Josef Štěpán

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SOME NOTES ON THE CONVOLUTION SEMIGROUP OF PROBABILITIES ON A METRIC GROUP

Josef ŠTĚPÁN, Praha

Summary: The present paper deals with probability measures, say P, on a complete separable metric abelian group such that there exists a nontrivial solution μ of the equation $P = P * \mu$. Such measures will be characterized in Section 2. We shall make use of these results in Section 3 finding extreme points of the closed convex hull of all translations of a probability measure P . Most of the methods which are used here are due to Parthasarathy [1967].

1. Introduction

Let G be a complete separable metric abelian group. Let us consider the space $M(G)$ of all probability measures which are defined on the σ-algebra $\mathcal{B}$ of Borel subsets of G . The space $M(G)$ is a commutative semigroup under the operation of convolution ($*$) which can be defined as

$$P * Q(A) = \int_G P(t^{-1}A)\,Q(dt)$$

for any two $P, Q \in M(G)$ and any $A \in \mathcal{B}$. Denote by ε_g the probability measure degenerated at a point

$g \in G$. Then ε_1 is the identity and the only regular element of $M(G)$.

Consider the family of sets

$$A_\mu(f_1 f_2, \ldots, f_n, \varepsilon) = \{\nu \in M(G) : |\mu(f_i) - \nu(f_i)| < \varepsilon, \quad i = 1, 2, \ldots, n\}$$

where $f_1 \ldots f_n$ are elements of $C(G)$ and $\varepsilon > 0$. This family is a base for a topology of $M(G)$ which is known as the weak topology.

The space $M(G)$ in the weak topology is a metrizable topological semigroup (see [1]) with the following properties:

1.1. Consider $P \in M(G)$ and $\mathcal{D} \subset M(G)$. Then the set $P * \mathcal{D}$ is relatively compact if and only if the set $\mathcal{D}$ is relatively compact (see [1], Chapter III, 2.1; [4]).

Put $D(P) = \overline{co}\{P_t : t \in G\}$ for each $P \in M(G)$, where the right-hand side is the closed convex hull of the set of translations of P. ($P_t(A) = P(t^{-1}A)$ for $t \in G$, $A \in \mathcal{B}$).

Then (see [4])

1.2. $D(P) = P * M(G)$ for every $P \in M(G)$.

The assertion is a not precisely easy consequence of the theorem on the separation of convex sets in linear topological spaces (see [1], V.III.10).

2. Invariant probability measures

Let us consider the ideal $J \subset M(G)$, $J = \{P \in M(G) : P = P * \mu$ for some $\mu \in M(G)$, $\mu \neq \varepsilon_1\}$. In this section we shall describe the elements of $\mathcal{J}$. For any $P \in M(G)$ we denote by A_P the set $A_P = \{t \in G : P_t \doteq P_1\}$. We shall say that the set A_P is <u>the maximal invariant set</u> of the measure $P \in M(G)$.

Now, we can prove the following

2.1. <u>Lemma</u>. The maximal invariant set A_P is a compact subgroup of G for every $P \in M(G)$.

<u>Proof</u>. Take two points $t, s \in A_P$. Then for any $A \in \mathcal{B}$, we have

$P_{t.s}(A) = P_1(t^{-1}s^{-1}A) = P_t(s^{-1}A) = P_1(s^{-1}A) = P_s(A) = P(A)$

and $P_{t^{-1}}(A) = P_1(tA) = P_t(tA) = P(A)$.

Hence A_P is a subgroup. Further, it is obvious that A_P is a closed set. To prove its compactness let us consider a sequence $\{t_n\}_1^\infty \subset A_P$. Then $P = P_{t_n} = P * \varepsilon_{t_n}$.

By 1.1 the sequence $\{\varepsilon_{t_n}\}_1^\infty$ is relatively compact and by a well-known theorem due to Prochorov (see Theorem 6.7, Chapter II in [1]) there is a compact set $K \subset G$ such that $\varepsilon_{t_n}(K) > \frac{1}{2}$ for all n. Hence $\{t_n\}_1^\infty \subset K$ and the set A_P is compact. This completes the proof.

In the case when G is a complete separable metric group we can characterize idempotent elements of $M(G)$. It is known that $h^2 = h$ for some $h \in M(G)$ just if there is a compact subgroup $S \subset G$ such that h is the normalized Haar measure of S ($h(S) = 1$, $h_t = h_1$ for $t \in S$).

Denote by $\mathcal{A}$ the family of all compact subgroups $S \subset G$, $S \neq \{1\}$ and by h^S the normalized Haar measure of S. Then $\{h^S ; S \in \mathcal{A}\} \subset \mathcal{J}$ holds and we shall show that the set on the left side is "a base" for $\mathcal{J}$.

The following lemma is very important for our purposes.

