Jiří Veselý

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ON THE LIMITS OF THE POTENTIAL OF THE DOUBLE DISTRIBUTION

(Preliminary communication)

Jiří VESELÝ, Praha

We identify the set of all finite complex numbers with the Euclidean plane E_2. Let K be a rectifiable curve in E_2 which is described by the complex-valued function ψ on a compact interval $\langle a, b \rangle$ and let

(1) $$\operatorname{var}_t [\psi; \langle a, b \rangle] < +\infty ,$$

$$(t_1, t_2 \in \langle a, b \rangle, 0 < |t_1 - t_2| < b - a) \Rightarrow \psi(t_1) \neq \psi(t_2) .$$

For $x \in E_2$, $r \in (0, +\infty \rangle$ we denote by $\mathcal{Y}_{r,x}$ the system of all components J of

$$\{t; t \in \langle a, b \rangle, 0 < |\psi(t) - x| < r\}$$

and by Θ_x^J a fixed continuous branch of the $\arg[\psi(t) - x]$ on J, $J \in \mathcal{Y}_{r,x}$. In case $r = +\infty$ we shall skip r in all symbols. Then we can define for $x \in E_2$

(2) $$v_r^K(x) = \sum_{J \in \mathcal{Y}_{r,x}} \operatorname{var}_t [\Theta_x^J; J] ,$$

$$\mu_r^K(x) = \sum_{J \in \mathcal{Y}_{r,x}} \operatorname{var}_t [|\psi(t) - x|; J] .$$

The functions $v^K(x)$, $u^K(x)$ are called cyclic and radial variations of K with respect to x. They were studied in [1] and [2] in connection with the behaviour of the logarithmic potential of the double distribution.

We shall describe the points in E_3 by pairs $[x; v]$, where $x \in E_2$, $v \in E_1$. Let us denote $H = K \times E_1 \subset E_3$ and define the measure μ on H as the product measure $\lambda \times \lambda$ of linear measures λ on K and E_1 respectively. Because of the rectifiability of K we can define the normal $n(\xi)$

$$n(\xi) = n_1(\xi) + i n_2(\xi) = i \cdot \frac{\psi'(t)}{|\psi(t)|}$$

at the point $\xi = \psi(t)$ for λ-almost every $\xi \in K$. The normal $\nu(R)$ at the point $R \in H$ with respect to H (where $R \equiv [\xi; v]$) can be defined by $\nu(R) = (n_1(\xi), n_2(\xi), 0)$. Then $\nu(R)$ is defined μ-almost everywhere on H.

If $C(H)$ is the Banach space of all bounded continuous functions F on H with the usual norm, $R \equiv [\xi; v]$, $Q \equiv [x; u]$

$$G(R, Q) = [|\xi - x|^2 + (v - u)^2]^{-1/2} \tag{3}$$

or

(4) $$G(R,Q)=(u-v)^{-1}\cdot \exp\left(-\frac{|\xi-z|^2}{4(u-v)}\right) \quad \text{for } v<u,$$
$$G(R,Q)=0 \quad \text{for } v\geq u,$$

we can define the functions $W^{\Psi}(F;Q)$ of Q for every $F\in C(H)$ by

(5) $$W^{\Psi}(F;Q)=\int_H F(R)\cdot\frac{\partial G(R;Q)}{\partial \nu(R)}\,d\mu(R)\,.$$

($G(R,Q)$ is given by (3) or (4).) Our main objective is the existence of the limit

$$\lim_{Q\to P} W^{\Psi}(F;G) \quad \text{for } F\in C(H),\ P\in H\,.$$

Since the functions $W^{\Psi}(F;Q)$ have similar properties, we denote them by the same symbol and the following theorem is valid for both cases:

<u>Theorem 1</u>: Let $P\equiv[\xi;v]\in H$, S be a segment with end points P, R. Suppose that there exist open spheres $K(P)$, $K(R)$ with centers P, R respectively so that for every $R'\in K(R)$ the straight line PR' and the set $K(P)\cap H$ have just one common point P. For the existence of

(6) $$\lim_{\substack{Q\to P\\ Q\in S}} W^{\Psi}(F;Q)$$

for every $F\in C(H)$ the following conditions are necessary and sufficient:

