\varphi}_i(U_i) = \mathring{\varphi}_i(\mathring{U}_i)$, $i \in I$.

Further, (1.5) implies

(1.10) Let $x, y \in U_i$, $i \in I$. Then $\pi(x) = \pi(y)$ iff $\tilde{\varphi}_i(x) = \tilde{\varphi}_i(y)$.

Hence

(1.11) if $(\tilde{u}, \hat{u}), (\tilde{u}, \hat{v}) \in \varphi_i(U_i \cap U_j)$ $i, j \in I$, then

$$\tilde{\varphi}_{j,i}(\tilde{u}, \hat{u}) = \tilde{\varphi}_{j,i}(\tilde{u}, \hat{v}),$$

that means $\varphi_{j,i}$ preserve fibres.

Define $\sigma : M \longrightarrow E$ as follows: for $x \in M$ find $i \in I$ such that $x \in \mathring{U}_i$ and put $\sigma(x) = \varphi_i^{-1}(\mathring{\varphi}_i(x), 0)$. It follows from (1.4) that $\mathring{\varphi}_i(x) = \mathring{\varphi}_i \circ \pi \circ \sigma(x)$, hence $x = \pi \circ \sigma(x)$. If $x \in \mathring{U}_j$, $j \in I$, then by (1.6) and (1.7) $\sigma(x) \in U_j$ and $\varphi_j \circ \sigma(x) = (\mathring{\varphi}_j(x), 0)$. Therefore $\sigma(x)$ does not depend on the choice of i. Obviously $\pi \circ \sigma = id$ and $\hat{P} \circ \varphi_j \circ \sigma(x) = 0$ if $x \in \mathring{U}_j$. Define $\mathring{U}_i^1 = \{x \in \mathring{U}_i \mid \mathcal{B}(\mathring{\varphi}_i(x), R_1) \subset \mathring{\varphi}_i(\mathring{U}_i)\}$, $U_i^1 = U_i \cap \pi^{-1}(\mathring{U}_i^1)$, $i \in I$. It is easy to see that

(1.12) $\mathring{\varphi}_i(\mathring{U}_i^1)$ is convex and for $x \in M$ there exists $i \in I$ such that $x \in \mathring{U}_i^1$ and $\mathcal{B}(\mathring{\varphi}_i(x), R_1) \subset \mathring{\varphi}_i(\mathring{U}_i^1)$,

(1.13) $\mathring{\varphi}_i(\mathring{U}_i^1) = \tilde{\varphi}_i(U_i^1)$ and $\mathring{\varphi}_i(\mathring{U}_i^1) \times \times \mathcal{B}(\hat{X}, 0, R_2) \subset \varphi_i(U_i^1)$,

(1.14) $\pi(U_i^1) = \mathring{U}_i^1$.

Put

$S = \{ s : M \to E \mid \pi \circ s = id$ and if $x \in \mathring{U}_i^1$, $i \in I$, then $s(x) \in U_i^1 \}$.

For $s \in S$, $i \in I$ let $s_i : \tilde{\varphi}_i(U_i^1) \to \hat{X}$ be defined by

$$(v, s_i(v)) = \varphi_i \circ s \circ (\mathring{\varphi}_i)^{-1}(v) \quad .$$

Elements of S are called sections.

For $\rho, L > 0$ let $S(Diff, \rho, L)$ be the set of such $s \in S$ that all s_i, $i \in I$ are continuously differentiable, $\|s_i(v)\| \leq \rho$, $\|Ds_i(v)\| \leq L$ for $v \in \tilde{\varphi}_i(U_i^1)$. If Ω is a modulus of continuity, let $S(Diff, \rho, L, \Omega)$ be the set of such $s \in S(Diff, \rho, L)$ that s_i admits Ω as a modulus of continuity, $i \in I$. Obviously $\sigma \in S(Diff, \rho, L, \Omega)$ for any ρ, L, Ω .

For $\rho > 0$ put

$E(\rho) = \{ x \in \bigcup_{i \in I} U_i^1 \mid \|\hat{\varphi}_j(x)\| \leq \rho$ if $x \in U_j^1$, $j \in I \}$.

For $f : E(R_2) \to E$ put $f_{j,i} = \varphi_j \circ f \circ \varphi_i^{-1}$, $\tilde{f}_{j,i} = \tilde{\varphi}_j \circ f \circ \varphi_i^{-1}$, $\hat{f}_{j,i} = \hat{\varphi}_j \circ f \circ \varphi_i^{-1}$, $i, j \in I$, (i.e. $f_{j,i}$ is defined on $\varphi_i(U_i \cap f^{-1}(U_j))$) and for $x = (\tilde{x}, \hat{x}) \in \varphi_i(U_i \cap f^{-1}(U_j))$ the value of $f_{j,i}$ will be denoted by $f_{j,i}(x)$ or $f_{j,i}(\tilde{x},\hat{x})$. $D_1 f_{j,i}$ is the differential of $f_{j,i}$ if $\hat{x}$ is kept fixed and $D_2 f_{j,i}$ is the differential of $f_{j,i}$, if $\hat{x}$ is kept fixed.

Assume that there is given a continuous map $f : E(R_2) \to E$ and positive $K_2, \eta, \varsigma, \xi$, $K_2 \geq 1, \eta \leq 1$, $\varsigma < 1, \xi < 1$, and a modulus of continuity ω_2 such that

(1.15) $f|_{\sigma(M)} : \sigma(M) \to E$ is a homeomorphism onto $\sigma(M)$;

(1.16) $Df_{j,i}$ exists at any $v \in \varphi_i(U_i \cap f^{-1}(U_j))$, $\|Df_{j,i}(v)\| \leq K_2$ and $Df_{j,i}$ admits ω_2 as a modulus of continuity;

(1.17) $D_1 \tilde{f}_{j,i}(\tilde{u}, 0)$ is a toplinear automorphism of $\tilde{X}$;

(1.18) $\|(D_1 \tilde{f}_{j,i}(\tilde{u}, 0))^{-1}\| \leq \eta^{-1}$;

(1.19) $\|D_2 \hat{f}_{j,i}(\tilde{u}, 0)\| \leq \varsigma$;

(1.20) $\|D_2 \hat{f}_{j,i}(\tilde{u}, 0)\| \, \|(D_1 \tilde{f}_{j,i}(\tilde{u}, 0))^{-1}\| \leq \xi$;

((1.17) - (1.20) for $(\tilde{u}, 0) \in \varphi_i(U_i \cap f^{-1}(U_j))$).

(1.15) implies that

(1.21) $\hat{f}_{j,i}(\tilde{u}, 0) = 0$ and $D_1 \hat{f}_{j,i}(\tilde{u}, 0) = 0$ for $(\tilde{u}, 0) \in \varphi_i(U_i \cap f^{-1}(U_j))$.

Let $g : E(R_2) \to E$ be continuous.

__Definition 1.2__. Let $\varepsilon > 0$. g is said to be ε-close to f ($\varepsilon - C^{(1)}$-close to f) if $x \in U_i^1 \cap$

$\cap E(R_2)$, $f(x) \in U_j^1$, $i, j \in I$ imply that $g(x) \in U_j$, $Dg_{j,i}$ exists at $v = \varphi_i(x)$ and

(1.22) $\|\varphi_j \circ g(x) - \varphi_j \circ f(x)\| \leqq \varepsilon$ (i.e. $\|g_{j,i}(v) - f_{j,i}(v)\| \leqq \varepsilon$),

(1.23) $\|Dg_{j,i}(v) - Df_{j,i}(v)\| \leqq \varepsilon$.

Main Theorem. There exists $L_1 > 0$ and to L, $0 < L \leqq L_1$ there exist $\varepsilon > 0$, $\rho > 0$ and a modulus of continuity Ω such that

if g is ε-close to f, then there exists $p = p(g) \in S(Diff, \rho, L, \Omega)$ such that $g|p(M) : p(M) \longrightarrow E$ is a diffeomorphism onto $p(M)$.

