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A SAALSCHUTZIAN THEOREM FOR TRIPLE SERIES

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1. Introduction. The object of the present paper is to obtain a Saalschutzian theorem for triple series. The Saalschutzian theorems for double series were obtained by Carlitz [1,2] and sum for double hypergeometric series of superior order was recently obtained by Jain [3]. It is interesting to note that results due to Carlitz [1,p.416 (9)], and Jain [3,p.300(1)] are particular cases of our theorem.

2. The following results [5,p.218(8.3.4)],[1,p.417 (12)] and [4] will be required in our investigations:

(1) $$F_1(a; b, b'; b+b'; x, y) =$$

$$= (1-x)^{-a} {}_2F_1(a, b'; b+b'; \frac{y-x}{1-x})$$

(2) $$= (1-x)^{b'-a}(1-y)^{b-a} \times$$

$$\times F_1(b+b'-a; b', b; b+b'; x, y),$$

(3) $F_T(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m (a_2)_{n+p} (b_1)_{m+p} (b_2)_n}{m!\, n!\, p!\, (c_1)_{m+n+p}} x^m y^n z^p ,$$

where $|x| < R$, $|y| < S$, $|z| < T$ such that $T = R - RS + S$.

3. The first formula to be established here is

(4) $F_T(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1; x, y, z)$

$$= (1-x)^{a_2-b_1} (1-y)^{b_1-a_2} (1-z)^{a_1-b_1} \times$$

$$\times F_T(a_2, a_1, a_1, b_2, b_1, b_2; c_1, c_1, c_1; x, y, z) ,$$

where $c_1 = a_1 + a_2 = b_1 + b_2$.

Proof. Expressing the series (3) for hypergeometric function of three variables F_T in terms of Appell's function F_1 , we have

(5)

$F_T(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1; x, y, z)$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m (b_1)_m}{m!\, (c_1)_m} F_1(a_2; b_2, b_1+m; c_1+m; y, z) x^m .$$

Now employing (1) in (5), we get

(6)

$F_T(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1; x, y, z)$

$$= (1-y)^{-a_2} F_1(b_1; a_1, a_2; c_1; x, \frac{z-y}{1-y})$$

provided $c_1 = b_1 + b_2$.

Again using (2) in (6), the result (4) follows immediately after little simplifications.

4. The main result that is the Saalschutzian theorem for triple series obtained here is

(7)
$$\sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{t=0}^{p} \frac{(-m)_r(-n)_s(-p)_t(a_2)_r(a_1)_{s+t}(b_2)_{r+t}(b_1)_s}{r!\,s!\,t!\,(d_1)_r(d_2)_s(d_3)_t(c_1)_{r+s+t}}$$

$$= \frac{(a_1)_m(a_2)_{n+p}(b_1)_{m+p}(b_2)_n}{(c_1)_{m+n+p}(b_1-a_2)_m(a_2-b_1)_n(b_1-a_1)_p}$$

provided $c_1 = b_1 + b_2 = a_1 + a_2$, $b_1 - a_2$ is not an integer, $1 + a_2 + b_2 - m = c_1 + d_1$, $1 + a_1 + b_1 - n = c_1 + d_2$ and $1 + a_1 + b_2 - p = c_1 + d_3$.

Proof. Employing the expansion of (3) in (4), we have

(8)
$$\sum_{m,n,p=0}^{\infty} \frac{(a_1)_m(a_2)_{n+p}(b_1)_{m+p}(b_2)_n}{m!\,n!\,p!\,(c_1)_{m+n+p}}\, x^m y^n z^p$$

$$= \sum_{m=0}^{\infty}\frac{(b_1-a_2)_m}{m!}x^m \cdot \sum_{n=0}^{\infty}\frac{(a_2-b_1)_n}{n!}y^n \cdot \sum_{p=0}^{\infty}\frac{(b_1-a_1)_p}{p!}z^p \times$$

$$\times \sum_{r,s,t=0}^{\infty} \frac{(a_2)_r(a_1)_{s+t}(b_2)_{r+t}(b_1)_s}{r!\,s!\,t!\,(c_1)_{r+s+t}}\, x^r y^s z^t$$

$$= \sum_{m,n,p=0}^{\infty}\sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{t=0}^{p} \frac{(b_1-a_2)_{m-r}(a_2-b_1)_{n-s}(b_1-a_1)_{p-t}(a_2)_r}{(m-r)!\,(n-s)!\,(p-t)!\,r!\,s!\,t!\,(c_1)_{r+s+t}} \times$$

$$\times (a_1)_{s+t}(b_2)_{r+t}(b_1)_s\, x^m y^n z^p .$$

$$= \sum_{m,n,p=0}^{\infty}\sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{t=0}^{p} \frac{(b_1-a_2)_m(a_2-b_1)_n(b_1-a_1)_{p-t}(-m)_r(-n)_s}{m!\,n!\,p!\,r!\,s!\,t!\,(1+a_2-b_1-m)_r(1+b_1-a_2-n)_s} \times$$

$$\times \frac{(-p)_t (a_2)_r (a_1)_{s+t} (b_2)_{r+t} (b_1)_s}{(1+a_1-b_1-p)_t \ (c_1)_{r+s+t}} x^m y^n z^p .$$

putting $1+a_2-b_1-m=d_1, 1+b_1-a_2-n=d_2, 1+b_1-a_1-p=d_3$ and equating the coefficient of $x^m y^n z^p$ on both side of (8), we obtain (7) under the conditions stated.

5. Particular Case. (i) Putting $m=0$ or $n=0$ in (7) we obtain a result due to Carlitz [1,p.416(9)].

(ii) When we put $p=0$ in (7), we get a result due to Jain [3,p.300(1)].

R e f e r e n c e s

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