2.2. __Lemma__. Suppose that P and μ are elements of $M(G)$ such that $\mu \neq \varepsilon_1$ and $P = P * \mu$. Then there exists an $S \in \mathcal{A}$, $S \subset A_P$ such that $P = P * h^S$ and $\mu(S) = 1$.

__Proof__. Take $P, \mu \in M(G)$, $\mu \neq \varepsilon_1$ such that $P = P * \mu$. It implies that $P = P * \nu_n$, where $\nu_n = \frac{1}{n}\sum_{k=1}^{n} \mu^k$ for $n = 1, 2, \dots$. By assertion 1.1 the sequence $\{\nu_n\}_1^\infty$ has an accumulation point $h \in M(G)$ and $P = P * h$ holds. Consider the subsequence $\{\nu_{n_k}\} \subset \{\nu_n\}$ such that $\nu_{n_k} \xrightarrow[k\to\infty]{} h$,

then

$$\| \nu_{n_k} * \mu - \nu_{n_k} \| = \left\| \frac{\mu^{n_k+1} - \mu}{n_k} \right\| \leq \frac{2}{n_k} \text{ for } k = 1, 2, \dots$$

(we have put $\| \mathcal{C} \| = \sup_{A \in \mathcal{B}} |\mathcal{C}(A)|$, where $\mathcal{C}$ is a set function on the σ-algebra $\mathcal{B}$)

which shows that $\nu_{n_k} * \mu \xrightarrow[k]{} h$ and, consequently, $h = h * \mu$. Therefore we can write $h = h * \nu_{n_k}$. Thus $h = h^2$ and h is a normalized Haar measure on a compact subgroup $S \subset G$. From the facts that $h = h * \mu$ and $\mu \neq \varepsilon_1$ we can easily deduce that $h \neq \varepsilon_1$. Hence $S \in \mathcal{A}$ and $h = h^S$. Since

$$1 = h(S) = \int_G h(t^{-1}S)\,\mu(dt) = \int_S h(t^{-1}S)\,\mu(dt) = \mu(S)$$

and $P_t = (P * h)_t = P * h_t^S = P * h = P$ for $t \in S$, the proof is completed.

2.3. Theorem. Let us suppose that G is a complete separable metric abelian group. Then $J = \bigcup_{S \in \mathcal{a}} D(h^S)$ holds. (We have employed the notation which was introduced in Section 1.)

Proof. According to Lemma 2.2 and the remark 1.2 we have $J \subset \bigcup_{S \in \mathcal{a}} D(h^S)$. On the contrary, let us suppose that $P \in D(h^S)$, where $S \in \mathcal{a}$. Then, again by the remark 1.2, there exists a μ such that $P = h^S * \mu$. We can write $P * h^S = (h^S)^2 * \mu = h^S * \mu = P$ and hence $P \in J$ as $h^S \neq \varepsilon_1$. The proof is completed.

The following assertion is an easy consequence of Theorem 2.3 and Corollary 6 in [5].

2.4. Corollary. Let us suppose that $P \in J$. Then P is an element of the ideal J if and only if there is $S \in \mathcal{a}$ such that $P(f) \leq \sup_{t \in G} h^S(f^t)$ for each $f \in C(G)$.

(We have used the notation $f^t(x) = f(t \cdot x)$ for t, $x \in G$.)

A slight reformulation of Theorem 2.3 is given in the following

2.5. Theorem. Let G be a complete separable metric abelian group. Then $P \in M(G)$ is an element of the ideal J if and only if the maximal invariant set of P, A_P, is an element of $\mathcal{a}$ ($A_P \neq \{1\}$). If $\mu \in M(G)$ is such that $P = P * \mu$ then $\mu(A_P) = 1$.

Proof. The second part of the theorem and the necessity of the first part follow easily from Lemma 2.2.

Conversely, let us suppose that $A_P \in \mathcal{A}$. Then there is $t \in A_P$, $t \neq 1$, and if we put

$\mu = \frac{1}{2}(\varepsilon_1 + \varepsilon_t)$ we have $\mu \neq \varepsilon_1$, $P = P * \mu$.
This implies that $P \in \mathcal{J}$ and the proof is completed.

The theorem which was just proved implies

2.6. Corollary. Suppose that G is a complete separable metric abelian group. Then the following statements are equivalent.

A) $\mathcal{J} \neq \emptyset$; B) $\mathcal{A} \neq \emptyset$.

C) The mapping $P_t : G \longrightarrow M(G)$ does not separate points of G for any $P \in M(G)$.

Elements of $\mathcal{J}$ have a simple description when G is a finite group:

2.7. Theorem. Suppose that G is a finite abelian group. Then $P \in \mathcal{J}$ if and only if there exists $S \in \mathcal{A}$ such that

(1) $P(\{x\}) = P(\{y\})$ holds for any two x, $y \in G$, $xy^{-1} \in S$.