Moreover,

(1.24) $g(E(\rho)) \subset E(\rho)$;

(1.25) if $x \in E(\rho) \cap U_i^1$, $y = g^k(x)$, $y \in U_j^1$, $i, j \in I$, k being a positive integer, then

$\|\hat{\varphi}_j(y) - p_j \circ \tilde{\varphi}_j(y)\| \leqq K_1 \zeta_1^k \|\hat{\varphi}_i(x) - p_i \circ \tilde{\varphi}_i(x)\|$,

$\zeta_1 = \zeta + \varepsilon + L(K_2 + \varepsilon)$.

Note 1.1. L_1 depends on $K_1, K_2, R_1, R_2, \eta, \zeta$, ξ, ω_1, ω_2 but not on M, E, f, g; ρ, ε and Ω depend, in addition, on L. Observe that ζ_1 is arbitrarily close to ζ for ε and L sufficiently small.

Corollary 1.1. If $y \in E(\rho)$, if k_ℓ are integers, $\ell = 1, 2, 3, \ldots$, $k_\ell \longrightarrow \infty$ with $\ell \longrightarrow \infty$ and if there exist $x_\ell \in E(\rho)$ such that $g^{k_\ell}(x_\ell) = y$, then (by (1.25)) $y \in p(M)$.

Corollary 1.2. p is unique in the following sense: if $s \in S$, $s(M) \subset E(\wp)$ and if for every $y \in s(M)$ there exists $x \in s(M)$ such that $g(x) = y$, then $s = p$.

This follows immediately from Corollary 1.1.

§ 2. Proof of Main Theorem

Let the assumptions introduced in § 1 be fulfilled. Let L, ρ, ε be positive. Conditions on L, ρ, ε will be introduced step by step. Let $g : E(R_2) \to \to E$ be continuous and ε-close to f.

Lemma 2.1. Let $0 < \alpha < \beta$, let Q be a topological space, Y a Banach space, $V \subset Q$ open, $\psi : V \to Y$, $h : \langle 0, 1 \rangle \to V$, ψ, h both continuous and suppose that

(2.1) $h(0) \in V$, $\mathcal{B}(\psi \circ h(0), \beta) \subset \psi(V)$,

(2.2) if $h(\lambda) \in V$ for some $\lambda \in (0, 1)$, then $D\psi \circ h$ exists at λ and $\| D\psi \circ h(\lambda) \| \leqq \alpha$,

(2.3) if $h(\lambda) \in V$ for $0 \leqq \lambda < \tau$ for some $\tau \leqq \leqq 1$ and if $H = \lim_{\lambda \to \tau} \psi \circ h(\lambda)$ exists and $H \in \in \psi(V)$, then $h(\tau) \in V$.

Then $h(\lambda) \in V$, $\psi \circ h(\lambda) \in \mathcal{B}(\psi \circ h(0), \beta)$ and $\| \psi \circ h(\lambda) - \psi \circ h(\lambda_1) \| \leqq \alpha |\lambda - \lambda_1|$ for $\lambda, \lambda_1 \in \langle 0, 1 \rangle$.

The proof is standard.

Note 2.1. Observe that (2.3) is fulfilled, if ψ is a homeomorphism.

Lemma 2.2. Let $\rho > 0$, $\beta > K_2\rho$, $i, j \in I$, $x \in U_i^1$, $\|\hat{\varphi}_i(x)\| \leqq \rho$, $f(x) \in U_j$, $\mathcal{B}(X, \varphi_j \circ f(x), \beta) \subset \varphi_j(U_j)$. Then

(2.4) $\|\hat{\varphi}_j \circ f(x)\| \leqq (\xi + \omega_2(\rho))\rho$.

Proof. Put $u = \varphi_i(x)$, $v = (\tilde{\varphi}_i(x), 0)$, $h(\lambda) = f \circ \varphi_i^{-1}(u + \lambda(v - u))$ for $\lambda \in \langle 0, 1\rangle$. Apply Lemma 2.1 ($\alpha = K_2$, $V = U_j$, $\psi = \varphi_j$, $\|v - u\| = \|\hat{\varphi}_i(x)\| \leqq \rho$, $\psi \circ h(\lambda) = f_{j,i}(u + \lambda(v - u))$). It follows that $f \circ \varphi_i(u + \lambda(v - u)) \in U_j$ for $\lambda \in \langle 0, 1\rangle$, i.e. $f_{j,i}(u + \lambda(v - u))$ is defined for $\lambda \in \langle 0, 1\rangle$. As $\varphi_i(v) \in \sigma(M)$, (1.15) implies that $f \circ \varphi_i(v) \in \sigma(M)$, i.e. $\hat{f}_{j,i}(v) = 0$

$\hat{\varphi}_j \circ f(x) = \hat{f}_{j,i}(u) = -\int_0^1 D\hat{f}_{j,i}(u + \lambda(v - u))\,d\lambda\,(v - u)$.

By (1.19) and (1.16) $\|D\hat{f}_{j,i}(v)\| \leqq \xi$,

$\|D\hat{f}_{j,i}(u + \lambda(v - u)) - D\hat{f}_{j,i}(v)\| \leqq \omega_2((1 - \lambda)\|u - v\|) \leqq$
$\leqq \omega_2(\rho)$ and (2.4) holds.

Assume that

(2.5) $K_2\rho + \varepsilon < \min(R_1, R_2)$, $2K_2\rho < R_2$.

Lemma 2.3. Let $x \in E(\rho)$. There exists $k \in I$ such that $f(x)$, $g(x) \in U_k^1$ and

(2.6) $\|\hat{\varphi}_k \circ f(x)\| \leqq (\xi + \omega(\rho))\rho$,

(2.7) $\|\hat{\varphi}_k \circ g(x)\| \leqq (\xi + \omega(\rho))\rho + \varepsilon$.

Proof. Choose $i \in I$ such that $x \in U_i^1$, put $u = \varphi_i(x)$, $v = (\tilde{\varphi}_i(x), 0)$. By (1.15)

$f \circ \varphi_i^{-1}(v) \in \sigma(M)$, as $\varphi_i^{-1}(v) \in \sigma(M)$.

Find $k \in I$ such that $\pi \circ f \circ \varphi_i^{-1}(v) \in \mathring{U}_k^1$ and $\mathcal{B}(\mathring{\varphi}_k \circ \pi \circ f \circ \varphi_i^{-1}(v), R_1) \subset \mathring{\varphi}_k(\mathring{U}_k^1)$ (cf.(1.12)). That means

$$f \circ \varphi_i^{-1}(v) \in U_k^1, \mathcal{B}(\mathring{\varphi}_k \circ f \circ \varphi_i^{-1}(v), R_1) \subset \tilde{\varphi}_k(U_k^1), \hat{\varphi}_k \circ f \circ \varphi_i^{-1}(v) = 0.$$

Put $h(\lambda) = f \circ \varphi_i^{-1}(v + \lambda(u - v)), \lambda \in \langle 0, 1 \rangle$ and apply Lemma 2.1. It follows that $f(x) = f \circ \varphi_i^{-1}(u) \in U_k^1$ and $\| \varphi_k \circ f(x) - \varphi_k \circ f \circ \varphi_i^{-1}(v) \| \leq K_2 \rho$. Therefore, by (2.5) and (1.6) $\mathcal{B}(X, \varphi_k \circ f(x), K_2 \rho + \sigma) \subset \subset \varphi_k(U_k)$ for some $\sigma > 0$ sufficiently small and (2.6) follows by Lemma 2.1. (2.7) follows by (2.6) and Definition 1.2 and finally $g(x) \in U_k^1$ by (2.7) and (2.5).

Assume that

(2.8) $(\varsigma + \omega(\rho))\rho + K_1 \varepsilon < \rho$,

(2.9) $K_1 \rho + K_2 \rho + 2K_1 \varepsilon < \min(R_1, R_2)$.