(7) $$v^K(\xi)<+\infty,\ \sup_{r>0} r^{-1}\mu_r^K(\xi)<+\infty$$

When both conditions (7) are fulfilled it is possible to express the limit (6) in the following way: for every $F \in C(H)$ we put

$$F(\psi(t), u) = f(t, u) \tag{8}$$

and denote by $C(\mathcal{H})$ the Banach space of all continuous functions f on $\mathcal{H} = \langle a, b \rangle \times E_1$ with the usual norm (in case $\psi(a) = \psi(b)$ we shall assume that $f(a, u) = f(b, u)$ for every $u \in E_1$ and $f \in C(\mathcal{H})$). Then (8) determines an isometric isomorphism between $C(H)$ and $C(\mathcal{H})$ and we can define for every $f \in C(\mathcal{H})$

$$w^{\psi}(f; x, u) = \tag{9}$$
$$= \sum_{J \in \mathcal{Y}_x} \int_J \int_{-\infty}^{+\infty} f(t, s)\, d_s \left[\frac{s - u}{(r_x^2(t) + (s-u)^2)^{1/2}} \right] d_t\, \Theta_x^J(t)$$

or by

$$w^{\psi}(f; x, u) = \tag{10}$$
$$= \sum_{J \in \mathcal{Y}_x} \int_J \int_{-\infty}^{u} f(t, s)\, d_s \left[-\exp\left(-\frac{r_x^2(t)}{4(u-s)}\right)\right] d_t\, \Theta_x^J(t) .$$

Assuming (8) we have for $Q \equiv [x; u]$, $W^{\psi}(F; Q) =$ $= w^{\psi}(f; x, u)$. Accordingly, we can study the limit of $w^{\psi}(f; x, u)$ instead of (6). For $P \equiv [\xi; v] \in$ $\in H$ the following theorem is valid:

<u>Theorem 2</u>: Assume (7) and suppose that $\psi^{-1}(\xi) =$ $= \{t_0\}$, $f \in C(\mathcal{H})$.

If $t_0 = a$ (or $t_0 = b$) then there exists the limit

$$\lim_{t \to a_+} \frac{\psi(t) - \psi(a)}{|\psi(t) - \psi(a)|} = \exp i\alpha$$

(or $$\lim_{t \to b_-} \frac{\psi(t) - \psi(b)}{|\psi(t) - \psi(b)|} = \exp i\alpha \quad)$$

and

$$\lim_{\rho \to 0_+} w^{\psi}(f; \xi + \rho \exp i\gamma, v + \rho \operatorname{tg} \gamma') =$$
$$= w^{\psi}(f; \xi, v) + 2f(a, v) \cdot (\pi + \alpha - \gamma)$$

(or

$$\lim_{\rho \to 0_+} w^{\psi}(f; \xi + \rho \exp i\gamma, v + \rho \operatorname{tg} \gamma') =$$
$$= w^{\psi}(f; \xi, v) + 2f(b, v) \cdot (\gamma - \alpha - \pi) \;)$$

uniformly for $\gamma \in \langle \alpha_1, \alpha_2 \rangle$, $\gamma' \in \langle \beta_1, \beta_2 \rangle$, where $\alpha < \alpha_1 \leq \alpha_2 < \alpha + 2\pi$, $-\frac{\pi}{2} < \beta_1 \leq \beta_2 < \frac{\pi}{2}$.

If $a < t_0 < b$, then there exist the limits

$$\lim_{t \to t_{0+}} \frac{\psi(t) - \psi(t_0)}{|\psi(t) - \psi(t_0)|} = \exp i\alpha_+ ,$$

$$\lim_{t \to t_{0-}} \frac{\psi(t) - \psi(t_0)}{|\psi(t) - \psi(t_0)|} = \exp i\alpha_- .$$

We can choose α_+, α_- so that $\alpha_+ \leq \alpha_- < \alpha_+ + 2\pi$ and put $\Delta = \pi - (\alpha_- - \alpha_+)$.

Then

$$\lim_{\rho \to 0_+} w^{\psi}(f; \xi + \rho \exp i\gamma, v + \rho \operatorname{tg} \gamma') =$$
$$= w^{\psi}(f; \xi, v) + 2f(t_0, v) \cdot (\pi + \Delta)$$

uniformly for $(\gamma, \gamma') \in F_1 \times F_2$, where F_1 is any compact in (α_+, α_-), F_2 is any compact in

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\lim_{\rho \to 0_+} w^{\Psi}(f; \xi + \rho \exp i\gamma, v + \rho \operatorname{tg} \gamma') =$$
$$= w^{\Psi}(f; \xi, v) - 2f(t_0, v) \cdot (\pi - \Delta)$$

uniformly for $(\gamma, \gamma') \in F_1' \times F_2'$, where F_1' is any compact in $(\alpha_-, \alpha_+ + 2\pi)$, F_2' is any compact in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Similar results can be obtained for the case when $H_0 = K \times \langle u_1, u_2 \rangle$, where $\langle u_1, u_2 \rangle$ is any compact interval in E_1. The proofs of theorems 1 and 2 together with further related results will be published elsewhere.

R e f e r e n c e s

[1] J. KRÁL: On the logarithmic potential of the double distribution, Czech.Math.J.14(89)(1964),306-321.

[2] J. KRÁL: Non-tangential limits of the logarithmic potential, Czech.Math.J.14(89)(1964),455-482.

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