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A NOTE ON UNIFORM BOUNDEDNESS PRINCIPLE FOR NONLINEAR OPERATORS

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Introduction. The uniform boundedness principle for operators is one of the powerful tools of Functional Analysis. This principle concerning linear operators was firstly introduced into Banach spaces by T.H. Hildebrandt [1] and has been extended by Banach and Steinhaus [2,p.53]for some further extensions to topological spaces, see for instance [3]. I.S. Gál [4] has generalized the uniform boundedness principle for a class of nonlinear operators. He has considered the operators U_n $(n = 1, 2 \dots)$ in Banach space X satisfying the following two conditions: (a) U_n $(n = 1, 2, \dots)$ are bounded and "homogeneous",i.e. there exist positive constants M_n $(n = 1, 2, \dots)$ such that $\| U_n(u) \| \leqq M_n \| u \|$ and $\| U_n(\lambda u) \| = |\lambda| \| U_n(u) \|$ for every $u \in X$, real λ and n $(n = 1, 2, \dots)$;

(b) U_n are asymptotically subadditive, i.e.

$$\| U_n(u+v) \| \leqq \| U_n(u) \| + O(|U_n| \| v \|)$$

uniformly in $u, v \in X$ and

$$\inf_{\|v\| \leqq 1} (\| U_n(u+v) \| + \| U_n(u) \| - \| U_n(v) \|) \geqq O(|U_n|)$$

for every $u \in X$, but not necessarily uniformly in X, where $|U_n| = \sup_{\|u\|=1} \|U_n(u)\|$.

I.S. Gál [4] has established the following result: If the mappings U_n $(n = 1, 2, \dots)$ satisfy the conditions (a), (b) and if

$$\overline{\lim_{n \to \infty}} \|U_n(u)\| = L(u) < +\infty$$

for every $u \in X$, then $|U_n| \leqq C < +\infty$ for every n $(n = 1, 2, \dots)$.

Another assertion concerning the uniform boundedness principle for nonlinear functionals has been proved by S. B. Stečkin [5]. His main result is as follows: Let the functionals $f_n(u)$ $(n = 1, 2, \dots)$ be defined on the closed ball $D_1(0) = \{u \in X : \|u\| \leqq 1\}$ and be such that the following conditions are satisfied: (1) $|f_n(u+v)| \leqq K(|f_n(u)| + |f_n(v)|)$, $(n = 1, 2, \dots)$ for each u, v, $u + v \in D_1(0)$, where a constant K does not depend on n $(n = 1, 2, \dots)$; (2) For any n $(n = 1, 2, \dots)$ there exists a number $\delta_n > 0$ so that $|f_n(u)| \leqq M$ for each $u \in X$ with $\|u\| \leqq \delta_n$ (M does not depend on n); (3) $u \in D_1(0) \Rightarrow \sup_{n=1,2,\dots} |f_n(u)| = L(u) < +\infty$.

Then

$$\sup_{\substack{n=1,2,\dots \\ u \in D_1(0)}} |f_n(u)| < +\infty .$$

The purpose of this note is to establish some further results concerning the uniform boundedness principle (Section 1) and the principle of condensation of singularities (Section 2) both for certain classes of nonlinear operators. The proofs of these assertions are based upon a category argument of Baire which was originally introduced into this domain of research by S. Saks. We point out there that the principle of condensation of singularities for nonlinear operators satisfying (a),(b) has been established by I.S. Gál [6].

1. The terminology and notation of Section 2 [7] is used. We prove the following

<u>Theorem 1</u>. Let X, Y be linear normed spaces, $M \subset X$ a subset of the second category in X, $F_n : X \to Y$ $(n = 1, 2, \dots)$ mappings of X into Y such that

(a) $\| F_n(u-v) \| \leqq f(\max(\|u\|, \|v\|)) \max(\|F_n(u)\|, \|F_n(v)\|)$

for every $u, v \in X$, n $(n = 1, 2, \dots)$, where the function $f(x)$ is defined in $J = \langle 0, \infty)$, is bounded and positive on each subinterval $\langle 0, a \rangle$ of J;

(b) $u \in M \Rightarrow \sup_{n=1,2,\dots} \| F_n(u) \| < +\infty$. If either

(c_1) $u_k \in X$, $u \in X$, $u_k \to u \Rightarrow \| F_n(u) \| \leqq \varliminf_{k\to\infty} \| F_n(u_k) \|, n = 1, 2, \dots,$

or (c_2) there exists a functional φ on M having the Baire property in M and is such that $g(u) \leqq |\varphi(u)|$ for each $u \in M$, where $g(u) =$

$= \sup_{n=1,2,\dots} \| F_n(u) \|$, then the sequence $\{F_n(u)\}$ is uniformly bounded on any closed ball $D_R(0) = \{u \in X : \|u\| \leqq R\}$ of X , i.e.

$$\sup_{\substack{n=1,2,\dots \\ u \in D_R(0)}} \| F_n(u) \| < +\infty$$

for arbitrary (but fixed) $R > 0$.