<u>Lemma 2.4</u>. Let $x \in E(\rho) \cap U_i^1$, $t = \varphi_i(x)$, $g(x) \in U_j^1$, $i, j \in I$. Then $f(x) \in U_j$ and

(2.10) $\| \varphi_j \circ f(x) - \varphi_j \circ g(x) \| \leq K_1 \varepsilon$,

(2.11) $\| Df_{j,i}(t) - Dg_{j,i}(t) \| \leq K_1 \varepsilon + K_2 \omega_1(\varepsilon)$,

(2.12) $\| \hat{\varphi}_j \circ g(x) \| < \rho$.

<u>Proof.</u> Find k according to Lemma 2.3 and put $u = (\tilde{u}, \hat{u}) = \varphi_k \circ g(x)$, $v = (\tilde{u}, 0)$,

$h(\lambda) = \varphi_k^{-1}(v + \lambda(v - u))$, $\lambda \in \langle 0, 1\rangle$.

As $g(x) = h(1) \in U_j^1$, it follows (cf.(1.14)) that $\pi \circ g(x) \in \mathring{U}_j^1$, by (1.10) $\pi \circ h(\lambda) = \pi \circ g(x)$ for $\lambda \in \langle 0, 1\rangle$, i.e. $\pi \circ h(0) \in \mathring{U}_j^1$. By definition of U_j^1, $\mathring{U}_j^1$ and by (1.7) $h(0) \in \mathring{U}_j^1$, $\hat{\varphi}_j \circ h(0) = 0$. Apply Lemma 2.1 (cf.(2.7),(2.8),(2.9),(1.8)). It follows $h(\lambda) \in U_j$ for $\lambda \in \langle 0, 1\rangle$ and moreover, $h(\lambda) \in U_j^1$ for $\lambda \in \langle 0, 1\rangle$, as $\pi \circ h(\lambda) = \pi \circ g(x) \in \mathring{U}_j^1$. Consequently $\hat{\varphi}_j \circ g(x) =$

$= \int_0^1 D\hat{\varphi}_j \circ h(\lambda)\, d\lambda = \int_0^1 D\hat{\varphi}_{j,k}(v + \lambda(u - v))\, d\lambda\, (u - v)$,

(2.13) $\quad \| \hat{\varphi}_j \circ g(x) \| \leq K_1(\rho + \varepsilon)$.

Put $w = \varphi_k \circ f(x)$, $h_1(\lambda) = \varphi_k^{-1}(u + \lambda(w - u))$, $\lambda \in \langle 0, 1\rangle$ (cf.(1.12)). Lemma 2.1 applies again (cf.(2.13),(2.9), (1.8), Definition 1.2). It follows that $f(x) = h_1(1) \in U_j$, and that (2.10) holds.

By (2.13),(2.10) and (2.9) Lemma 2.2 applies and (2.12) follows from (2.4),(2.10) and (2.8).

$Df_{j,i}(t) = D\varphi_{j,k}(f_{k,i}(t)) \circ Df_{k,i}(t)$ and an analogous formula is valid for $Dg_{i,j}(t)$. (2.11) follows from Definition 1.2, (1.8) and (1.16).

An immediate consequence of Lemma 2.4 is

(2.14) $$g(E(\rho)) \subset E(\rho).$$

Assume that

(2.15) $\quad \omega_2(\rho) + K_2 L + (K_1 \varepsilon + K_2 \omega_1(\varepsilon)) < \eta$.

Lemma 2.5. For $\kappa > 0$ there exists $\varkappa(\kappa) > 0$ having the following property:

Let $s \in \mathcal{S}(\mathrm{Diff}, \rho, L)$, $i, j \in I$ and define the map θ by $\theta(\tilde{y}) = \tilde{g}_{j,i}(\tilde{y}, s_i(\tilde{y}))$ for all $\tilde{y}$ that the right hand side makes sense. Let $\tilde{u} \in \tilde{\varphi}_j(U_j^1)$, $\mathcal{B}(\tilde{u}, \kappa) \subset \tilde{\varphi}_j(U_j^1)$, $\tilde{v} \in \tilde{\varphi}_i(U_i^1)$, $\mathcal{B}(\tilde{v}, \kappa) \subset \tilde{\varphi}_i(U_j^1)$, $\tilde{u} = \theta(\tilde{v})$.

Then θ^{-1} exists on $\mathcal{B}(\tilde{u}, \varkappa(\kappa))$ and is continuously differentiable.

Note 2.1. The actual value of $\varkappa(\kappa)$ is irrelevant in what follows. It is sufficient to know that $\varkappa(\kappa)$ is positive and independent of $i, j, \tilde{v}$.

Proof. Put $\theta_1 = (D_1 \tilde{f}_{j,i}(\tilde{v}, 0))^{-1} \circ \theta$, $\tilde{z} = (D_1 f_{j,i}(\tilde{v}, 0))^{-1} \tilde{x}$ and solve

$$\tilde{z} = \theta_1(\tilde{y}) = \theta_1(\tilde{v}) + \tilde{y} - \tilde{v} + \Xi(\tilde{y}) \tag{2.16}$$

for $\tilde{z}$ close to $(D_1 \tilde{f}_{j,i}(\tilde{v}, 0))^{-1} \tilde{u}$ instead of $\tilde{x} = \theta(\tilde{y})$.

Write

$$\begin{aligned} \theta(\tilde{y}) = {} & \theta(\tilde{v}) + D_1 \tilde{f}_{j,i}(\tilde{v}, 0)(\tilde{y} - \tilde{v}) + \\ & + [\tilde{f}_{j,i}(\tilde{y}, s_i(\tilde{y})) - \theta(\tilde{v}) - D_1 \tilde{f}_{j,i}(\tilde{v}, 0)(\tilde{y} - \tilde{v})] + \\ & + \tilde{g}_{j,i}(\tilde{y}, s_i(\tilde{y})) - \tilde{f}_{j,i}(\tilde{y}, s_i(\tilde{y})) . \end{aligned} \tag{2.17}$$

It may be shown from Lemma 2.4, Definition 1.2 and (1.8) that there exists $\delta_1(\kappa) > 0$ such that (2.17) holds for $\tilde{y} \in \mathcal{B}(\tilde{v}, \delta(\kappa))$. From (2.17) and (2.16) it may be found that $\eta^{-1}(\omega_2(\rho) + K_2 L + (K_1 \varepsilon + K_2 \omega_1(\varepsilon))$ is a Lipschitz constant for Ξ. $\Xi(\tilde{v}) = 0$

and Implicit Function Theorem applied to (2.16) makes the proof complete (cf.(2.15)).

Lemma 2.6. Let $0 < r < R_1$, $s \in S(\mathcal{D}iff, \rho, L)$, $j \in I$, $\tilde{u} \in \tilde{\varphi}_j(\mathcal{U}_j^1 \cap q \circ s(M))$, $\mathcal{B}(\tilde{u}, r) \subset \tilde{\varphi}_j(\mathcal{U}_j^1)$. Then $\mathcal{B}(\tilde{u}, \varkappa(r)) \subset \tilde{\varphi}_j(\mathcal{U}_j^1 \cap q \circ s(M))$, $\varkappa(r)$ being the same as in Lemma 2.5.

Proof. Find $x \in s(M)$ such that $\tilde{u} = \tilde{\varphi}_j \circ q(x)$ and $i \in I$ such that $x \in \mathcal{U}_i^1$, $\mathcal{B}(\tilde{\varphi}_1(x), R_1) \subset \varphi_i(\mathcal{U}_i^1)$ (cf.(1.12)). Lemma 2.6 follows from Lemma 2.5, as $\tilde{x} \in \tilde{\varphi}_j(\mathcal{U}_j^1 \cap q \circ s(M))$ if there exists $\tilde{y}$ such that $\tilde{x} = \Theta(\tilde{y})$.

Lemma 2.7: Let $s \in S(\mathcal{D}iff, \rho, L)$. Then $\pi \circ q \circ s(M) = M$.

