Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 10 (1969), No. 4, 567--569

Persistent URL: http://dml.cz/dmlcz/105252

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A NOTE ON FREE ABELIAN GROUPS

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In this note we shall give a simple generalization of a theorem of A. Ehrenfeucht (see [1]) concerning free abelian groups. All groups considered here are also abelian.

We begin with the formulation of the following statement.

Proposition 1. A torsion free group G is free if and only if $Ext(G, C(\infty)) = 0$ (such group G is called W-group) and G belongs to some Baer's class Γ_α.

Proof. If G is free then evidently both conditions of our proposition are fulfilled. Conversely, suppose that G is a W-group and simultaneously $G \in \Gamma_\alpha$ for certain ordinal α. For the freeness of G we shall give two different proofs: 1) By [3, Lemma 0 and Theorem 2] G is $\aleph_1$-free and hence G is homogeneous of the same type as $C(\infty)$. In view of [3, Corollary 1] G is also separable; thus by a Baer's theorem (see [2], Theorem 49.2) G is completely decomposable, therefore, it is free. 2) Let H be a free subgroup of G generated by any maximal independent set in G; therefore,

G/H is torsion. G as a W-group is $\aleph_1$-free. This implies that if S is an arbitrary pure subgroup in G of finite rank, then S is free and hence $S/(S \cap H)$ is finite. Furthermore from the $\aleph_1$-freeness of G we conclude $\kappa_p(G) = 0$ (see [4],Lemma 1) for all primes p. Thus we may apply [4,Theorem 1] and we get $G \cong H$. Therefore, G is free. (This proof does not use the separability of W-groups.)

From this proposition we conclude easily the following equivalent statement generalizing theorem from [1].

Proposition 2. Let H be a subgroup of a free group F. Then H is a direct summand of F if and only if every homomorphism of H into $C(\infty)$ can be extended to a homomorphism of F into $C(\infty)$ and F/H is a torsion free group belonging to some class Γ_α.

Proof. Evidently both above conditions are necessary for H to be a direct summand of F. Thus we shall suppose that F/H lies in some class Γ_α and

(1) $$Hom(H, C(\infty)) = Hom(F|H, C(\infty)).$$

(The symbol $Hom(F|H, C(\infty))$ denotes here the set of all homomorphisms in $Hom(H, C(\infty))$ which can be extended to a homomorphism of F into $C(\infty)$.) The exact sequence

$$0 \longrightarrow H \xrightarrow{\mu} F \longrightarrow F/H \longrightarrow 0$$

induces the exactness of the sequence

(2) $$0 \longrightarrow Hom(F/H, C(\infty)) \longrightarrow Hom(F, C(\infty)) \xrightarrow{\mu^*} \\ \xrightarrow{\mu^*} Hom(H, C(\infty)) \xrightarrow{E^*} Ext(F/H, C(\infty)) \longrightarrow 0.$$

Since μ is the immersion of H into F, we conclude that the image of any $\eta \in Hom(F, C(\infty))$ under μ^* is the restriction $\eta | H$ of η to H. Thus in view of (1) μ^* is an epimorphism and this implies that E^* is zero-homomorphism; but by the exactness of (2) E^* is likewise an epimorphism, therefore, $Ext(F/H, C(\infty)) = 0$. Hence by Proposition 1 F/H is free and H is a direct summand of F.

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(Oblatum 8.10.1969)