Proof. Suppose $P \in \mathcal{J}$. It follows from 2.5 that $A_P \in \mathcal{A}$. Take x, $y \in G$ such that $t = xy^{-1} \in A_P$. Hence

$$P(\{x\}) = P_t(\{x\}) = P(\{y\}) .$$

Conversely, let $S \in \mathcal{A}$ be a subgroup such that the condition (1) holds. Then we can write

$P_t(\{x\}) = P(\{t^{-1}x\}) = P(\{x\})$ for each $(t, x) \in (S \times G)$.

Therefore $A_P \supset S \in \mathcal{A}$ and it follows from 2.5 that $P \in \mathcal{J}$. This completes the proof.

Now we shall examine the special case when G has a σ-finite Haar measure h ($h_t = h$ for all $t \in G$). Denote $\mathcal{J}_a = \{P \in \mathcal{J} : P \ll h\}$, $\mathcal{J}_s = \{P \in \mathcal{J} : P \perp h\}$ where $P \perp h$ signifies the fact that the measure P is h-singular. We can prove the following "decomposition theorem":

2.8. Theorem. Let G be a complete separable metric abelian group with a σ-finite Haar measure h. Suppose $P \in \mathcal{J} - (\mathcal{J}_a \cup \mathcal{J}_s)$. Then there exists unique $(\alpha, Q, R) \in (0,1) \times \mathcal{J}_a \times \mathcal{J}_s$ such that $P = \alpha Q + (1-\alpha)R$. Moreover, $A_P = A_Q \cap A_R$ holds.

Proof. Consider $P \in \mathcal{J} - (\mathcal{J}_a \cup \mathcal{J}_s)$. Then (see [2]) there are nonnegative finite measures A, S which are defined on $\mathcal{B}$ such that

(2) $\quad P = A + S, \quad A \ll h, \quad S \perp h, \quad A, S \neq \vartheta$.

The measures A, S are uniquely determined.

It is quite clear that $A_t \ll h$ for each $t \in G$. Since $S \perp h$, there is a $C \in \mathcal{B}$ such that $h(C) = 0$ and $S(B \cap C^c) = 0$ for all $B \in \mathcal{B}$. (We have denoted $C^c = G - C$.) Hence $h(t^{-1}C) = 0$ and

$S_t(B \cap (t^{-1}C)^c) = S(t^{-1}B \cap C^c) = 0$ for all $B \in \mathcal{B}$ and $t \in G$. Thus $S_t \perp h$ for every $t \in G$. Therefore we have $P = P_t = A_t + S_t$ for each $t \in A_P$. It follows from the uniqueness of the decomposition (2) that

(3) $\quad A_t = A, \quad S_t = S \qquad$ for $t \in A_P$.

If we put $\alpha = A(G)$ then $0 < \alpha < 1$ and

(4) $\qquad P = \alpha Q + (1-\alpha) R$

where $Q = \frac{A}{\alpha}$, $R = \frac{S}{1-\alpha}$. It follows from (3) and Theorem 2.5 that $Q \in \mathcal{J}_{a}$, $R \in \mathcal{J}_{S}$ and $A_P \subset A_Q \cap A_R$. The relation (4) implies that $A_Q \cap A_R \subset A_P$.

The uniqueness of our decomposition is an easy consequence of the fact that the measures A, S in (2) are uniquely determined. The proof is completed.

It is quite easy to characterize elements of the set $\mathcal{J}_a$.

2.9. <u>Theorem</u>. Let G be a complete separable metric abelian group with a σ-finite Haar measure h . Then $P \in \mathcal{J}_a$ if and only if there is $S \in \mathcal{a}$ such that

$$h(\{x: \frac{dP}{dh}(t^{-1}x) = \frac{dP}{dh}(x)\}) = 0 \text{ holds for each } t \in S .$$

The assertion of the theorem is a consequence of Theorem 2.5 and Radon-Nikodim's theorem if we realize that

$$\frac{dP_t}{dh} = \left(\frac{dP}{dh}\right)^{t^{-1}} \quad \text{for } t \in G$$ using the

same notation as in Corollary 2.4.

3. <u>Extreme points of the set $D(P)$</u>

The aim of this section is to find extreme points of the convex set $D(P) = \overline{co} \{P_t : t \in G\}$.
We shall have occasion to use the result of the section 2. Denote by $ex\, A$ the set of extreme points of a convex set A . First of all we note that the space $M(G)$

with the weak topology can be topologically imbedded into the space $C^*(G)$ of all continuous linear functionals on $C(G)$ with the weak*topology (see [3],Chapter V). (By the Riesz reprezentation theorem we can consider elements of $C^*(G)$ as regular additive set functions on the algebra $\mathcal{B}_0 \subset \mathcal{B}$ which is generated by all the open sets of G .)