Proof. Let $x \in M$. Find $j \in I$ such that $x \in \mathring{\mathcal{U}}_j^1$, $\mathcal{B}(\mathring{\varphi}_j(x), R_1) \subset \mathring{\varphi}_j(\mathring{\mathcal{U}}_j^1)$, i.e. $\sigma(x) \in \mathcal{U}_j^1$, $\mathcal{B}(\varphi_j \circ \sigma(x), R_1) \subset \varphi_j(\mathcal{U}_j^1)$, $\hat{\varphi}_j \circ \sigma(x) = 0$. There exists $w \in \sigma(M)$ such that $f(w) = \sigma(x)$ (cf.(1.15)). Find $i \in I$ such that $w \in \mathcal{U}_i^1$ and put $v = \varphi_i(w) = (\tilde{v}, 0)$, $u = (\tilde{v}, s_i, (\tilde{v}))$, $x = \varphi_i^{-1}(u)$, $h(\lambda) = f \circ \varphi_i^{-1}(v + \lambda(u - v))$ for $\lambda \in \langle 0, 1 \rangle$. Obviously $x \in s(M)$. Apply Lemma 2.1 (cf.(2.9)). It follows that $f(x) \in \mathcal{U}_j$, $\|\varphi_j \circ f(x) - \varphi_j \circ f(w)\| \leq K_2 \rho$. Hence $f(x) \in \mathcal{U}_j^1$ and $\mathcal{B}(\varphi_j \circ f(x), 2\varepsilon) \subset \varphi_j(\mathcal{U}_j^1)$. By Definition 1.2 $q(x) \in \mathcal{U}_j$, $\|\varphi_j \circ q(x) - \varphi_j \circ f(x)\| \leq \varepsilon$. Therefore $q(x) \in \mathcal{U}_j^1$, $\mathcal{B}(\varphi_j \circ q(x), \varepsilon) \subset \varphi_j(\mathcal{U}_j^1)$, $\mathcal{B}(\tilde{\varphi}_j \circ q(x), \varepsilon) \subset \tilde{\varphi}_j(\mathcal{U}_j^1)$, $\tilde{\varphi}_j \circ q(x) \in \tilde{\varphi}_j(\mathcal{U}_j^1 \cap q \circ s(M))$.

Apply Lemma 2.6 (step by step on

$\{\tilde{\varphi}_j \circ g(x) + \lambda(\tilde{\varphi}_j \circ \sigma(x) - \tilde{\varphi}_j \circ g(x)) \mid \lambda \in \langle 0, 1 \rangle \})$.

It follows that $\tilde{\varphi}_j \circ \sigma(x) \in \tilde{\varphi}_j(U_j^1 \cap g \circ s(M))$.

That means that there exists $t \in s(M)$ such that

$\tilde{\varphi}_j \circ \sigma(x) = \tilde{\varphi}_j \circ g(t)$ and $x = \pi \circ g(t)$, as $\tilde{\varphi}_j = \mathring{\varphi}_j \circ \pi$

and $\pi \circ \sigma = id$.

Assume that

(2.18) $2K_2(1+L)(K_2\rho + \varepsilon) < \eta R_1$,

(2.19) $K_2 L + (1+L)\omega_2((1+L)\eta^{-1} 2(K_2\rho + \varepsilon)) + \varepsilon(1+L) < \eta$.

<u>Lemma 2.8</u>. Let $s \in S(Diff, \rho, L)$, $x_1, x_2 \in s(M)$,

$\pi \circ g(x_1) = \pi \circ g(x_2)$. Then $x_1 = x_2$.

<u>Proof</u>. Find $j \in I$ such that $\pi \circ g(x_1) \in \mathring{U}_1$,

$B(\mathring{\varphi}_j \circ \pi \circ g(x_1), R_1) \subset \mathring{\varphi}_j(\mathring{U}_j^1)$ (cf.(1.12)), i.e.

$g(x_1), g(x_2) \in U_j^1$ (cf.(2.14) and (1.13))

and $B(\tilde{\varphi}_i \circ g(x_1), R_1) \subset \tilde{\varphi}_j(U_j^1)$, $\tilde{\varphi}_j \circ g(x_1) =$

$= \tilde{\varphi}_j \circ g(x_2)$ (cf.(1.5)). Similarly, find $i \in I$ such that $x_1 \in U_i^1$, $B(\tilde{\varphi}_i(x_1), R_1) \subset \tilde{\varphi}_i(U_i^1)$ and put

$u_1 = (\tilde{u}_1, \hat{u}_1) = \varphi_i(x_1)$, $y_1 = \sigma \circ \pi(x_1)$, $y_2 = \sigma \circ \pi(x_2)$, $v_1 = \varphi_i(y_1)$.

It follows that $\hat{u}_1 = s_i(\tilde{u}_1)$, $v_1 = (\tilde{u}_1, 0)$. According to Lemma 2.4 $f(x_1) \in U_j$, $\| \varphi_j \circ f(x_1) -$

$- \varphi_j \circ g(x_1) \| \leq K_1 \varepsilon$, hence (cf.(2.9),(2.14))

$f(x_1) \in U_j^1$, $\| \varphi_j \circ f(x_1) - \varphi_j \circ g(x_1) \| \leq \varepsilon$ by Definition 1.2 and $B(\tilde{\varphi}_j \circ f(x_1), K_2\rho) \subset \varphi_j(U_j^1)$

by (2.9). Put $h(\lambda) = f \circ \varphi_i^{-1}(u_1 + \lambda(v_1 - u_1))$ for $\lambda \in \langle 0, 1 \rangle$ and apply Lemma 2.1 ($V = U_j^1$, $\psi = \tilde{\varphi}_j$, $\alpha = K_2$).

It follows that $f(y_1) \in U^1_j$ and $\| \tilde{\varphi}_j \circ f(y_1) -$
$- \tilde{\varphi}_j f(x_1) \| \leqq K_2 \rho$, $\| \tilde{\varphi}_j \circ f(y_1) - \tilde{\varphi}_j \circ g(x_1) \| \leqq K_2 \rho + \varepsilon$.
Similarly $f(y_2) \in U^1_j$ and $\| \tilde{\varphi}_j \circ f(y_2) - \tilde{\varphi}_j \circ g(x_2) \| \leqq$
$\leqq K_2 \rho + \varepsilon$ and $\| \tilde{\varphi}_j \circ f(y_2) - \tilde{\varphi}_j \circ f(y_1) \| \leqq 2(K_2 \rho + \varepsilon)$
(as $\tilde{\varphi}_j \circ g(x_2) = \tilde{\varphi}_j \circ g(x_1)$). Apply Lemma 2.1 putting
$Q = \sigma(M)$, $V = U^1_i \cap \sigma(M)$, $\psi = \varphi_i$, $h_1(\lambda) = (f|_{\sigma(M)})^{-1} \circ$
$\circ \varphi_j^{-1}(\varphi_j \circ f(y_1) + \lambda(\varphi_j \circ f(y_2) - \varphi_j \circ f(y_1)))$, $\lambda \in \langle 0, 1 \rangle$
(h_1 is well defined, as $\tilde{\varphi}_j(U^1_j)$ is convex, cf. (1.12) and (1.13)). $D\varphi_i \circ h_1(\lambda) = (D_1 \tilde{f}_{j,i}(w(\lambda)))^{-1}$,
$w(\lambda) = (\tilde{\varphi}_j \circ f(y_1) + \lambda(\tilde{\varphi}_j \circ f(y_2) - \tilde{\varphi}_j f(y_1)), 0)$.
Therefore by (1.18),(2.18) and Lemma 2.1

$$\tilde{u}_2 = \tilde{\varphi}_i(y_2) \in \mathcal{B}(\tilde{u}_1, R_1) ,$$

$$\| \tilde{u}_2 - \tilde{u}_1 \| \leqq \eta^{-1} 2(K_2 \rho + \varepsilon) . \tag{2.20}$$