<u>Proof</u>. Suppose (a),(b),(c_1) are fulfilled and denote $X_n = \{u \in X : \sup_{k=1,2,\dots} \| F_k(u) \| \leqq n\}$ $(n = 1, 2, \dots)$. Then X_n are closed and $M \subseteqq \bigcup_{n=1}^{\infty} X_n$. Since M is of the second category in X at least one of X_n , say X_{n_0} , must contain a closed ball. Hence there exist $\delta > 0$ and $u_0 \in X$ such that $D(u_0, \delta) = \{u \in X : \|u - u_0\| \leqq \delta\} \subset X_{n_0}$. Then $u \in D(u_0, \delta) \Rightarrow \Rightarrow \| F_k(u) \| \leqq n_0$, $(k = 1, 2, \dots)$. For any $v \in X$ with $\|v\| \leqq \delta$ we have that $v + u_0 \in D(u_0, \delta)$ and

$$\begin{aligned} \| F_n(v) \| &= \| F_n((v + u_0) - u_0) \| \leqq \\ &\leqq K \max (\| F_n(v + u_0) \|, \| F_n(u_0) \|) \leqq \\ &\leqq K n_0 , \quad (n = 1, 2, \dots) , \end{aligned}$$

where K is a constant from the boundedness of the function $f(r)$ on the interval $\langle 0, \|u_0\| + \delta \rangle$. Hence $\sup_{n=1,2,\dots} \| F_n(v) \| \leqq K n_0$ for each $v \in X$, $\|v\| \leqq \delta$. Now we show that $\{F_n(u)\}$ is uniformly bounded on any closed ball $D_R(0)$. Let $R > 0$

be fixed, we note that the hypothesis (a) implies the following inequality

(a') $\| F_n(u+v)\| \leqq C \max(\| F_n(u)\|, \| F_n(v)\|)$

$(n = 1, 2, \ldots)$, which is valid for each $u, v \in D_R(0)$, $u + v \in D_R(0)$ and where C is some positive constant.

In fact, consider $u, v \in D_R(0)$ such that $u + v \in D_R(0)$ and denote $N = \sup_{r \in \langle 0, R\rangle} f(r)$. According to hypothesis (a) for any $u \in D_R(0)$ we have that $\| F_n(0)\| \leqq N \| F_n(u)\|$ and $\| F_n(-u)\| \leqq N \max(\| F_n(0)\|, \| F_n(u)\|) \leqq C_0 \| F_n(u)\|$, where $C_0 = N \max(N, 1)$. Thus for $u, v \in D_R(0)$, $u + v \in D_R(0)$ we obtain

$$\| F_n(u+v)\| \leqq N \max(\| F_n(u)\|, \| F_n(-v)\|) \leqq$$

$$\leqq C \max(\| F_n(u)\|, \| F_n(v)\|),$$

where $C = N \max(C_0, 1)$. Furthermore, there exists an integer n_1 such that $\frac{R}{n_1} \leqq \delta$. Then for any $u \in D_R(0)$ and n $(n = 1, 2, \ldots)$ we have

$$\| F_n(u)\| = \| F_n(\frac{u}{n_1} \cdot n_1)\| \leqq$$

$$\leqq C \max(\| F_n(\frac{u}{n_1}(n_1 - 1))\|, \| F_n(\frac{u}{n_1})\|) \leqq$$

$$\leqq \ldots \leqq C_1 \| F_n(\frac{u}{n_1})\| \leqq C_1 K n_0 ,$$

where C_1 is some positive constant. Thus $\{ F_n(u)\}$ is uniformly bounded on $D_R(0)$ and the first part of our theorem is proved. Assume (a),(b),(c_2) are fulfilled and use the same arguments as in the proof of Th.9 [7] ; we see that $\{F_n(u)\}$ is uniformly bounded on some neighbourhood W of 0 . Hence there exists $\delta > 0$ such that $\|u\| \leqq \delta \Rightarrow u \in W$. Employing the above considerations we observe that $\{F_n(u)\}$ is uniformly bounded on any closed ball $D_R(0)$. Theorem is proved.

Using the properties of Baire sets of the second category and category arguments one may prove the following

Theorem 2. Let X, Y be linear normed spaces, X of the second category in itself, $F_n : X \to Y$ $(n = 1, 2, \ldots)$ mappings from X into Y satisfying the condition (a) of Theorem 1. Suppose the following assumptions are fulfilled: (1) For every n $(n = 1, 2, \ldots)$ there exists $u_n \in X$ and the open neighborhood $V(u_n) \subset X$ of u_n such that $u \in V(u_n) \Rightarrow \|F_n(u)\| \leqq N$, where a positive number N does not depend on n . (2) There exists a subset $M \subset X$ such that M is elsewhere dense in X , is not of the first category in some open ball $B_{R_0}(0) = \{u \in X : \|u\| < R_0\}$ and $u \in M \Rightarrow$ $\Rightarrow \sup_{n=1,2,\ldots} \|F_n(u)\| < +\infty$.