Denote the closure of a set $A \subset C^*(G)$ by $\overline{A}^*$.

3.1. Let $K \subset G$ be a compact set. Then $\{P_t : t \in K\}$ and $\overline{co}\{P_t : t \in K\}$ are compact subsets of $C^*(G)$.

To prove the assertion it is sufficient to show that both sets are compact in the weak topology of $M(G)$ and this is an easy consequence of the relation (see [4]),

(5) $\overline{co}\{P_t : t \in K\} = \{Q \in M(G) : Q = P * \mu$, where $\mu \in M(G)$ and $\mu(K) = 1\}$.

An easy consideration together with one of the condequences of Krein-Milman theorem (see [3],V,8.5) shows us that

3.2 $ex\, \overline{co}\{P_t : t \in K\} = \{P_t : t \in K\}$ for each compact subgroup $K \subset G$.

Now we are able to prove the following theorem.

3.3. __Theorem__. Let G be a complete separable metric abelian group. Then the equality

$$ex\, D(P) = \{P_t : t \in G\} \text{ holds for every } P \in M(G).$$

__Proof__. Have a $P \in M(G)$. First of all we shall show that $P \in ex\, D(P)$. Consider $(\alpha, R, Q) \in (0,1) \times \times D(P) \times D(P)$ such that $P = \alpha R + (1-\alpha)\ Q$. By 1.2 there exist $\mu, \nu \in M(G)$ such that $R = P * \mu$, $Q = P * \nu$. Putting $\alpha\mu + (1-\alpha)\nu = \eta$ we can write $P = P * \eta$. It follows from Lemma 2.2 that there is a compact subgroup $S \subset G$ such that $\eta(S) = 1$. Hence $\mu(S) = \nu(S) = 1$ and according to (5) we can see that $P, Q, R \in \overline{co}\{P_t : t \in S\}$. It follows from 3.2 that P is an extreme point of the set $\overline{co}\{P_t : t \in S\}$ and hence $P = Q = R$. Therefore $P \in \in ex\, D(P)$. Now, an easy consideration will show that $\{P_t : t \in G\} \subset ex\, D(P)$. Let us prove that $ex\, D(P) \subset \subset \{P_t : t \in G\}$. The set $\overline{D(P)}^*$ is a closed bounded subset of $C^*(G)$. Thus $\overline{D(P)}^*$ is weakly compact (see [3], V. 4.2).

Therefore by Krein-Milman theorem

(6) $\qquad ex\, \overline{D(P)}^* \subset \overline{\{P_{t}, t \in G\}}^*$

Take $Q \in ex\, D(P)$ and consider $(\alpha, \varkappa_1, \varkappa_2) \in (0,1) \times \times \overline{D(P)}^* \times \overline{D(P)}^*$ such that $Q = \alpha\varkappa_1 + (1-\alpha)\varkappa_2$ (this means that $Q(B) = \alpha\varkappa_1(B) + (1-\alpha)\varkappa_2(B)$ for all $B \in \mathcal{B}_0$). Since $Q(B) \geq \alpha\varkappa_1(B)$, $Q(B) \geq (1-\alpha)\varkappa_2(B)$ for all $B \in \mathcal{B}_0$, the set functions $\varkappa_i$ $(i = 1, 2)$ are σ-additive on $\mathcal{B}_0$. Therefore they have extensions to the σ-algebra $\mathcal{B}$. Denote them R_1, R_2. Obviously $R_1, R_2 \in D(P)$ and $P(A) = \alpha R_1(A) + (1-\alpha) R_2(A)$ holds for each $A \in \mathcal{B}$. It follows from our assumption

$(Q \in ex\, D(P))$ that $R_1 = R_2$ and consequently $\varkappa_1 = \varkappa_2$. Therefore we have $Q \in ex\, \overline{D(P)}^*$. According to (6) and the fact that $Q \in M(G)$ it is clear that $Q \in \overline{\{P_t : t \in G\}}$. Since $M(G)$ is a metrizable topological semigroup, there exists a sequence $\{t_n\} \subset G$ such that $P_{t_n} = P * \varepsilon_{t_n} \xrightarrow[n \to \infty]{} Q$. It follows from 1.1 that the sequence $\{\varepsilon_{t_n}\}_1^\infty$ is relatively compact. Using the same argument as that in the proof of Lemma 2.1 we can show that the sequence $\{t_n\}_1^\infty$ is relatively compact. Hence $Q = P_{t_0}$ for each accumulation point t_0 of the sequence $\{t_n\}_1^\infty$. Therefore $Q \in \{P_t : t \in G\}$ and the proof is completed.

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Matematicko-fyzikální fakulta
Karlovy university
Sokolovská 83, Praha Karlín

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