Find $\ell \in I$ such that $f(x_1) \in U^1_\ell$, $\mathcal{B}(\tilde{\varphi}_\ell \circ f(x_1), R_1) \subset$
$\subset \tilde{\varphi}_\ell(U^1_\ell)$. Put $h_2(\lambda) = f \circ \varphi_i^{-1}(u_1 + \lambda(u_2 - u_1))$
and apply Lemma 2.1 with $Q = E$, $V = U^1_\ell$, $\psi = \tilde{\varphi}_\ell$, $\alpha = K_2(1+L)$ (cf. Lemma 2.2, (2.8),(2.5) and (2.18)). It follows that
$h_2(\lambda) \in U^1_\ell$ for $\lambda \in \langle 0, 1 \rangle$ and by Definition 1.2
$g \circ \varphi_i^{-1}(u_1 + \lambda(u_2 - u_1)) \in U_\ell$ for $\lambda \in \langle 0, 1 \rangle$ and

$$\| Dg_{\ell,i}(u_1 + \lambda(u_2 - u_1)) - Df_{\ell,i}(u_1 + \lambda(u_2 - u_1)) \| \leqq \varepsilon .$$

$$\tilde{\varphi}_\ell \circ g(x_2) - \tilde{\varphi}_\ell \circ g(x_1) = \int_0^1 D\tilde{g}_{\ell,i}(u_1 + \lambda(u_2 - u_1))(u_2 - u_1) d\lambda = D_1 \tilde{f}_{\ell,i}(v_1)(\tilde{u}_2 - \tilde{u}_1) + \Xi , \tag{2.21}$$

$$D\tilde{g}_{\ell,i}(u_1+\lambda(u_2-u_1))(u_2-u_1) = D_1\tilde{f}_{\ell,i}(v_1)(\tilde{u}_2-\tilde{u}_1) +$$
$$+D_2\tilde{f}_{\ell,i}(v_1)(s_i(\tilde{u}_2)-s_i(\tilde{u}_1))+[D\tilde{f}_{\ell,i}(u_1+\lambda(u_2-u_1))-D\tilde{f}_{\ell,i}(v_1)].$$
$$(u_2-u_1)+[D\tilde{g}_{\ell,i}(u_1+\lambda(u_2-u_1))-$$
$$-D\tilde{f}_{\ell,i}(u_1+\lambda(u_2-u_1))](u_2-u_1) \quad ,$$

hence (cf.(1.16))

$$\|\Xi\| \leqq [K_2 L+(1+L)\omega_2((1+L)\eta^{-1}2(K_2\varphi+\varepsilon)) +$$
$$+\varepsilon(1+L)]\|\tilde{u}_2-\tilde{u}_1\| < \eta\|\tilde{u}_2-\tilde{u}_1\| \quad .$$

By (1.18) $\|D_1\tilde{f}_{\ell,i}(v_1)(\tilde{u}_2-u_1)\geqq \eta\|\tilde{u}_2-\tilde{u}_1\|$, therefore by (2.21) $\tilde{\varphi}_\ell\circ g_2(x)\neq\tilde{\varphi}_\ell\circ g_1(x)$ and $\pi\circ g_2(x)\neq$ $\neq\pi\circ g_1(x)$, which contradicts the assumptions of Lemma 2.8.

Lemma 2.9. There exists a unique map $g^*: S(Diff,\rho,L)\longrightarrow S$ such that $g^*(s)(M)=g\circ s(M)$.

Proof. Let $s\in S(Diff,\rho,L)$. Lemma's 2.7 and 2.8 imply that for $x\in M$ the intersection $\pi^{-1}(x)\cap g\circ s(M)$ contains just a single point which will be denoted by $z(x)$. Thus $z: M\longrightarrow E(\rho)$ is defined, $z\in S$ (cf. (1.13) and (2.9)) and $g^*(s)=z$.

Assume that

(2.22) $(K_1\varepsilon+K_2\omega_1(\varepsilon)+\omega_2(\rho))(1+L)^2+K_2L^2\leqq\eta(1-\xi)L$.

Lemma 2.10. Let $s\in S(Diff,\rho,L)$. Then $g^*(s)\in$ $\in S(Diff,\rho,L)$.

Proof. Put $g^*(s)=z$. Let $j\in I$, $\tilde{u}\in\tilde{\varphi}_j(U_j^1)$, $x=\varphi_j^{-1}(\tilde{u},z_j(\tilde{u}))$. As $x\in z(M)=g\circ s(M)$, there exists $y\in s(M)$ such that $g(y)=x$. Find

$i \in I$ such that $y \in U_i^1$ and put $v = (\tilde{v}, \hat{v}) = \varphi_i(y)$; $\hat{v} = s_i(\tilde{v})$, as $y \in s(M)$. Hence

(2.23) $\tilde{u} = \tilde{g}_{j,i}(\tilde{v}, s_i(\tilde{v})),\ x_j(\tilde{u}) = \hat{g}_{j,i}(\tilde{v}, s_i(\tilde{v}))$.

By Lemma 2.5 x_j is continuously differentiable on some neighbourhood of $\tilde{u}$. By (2.14), $\| x_j(\tilde{u}) \| \leqq \rho$. It remains to prove that

(2.24) $\| D x_j(\tilde{u}) \| \leqq L$.

By (2.20)

(2.25) $D x_j(\tilde{u}) = [D_1 \hat{g}_{j,i}(\tilde{v}, s_i(\tilde{v})) + D_2 \hat{g}_{j,i}(\tilde{v}, s_i(\tilde{v})) \circ$
$\circ D s_i(\tilde{v})] \circ [D_1 \tilde{g}_{j,i}(\tilde{v}, s_i(\tilde{v})) + D_2 \tilde{g}_{j,i}(\tilde{v}, s_i(\tilde{v})) \circ D s_i(\tilde{v})]^{-1} =$
$= D_2 \hat{f}_{j,i}(\tilde{v}, 0) \circ D s_i(\tilde{v}) \circ [D_1 \tilde{f}_{j,i}(\tilde{v}, 0)]^{-1} + \Xi$.

By (1.20)

(2.26) $\| D_2 \hat{f}_{j,i}(\tilde{v}, 0) \circ D s_i(\tilde{v}) \circ [D_1 \tilde{f}_{j,i}(\tilde{v}, 0)]^{-1} \| \leqq \xi L$.

To estimate Ξ, put

$A = D_2 \hat{f}_{j,i}(\tilde{v}, 0) \circ D s_i(\tilde{v}),\ B = D_1 \tilde{f}_{j,i}(\tilde{v}, 0)$,

$A + E = D_1 \hat{g}_{j,i}(\tilde{v}, s_i(\tilde{v})) + D_2 \hat{g}_{j,i}(\tilde{v}, s_i(\tilde{v})) \circ D s_i(\tilde{v})$,

$B + H = D_1 \tilde{g}_{j,i}(\tilde{v}, s_i(\tilde{v})) + D_2 \tilde{g}_{j,i}(\tilde{v}, s_i(\tilde{v})) \circ D s_i(\tilde{v})$.

It follows that

$\| A \circ B^{-1} \| \leqq \xi L$ (cf. (2.26)), $\| B^{-1} \| \leqq \eta^{-1}$ (cf. (1.18)),

$\| E \| \leqq (K_1 \varepsilon + K_2 \omega_1(\varepsilon) + \omega_2(\rho))(1 + L) = \varkappa_1$ (cf. (1.21), (1.16), Lemma 2.4),

$$(2.27)\quad \|H\| \leqq K_2 L + (K_1\varepsilon + K_2\omega_1(\varepsilon) + \omega_2(\rho))(1+L) = \varkappa_2,$$

$$\Xi = (A+E)\circ(B+H)^{-1} - A\circ B^{-1} =$$
$$= A\circ B^{-1}\circ[(id + H\circ B^{-1})^{-1} - id] + E\circ B^{-1}\circ(id+H\circ B^{-1})^{-1},$$

$$(2.28)\quad \|\Xi\| \leqq \xi L\eta^{-1}\varkappa_2(1-\eta^{-1}\varkappa_2)^{-1} + \varkappa_1\eta^{-1}(1-\eta^{-1}\varkappa_2)^{-1},$$

(2.24) is implied by (2.25),(2.26),(2.28) provided that

$$\xi L + \xi L\eta^{-1}\varkappa_2(1-\eta^{-1}\varkappa_2)^{-1} + \eta^{-1}\varkappa_1(1-\eta^{-1}\varkappa_2)^{-1} \leqq L,$$

i.e. $\varkappa_1 + L\varkappa_2 \leqq (1-\xi)\eta L$, which is (2.22).