Then $\{F_n(u)\}$ is uniformly bounded on any closed ball $D_R(0)$ of X .

Remarks. One may replace the condition (a) in Theorems 1,2 by the following one:

$$\|F_n(u+v)\| \leqq f(\max(\|u\|, \|v\|)) \max(\|F_n(u)\|, \|F_n(v)\|)$$

$(n=1,2,\dots), u, v \in X$, where the function $f(\varkappa)$ has the same properties as in the assumption (a). However, we must add that either (a_1) $F_n(-u) = -F_n(u)$, $(n = 1, 2, \dots)$ for every $u \in X$, or (a_2) $\| F_n(\lambda u) \| = |\lambda|^p \| F_n(u) \|$ for every $u \in X$ and real λ, where p is some positive number. Moreover, in case of Th.2 one can require only the satisfying of the condition (a_1) for each $u \in V(u_n)$, $(n = 1, 2, \dots)$. Each demicontinuous map F_n satisfies (c_1). Indeed, if $u_k \to u$ in X, then demicontinuity of F_n implies the weak convergence of $\{ F_n(u_k) \}_{k=1}^{k=\infty}$ to $F_n(u)$. Hence

$$\| F_n(u) \| \leqq \varliminf_{k \to \infty} \| F_n(u_k) \| \leqq \varlimsup_{k \to \infty} \| F_n(u_k) \| .$$

The assumption (1) of Theorem 2 is fulfilled for instance if each F_n $(n = 1, 2, \dots)$ has at least one point of continuity in X. We point out that Theorems 1,2 are valid for linear metric spaces X, Y, where Y is the space of the second category in itself.

2. In this section we make some notes concerning the principle of condensation of singularities for nonlinear operators. This principle for linear case is well-known;see for example Banach-Steinhaus [8], Yoshida [9]. W. Orlicz [10] has established this principle for double sequence of linear operations depending on some parameter, meanwhile I.S. Gál [6] has considered this one for nonlinear operators. We state the following

<u>Theorem 3</u>. Let X, Y_k $(k = 1, 2, \dots)$ be linear normed spaces, X of the second category in itself,

$F_k : X \to Y_k$ $(k = 1, 2, \dots)$ mappings of X into Y_k so that: (a) $\| F_k(u + v) \| \leq$
$\leq K \max(\| F_k(u) \|, \| F_k(v) \|)$ for every $u, v \in$
$\in X$, $(k = 1, 2, \dots)$, where K is some positive constant; (b) $u_n \to u$, $u_n, u \in X \Rightarrow \| F_k(u) \| \leq$
$\leq \overline{\lim_{n \to \infty}} \| F_k(u_n) \|$, $(k = 1, 2, \dots)$.

Furthermore, let one of the following conditions be fulfilled: (1) $F_k(-u) = -F_k(u)$ for every $u \in X$, $(k = 1, 2, \dots)$; (2) $\| F_k(\lambda u) \| =$
$= |\lambda|^p \| F_k(u) \|$ for every $u \in X$, $(k =$
$= 1, 2, \dots)$ and λ real (p is some positive number).

Then the set $M = \{u \in X : \overline{\lim_{k \to \infty}} \| F_k(u) \| < +\infty\}$ is either equal to X, or M is a set of the first category in X.

From Theorem 3 we conclude

<u>Theorem 4</u> (Principle of condensation of singularities). Let X, Y_k have the same properties as in Th.3, $\{F_{kn}\}$, $(n = 1, 2, \dots)$ a sequence of nonlinear operators from X into Y_k satisfying the assumptions of Th.3 for each fixed k $(k = 1, 2, \dots)$. Suppose that for each k $(k = 1, 2, \dots)$ there exists $u_k \in X$ so that $\overline{\lim_{n \to \infty}} \| F_{kn}(u_k) \| = +\infty$.

Then $M = \{u \in X : \overline{\lim_{n \to \infty}} \| F_{kn}(u) \| = +\infty$ for every $k = 1, 2, \dots\}$ is the set of the second category in X

Recall that some related close theorems to Th.3,4 were obtained by A. Alexiewicz [11]. He has considered continuous operations U_n having some further properties quite different than ours. Now we make the following note (compare [10]). Let X be a metric space of the second category in itself, Y a linear normed space, $F_n : X \to Y$, $(n = 1, 2, \dots)$ mappings of X into Y so that $u_k \in X, u \in X, u_k \to u \Rightarrow \|F_n(u)\| \leq \overline{\lim_{k\to\infty}} \|F_n(u_k)\|$,

$(n = 1, 2, \dots)$. If $u \in A \Rightarrow \overline{\lim_{n\to\infty}} \|F_n(u)\| = +\infty$, where A is some subset of X everywhere dense in X, then $M = \{u \in X : \overline{\lim_{n\to\infty}} \|F_n(u)\| = +\infty\}$ is the set of the second category in X.

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