<u>Note 2.2</u>. If $s \in \mathcal{S}(\mathcal{D}iff, \rho, L)$, $i, j \in I$, $\tilde{u} \in \tilde{\varphi}_j(U_j^1)$, $\tilde{v} \in \tilde{\varphi}_i(U_i^1)$, $\tilde{u} = q_{j,i}(\tilde{v}, s_i(\tilde{v}))$ then

$$(2.29)\quad \|[D_1\tilde{q}_{j,i}(\tilde{v}, s_i(\tilde{v})) + D_2\tilde{q}_{j,i}(\tilde{v}, s_i(\tilde{v}))\circ Ds_i(\tilde{v})]^{-1}\| \leqq (\eta - \varkappa_2)^{-1},$$

$\varkappa_2$ being defined in (2.27).

This is a by-product of the proof of Lemma 2.10, as the right hand side in (2.29) may be written as $\|B^{-1}(id + H\circ B^{-1})^{-1}\|$.

For $r > 0$ put $\mathcal{S}(r) = \{s \in \mathcal{S} \mid s(M) \subset E(r)\}$. Observe that if $x \in E(r)$, $x \in U_i^1$, $s \in \mathcal{S}(r)$, $0 < r < R_2$, then $s_0\pi(x) \in U_i^1$ so that $s_i \circ \tilde{\varphi}_i(x)$ is defined (cf. (1.13)).

<u>Definition 2.1</u>. For $x \in E(\rho)$, $s, r \in \mathcal{S}(\rho)$ put

$$\|x, s\| = \inf\{\|\hat{\varphi}_i(x) - s_i\circ\tilde{\varphi}_i(x)\| \mid i \in I,\ x \in U_i^1\},$$

$$(2.30)\quad \|s, z\| = \max(\sup_{x\in s(M)}\|x, z\|,\ \sup_{x\in z(M)}\|x, s\|).$$

Lemma 2.11. (2.30) defines a metric on $S(\varphi)$.

Note 2.2. If $x \in U_i^1 \cap E(\varphi)$, $s \in S(\varphi)$, then $\|x, s\| \leq \|\hat{\varphi}_i(x) - s_i \circ \tilde{\varphi}_i(x)\| \leq K_1 \|x, s\|$. This follows by (1.8) and (1.11).

Assume that

(2.31) $\quad \xi_1 = \xi + \varepsilon + L(K_2 + \varepsilon) < 1$.

Lemma 2.12. Let $x \in E(\varphi)$, $s \in S(Diff, \varphi, L)$. Then

$$\|g(x), g^*(s)\| \leq \xi_1 \|x, s\| .$$

Proof. Put $g^*(s) = z$. Find $j \in I$ such that $g(x) \in U_j^{(1)}$, $\mathcal{B}(\tilde{\varphi}_j \circ g(x), R_1) \subset \tilde{\varphi}_j(U_j^1)$. Choose $i \in I$ such that $x \in U_i^1$ and put $u = (\tilde{u}, \hat{u}) = \varphi_i(x)$, $v = (\tilde{u}, s_i(\tilde{u}))$, $h(\lambda) = f \circ \varphi_i^{-1}(u + \lambda(v - u))$.

By Lemma 2.4 $h(0) = f(x) \in U_j$, $\|\varphi_j \circ h(0) - \varphi_j \circ g(x)\| \leq K_1 \varepsilon$, hence $\varphi_j \circ f(x) \in U_j^1$, $\|\varphi_j \circ f(x) - \varphi_j \circ g(x)\| \leq \varepsilon$ by Definition 1.2 and $\mathcal{B}(\tilde{\varphi}_j \circ f(x), R_1 - \varepsilon) \subset \tilde{\varphi}_j(U_j^1)$ (cf. (2.18)).

Lemma 2.1 ($V = U_j^1$, $\psi = \tilde{\varphi}_j$, $\alpha = K_2$, cf. Lemma 2.2, (2.8)) implies that $h(\lambda) \in U_j^1$,

$\mathcal{B}(\tilde{\varphi}_j \circ h(\lambda), R_1 - \varepsilon - 2K_2\varphi) \subset \tilde{\varphi}_j(U_j^1)$ for $\lambda \in \langle 0, 1 \rangle$.

Therefore (cf. Definition 1.2 and (1.15))

$g \circ \varphi_i^{-1}(u + \lambda(v - u)) \in U_j^1$, $\|Dg_{j,i}(u + \lambda(v - u)) - Df_{j,i}(u + \lambda(v - u))\| \leq \varepsilon$, $\|Dg_{j,i}(u + \lambda(v - u))\| \leq K_2 + \varepsilon$. $g \circ \varphi_i^{-1}(v) \in z(M)$,

as $\varphi_i^{-1}(v) \in s(M)$; therefore $\hat{g}_{j,i}(v) = z_j \circ \tilde{g}_{j,i}(v)$.

Observe that $\varphi_j \circ q(x) = q_{j,i}(u)$. Hence

$$(2.32)\ \|q(x), z\| \le \|\hat{\varphi}_j \circ q(x) - z_j \circ \tilde{\varphi}_j \circ q(x)\| =$$
$$= \|\hat{q}_{j,i}(u) - z_j \circ \tilde{q}_{j,i}(u)\| \le \|\hat{q}_{j,i}(u) - \hat{q}_{j,i}(v)\| +$$
$$+ \|z_j \circ \tilde{q}_{j,i}(v) - z_j \circ \tilde{q}_{j,i}(u)\|, \tilde{q}_{j,i}(v) - \tilde{q}_{j,i}(u) =$$
$$= \int_0^1 D\tilde{q}_{j,i}(u + \lambda(v-u))(v-u)\,d\lambda \ ,$$
$$\|\tilde{q}_{j,i}(v) - \tilde{q}_{j,i}(u)\| \le (K_2 + \varepsilon)\|v-u\| = (K_2+\varepsilon)\|\hat{\varphi}_i(x) - s_i \circ \tilde{\varphi}_i(x)\|$$

$$(2.33)\ \|z_j \circ \tilde{q}_{j,i}(v) - z_j \circ \tilde{q}_{j,i}(u)\| \le L(K_2+\varepsilon)\|\hat{\varphi}_i(x) -$$
$$- s_i \circ \tilde{\varphi}_i(x)\|, \hat{q}_{j,i}(v) - \hat{q}_{j,i}(u) = \int_0^1 D\hat{q}_{j,i}(u + \lambda(v-u))(v-$$
$$- u)\,d\lambda = \int_0^1 D_1 \hat{q}_{j,i}(u + \lambda(v-u))(\hat{v} - \hat{u})\,d\lambda$$

$$\|D_1 \hat{q}_{j,i}(u + \lambda(v-u))\| \le \|D_1 f_{j,i}(v)\| +$$
$$+ \|D_1 \hat{q}_{j,i}(u+\lambda(v-u)) - D_1 f_{j,i}(v)\| \le \xi + \omega_2(2\rho) \ ;$$

$$(2.34)\ \|\hat{q}_{j,i}(v) - \hat{q}_{j,i}(u)\| \le (\xi + \omega_2(2\rho))\|\hat{\varphi}_i(x) - s_i \circ \tilde{\varphi}_i(x)\| .$$

(2.31) - (2.34) imply that $\|q(x), q^*(s)\| \le$ $\le \xi_1 \|\hat{\varphi}_i(x) - s_i \circ \tilde{\varphi}_i(x)\|$ and Lemma 2.12 holds, as i fulfils no other condition than $x \in U_i^1$.

<u>Lemma 2.13</u>. Let $s, z \in S(Diff, \rho, L)$. Then $\|q^*(s), q^*(z)\| \le \xi_1 \|s, z\|$.

This follows immediately from Lemma 2.10 and Definition 2.1.

<u>Lemma 2.14</u>. Let $0 < \alpha < 1$, $0 < \beta < 1$, let ω_3 be a modulus of continuity, $\gamma > 0$, $\omega_3(\gamma) = 2L(1-\alpha)$. Then there exists a modulus of continuity Ω such that

(2.35) $\Omega(\lambda) = 2L$ for $\lambda \geq \beta\gamma$, $\Omega(\beta\lambda) \geq$
$\geq \alpha\,\Omega(\lambda) + \omega_3(\lambda)$ for $0 \leq \lambda \leq \beta\gamma$.

To prove Lemma 2.14 it is sufficient to put $\gamma_k = \beta^k\gamma$, $k = 0, 1, 2, \dots$ and to define Ω by $\Omega(\lambda) = 2L$ for $\lambda \geq \gamma_1$ and then by $\Omega(\beta\lambda) = \alpha\,\Omega(\lambda) + \omega_3(\lambda)$ step by step on $\langle \gamma_2, \gamma_1 \rangle$, $\langle \gamma_3, \gamma_2 \rangle$, etc.

Assume that

(2.36) $\varkappa_1 + \varkappa_2 + L(\xi + \varkappa_1) < \eta(1 - \xi)$,
$\varkappa_1 = \omega_2(\rho) + K_1\varepsilon + K_2\omega_1(\varepsilon)$, $\varkappa_2 = \varkappa_1 + K_2 L$.

Lemma 2.15. There exists such a modulus of continuity Ω that

$$g^*: S(\mathrm{Diff}, \rho, L, \Omega) \rightarrow S(\mathrm{Diff}, \rho, L, \Omega).$$

Proof. Let Ω be a modulus of continuity, $s \in S(\mathrm{Diff}, \rho, L, \Omega)$, $g^*(s) = z$. Let $j \in I$, $\tilde{u}_1, \tilde{u}_2 \in \tilde{\varphi}_j(U_j^1)$, $\|\tilde{u}_1 - \tilde{u}_2\| < (\eta - \varkappa_2)(1+L)^{-1}R_1$. As $g|_{s(M)}: s(M) \longrightarrow z(M)$ is bijective (cf. Lemmas 2.8 and 2.9), there exist $x_1, x_2 \in s(M)$ such that $\tilde{u}_k = \tilde{\varphi}_j \circ g(x_k)$, $k = 1, 2$. Find $i \in I$ such that $x_1 \in U_i^1$, $B(\tilde{v}_1, R_1) \subset \tilde{\varphi}_i(U_i^1)$, $v_1 = (\tilde{v}_1, \hat{v}_1) = \varphi_i(x_1)$. Denote by $G: z(M) \rightarrow s(M)$ the inverse to $g|_{s(M)}: s(M) \longrightarrow z(M)$ and put

$$h(\lambda) = G \circ \varphi_j^{-1}(\tilde{u}_1 + \lambda(\tilde{u}_2 - \tilde{u}_1), z_j(\tilde{u}_1 + \lambda(\tilde{u}_2 - \tilde{u}_1))), \lambda \in \langle 0, 1 \rangle.$$

If $h(\lambda) \in U_i^1$ for some $\lambda \in \langle 0, 1 \rangle$, then

$$g_{j,i} \circ \varphi_i \circ h(\lambda) = (\tilde{u}_1 + \lambda(\tilde{u}_2 - \tilde{u}_1), z_j(\tilde{u}_1 + \lambda(\tilde{u}_2 - u_1))),$$

and by Note 2.2

$\| D\varphi_i \circ h(\lambda) \| \leq (\eta - \varkappa_2)^{-1}(1+L)\| \tilde{u}_2 - \tilde{u}_1 \|$,

hence by Lemma 2.1 $h(\lambda) \in \mathcal{U}^1_i$ for $\lambda \in \langle 0, 1 \rangle$.

Put $\tilde{v}_2 = \tilde{\varphi}_i \circ h(1)$; $h(1) \in s(M)$, therefore

$\varphi_i \circ h(1) = (\tilde{v}_2, s_i(\tilde{v}_2))$,

$g_{j,i}(\tilde{v}_2, s_i(\tilde{v}_2)) = (\tilde{u}_2, x_j(\tilde{u}_2))$ and $\varphi_i^{-1}(\tilde{v}_2$, $s_i(\tilde{v}_2)) = x_2$. By Lemma 2.1

(2.37) $\| \tilde{v}_2 - \tilde{v}_1 \| \leq (\eta - \varkappa_2)^{-1}(1+L)\| \tilde{u}_2 - \tilde{u}_1 \|$.

By (2.25),

$$D x_j(\tilde{u}_k) = [(D_1 \hat{g})_k + (D_2 \hat{g})_k \circ D s_i(\tilde{v}_k)] \circ [(D_1 \tilde{g})_k + (D_2 \tilde{g})_k \circ D s_i(\tilde{v}_k)], \; (D_1 \hat{g})_k = D_1 \hat{g}_{j,i}(\tilde{v}_k, s_i(\tilde{v}_k))$$

etc., $k = 1, 2$ and

$$(2.38) \quad D x_j(\tilde{u}_2) - D x_j(\tilde{u}_1) = (D_2 \hat{g})_2 \circ [D s_i(\tilde{v}_2) - D s_i(\tilde{v}_1)] \circ [(D_1 \tilde{g})_2 + (D_2 \tilde{g})_2 \circ D s_i(\tilde{v}_2)]^{-1} - D x_j(\tilde{u}_1) \circ (D_2 \hat{g})_1 \circ [D s_i(\tilde{v}_2) - D s_i(\tilde{v}_1)] \circ [(D_1 \tilde{g})_2 + (D_2 \tilde{g})_2 \circ D s_i(\tilde{v}_2)]^{-1} + \Xi$$

and it is not difficult to see that Ξ may be estimated by

$$\| \Xi \| \leq \omega_3 (\| \tilde{u}_2 - \tilde{u}_1 \|) ,$$

ω_3 being a modulus of continuity. The first term on the right hand side of (2.38) may be estimated in a similar manner as the right hand side of (2.25). Put

$A_1 = D_2 \hat{f}_{j,i}(\tilde{v}_2, 0)$, $A_1 + E_1 = (D_2 \hat{g})_2$,
$B_1 = D_1 \tilde{f}_{j,i}(\tilde{v}_2, 0)$, $B_1 + H_1 = (D_1 \tilde{g})_2 \circ D s_i(\tilde{v}_2)$.

Then

$\|A_1\|\cdot\|B_1^{-1}\| \leqq \xi$, $\|E_1\| \leqq (\omega_2(\rho)+K_1\varepsilon+K_2\omega_1(\varepsilon))(1+L)=\varkappa_1$,
$\|B_1^{-1}\| \leqq \eta^{-1}$, $\|H_1\| \leqq (\omega_2(\rho)+K_1\varepsilon+K_2\omega_1(\varepsilon))(1+L)+K_2L=\varkappa_2$.
Therefore, the first term on the right hand side of (2.38) may be estimated by

$$(2.39)\quad \Omega(\|\tilde v_2-\tilde v_1\|)\cdot\|A_1+E_1\|\cdot\|(B_1+H_1)^{-1}\| \leqq$$
$$\leqq \Omega(\|\tilde v_2-\tilde v_1\|)\cdot(\|A_1\|\cdot\|B_1^{-1}\|+\|E_1\|\cdot\|B_1^{-1}\|)\|(id+ +H_1\circ B_1^{-1})\| \leqq \Omega(\|\tilde v_2-\tilde v_1\|)(\xi+\varkappa_1\eta^{-1})(1-\varkappa_2\eta^{-1})^{-1} .$$

In order to estimate the second term, observe that

$$\|Dx_j(\tilde u_1)\leqq L,\ \|(D_2 g)_1\|\leqq \xi+\omega_2(\rho)+K_1\varepsilon+K_2\omega_1(\varepsilon)=\xi+\varkappa_1$$
$$\|[(D_1 g)_2+(D_2 g)_2\circ Ds_1(\tilde v_2)]^{-1}\|\leqq\|B_1^{-1}\|\cdot\|(id+H_1 B_1^{-1})^{-1}\|\leqq$$
$$\leqq \eta^{-1}(1-\eta^{-1}\varkappa_2)=(\eta-\varkappa_2)^{-1} .$$

Therefore, the second term in (2.38) may be estimated by

$$(2.40)\quad \Omega(\|\tilde v_2-\tilde v_1\|)\cdot L\cdot(\xi+\varkappa_1)(\eta-\varkappa_2)^{-1} ;$$

(2.38) together with (2.39) and (2.40) imply

$$(2.41)\quad \|Dx_j(\tilde u_2)-Dx_j(\tilde u_1)\|\leqq \omega_3(\|\tilde v_2-\tilde v_1\|)+\Omega(\|\tilde v_2-\tilde v_1\|)\cdot\alpha ,$$
$$\alpha = (\xi\eta+\varkappa_1+L(\xi+\varkappa_1))(\eta-\varkappa_2)^{-1} .$$

By (2.36), $\alpha<1$. Put $\beta=(\eta-\varkappa_2)(1+L)^{-1}$. Without loss of generality it may be assumed that $\omega_3(\gamma)= =2L$ for some γ, $0<\gamma<(\eta-\varkappa_2)(1+L)^{-1}R_1$. It follows from (2.41) and (2.37) that Dx_j admits Ω as a modulus of continuity provided that Ω fulfils (2.35) and the existence of such Ω is guaranteed by Lemma 2.14.

The following lemma is standard.

Lemma 2.16. $S(\mathit{Diff}, \rho, L, \Omega)$ is a complete nonempty space.

To finish the proof of Main Theorem, observe that L, ρ and ε have to fulfil (2.5),(2.8),(2.9),(2.15),(2.18), (2.19),(2.22),(2.31) and (2.36). The special role of L is due to (2.22), all other inequalities require that L, ρ, ε are sufficiently small. Therefore, there exists $L_1 > 0$ and to every L, $0 < L \leq L_1$ there exist $\rho > 0$ and $\varepsilon > 0$ such that the above conditions are fulfilled. Find Ω by Lemma 2.15. By Lemmas 2.10, 2.13 and 2.16 there exists $r \in S(\mathit{Diff}, \rho, L, \Omega)$ such that $g^*(r) = r$, i.e. (Lemma 2.9) $r(M) = g \circ r(M)$; $g|_{r(M)}$ is bijective by Lemmas 2.7 and 2.8 and it is a diffeomorphism by Lemma 2.5 and (2.23). (1.24) holds by (2.14) and (1.25) follows by Lemma 2.12, (1.18),(1.6) and (2.5). The proof of Main Theorem is complete.

§ 3. Flows

Main Theorem may be modified so that it may be applied directly to differential equations, functional differential equations etc.

Assume that $T > 0$ and that for $t \in \langle 0, 2T \rangle$ there exists $f_t : E(R_2) \longrightarrow E$ fulfilling (1.15) and (1.16); in addition, let (1.17) - (1.20) be fulfilled for $t \in \langle T, 2T \rangle$. Let $g_t : E(R_2) \rightarrow E$ be continuous for $t \in \langle 0, 2T \rangle$.

Definition 1.3. Let $\varepsilon > 0$. $\{g_t\}$ is said to be ε-close to $\{f_t\}$, if g_t is ε-close to f_t for any $t \in \langle 0, 2T \rangle$.

Assume that

(3.1) if $x \in E(R_2)$, $t_1, t_2, t_3, t_4 \in \langle 0, 2T \rangle$, $t_1 + t_2 = t_3 + t_4$, $\varphi_{t_1}(x)$, $\varphi_{t_3}(x) \in E(R_2)$, then

$$\varphi_{t_2} \circ \varphi_{t_1}(x) = \varphi_{t_4} \circ \varphi_{t_3}(x).$$

Theorem 3.1. There exists $L_1 > 0$ and to L, $0 < L \leq L_1$ there exist $\rho > 0$, $\varepsilon > 0$ and a modulus of continuity Ω such that if $\{\varphi_t\}$ is ε-close to $\{f_t\}$, then there exists $p \in S(Diff, \rho, L, \Omega)$ and $\varphi_t|_{p(M)} : p(M) \longrightarrow p(M)$ is a diffeomorphism onto $p(M)$ for $t \in \langle 0, 2T \rangle$. Moreover,

(3.2) $\varphi_t(E(\rho)) \subset E(\rho)$ for $t \in \langle T, 2T \rangle$,

and

(3.3) if $i, j \in I$, $x \in E(\rho) \cap U_i^1$, $z \in U_j^1$, $z = \varphi_t \circ \varphi_T^k(x)$ for some $t \in \langle T, 2T \rangle$, $k = 0, 1, 2, \dots$, then

$$\|\hat{\varphi}_j(z) - p_j \circ \tilde{\varphi}_j(z)\| \leq K_1 \xi_1^{k+1} \|\hat{\varphi}_i(x) - p_i \circ \tilde{\varphi}_i(x)\|.$$

Note 3.1. If there exists a vector field Y on E such that $\varphi_t(x)$ is the value at t of the solution y of

(3.4) $$\frac{dy}{dt} = Y,$$

$y(0) = x$ (assume local existence and uniqueness for solutions of (3.4)), then (3.1) is fulfilled and the following assertions are consequences of Theorem 3.1.

(3.5) if $x \in p(M)$, then there exists a solution y

of (3.4) such that $y(0)=x$, $y(t)\in p(M)$ for $t\in R^1$;

(3.6) If y is a solution of (3.4) and $y(t)\in E(\varphi)$ for $t\in R^1$, then $y(t)\in p(M)$ for $t\in R^1$.

Proof. For $\lambda\in\langle T,2T\rangle$ denote by p_λ the element of $S(\mathcal{D}iff,\varphi,L,\Omega)$ which exists according to Main Theorem, if g is replaced by g_λ. (3.2) is fulfilled by (1.24). By (3.2) $g_\tau\circ g_\lambda(x)$, $g_\lambda\circ g_\tau(x)$ are defined for $x\in E(\varphi),\lambda,\tau\in\langle T,2T\rangle$ and by (3.1) $g_\lambda\circ g_\tau(x)=g_\tau\circ g_\lambda(x)$. By Main Theorem $g_\tau|_{p_\tau(M)}$ is a bijection onto $p_\tau(M)$; denote by G_τ its inverse. Let $x\in p_\tau(M)$, $y=G_\tau^k x$ for some positive integer k. Then $g_\tau^k\circ g_\lambda(y)=g_\lambda\circ g_\tau^k(y)=g_\lambda(x)$ and $g_\lambda(x)\in p_\tau(M)$ by Corollary 1.1, i.e. $g_\lambda|_{p_\tau(M)}: p_\tau(M)\longrightarrow p_\tau(M)$. By Lemma 2.7 g_λ maps $p_\tau(M)$ onto $p_\tau(M)$ and by Corollary 1.1 $p_\tau(M)=p_\lambda(M)$, i.e. $p_\tau=p_\lambda$. Write p instead of p_λ. If $t\in\langle T,2T\rangle$, then $g_t|_{p(M)}: p(M)\longrightarrow p(M)$ is a diffeomorphism onto $p(M)$ by Main Theorem. Put $G_T=(g_T|_{p(M)})^{-1}$. If $t\in\langle 0,T\rangle$, then $g_t|_{p(M)}=g_{t+T}|_{p(M)}\circ G_T$ and $g_t|_{p(M)}$ is a diffeomorphism onto $p(M)$. (3.2) follows by (1.25).

It may be useful to have an estimate analogous to the one from (3.3) for $t\in\langle 0,T\rangle$; in a similar manner as in § 2 it may be proved that

(3.7) $q_t(E(\rho)) \subset E(K_2\rho+\varepsilon)$ for $t \in \langle 0, T\rangle$,

(3.8) if $t \in \langle 0, T\rangle$, $i, j \in I$, $x \in U_i^1 \cap E(\rho)$, $z = q_t(x) \in U_j^1$,

then

$\|\hat{\varphi}_j(z) - p_j \circ \tilde{\varphi}_j(z)\| \leq K_1(K_2+\varepsilon)(1+L)\|\hat{\varphi}_i(x) - p_i \circ \tilde{\varphi}_i(x)\|$.

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