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ALMOST OPTIMAL APPROXIMATIONS OF COMPACT SETS IN HILBERT SPACE

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1. Let H denote a Hilbert space which is supposed to be separable. Let $T: H \rightarrow H$ be a completely continuous operator and let $\mathcal{R}(T) = T(H)$ denote its range. Then the operator $A = [T^* T]^{\frac{1}{2}}$ (T^* is the adjoint operator to T) is a completely continous positive operator. Therefore A has the non-increasing sequence (λ_n) of positive eigenvalues and there exists the orthonormal sequence (in H) of its eigenfunctions (e_n) (See [1],pp.189-191). If we denote $Ug = Tf$ for $g = Af$ then U is a unitary operator and $T = UA$. Setting $h_n = Ue_n$, (h_n) is an orthonormal sequence in H and

(1.1) $$Tf = \sum_{n=1}^{+\infty} \lambda_n (f, e_n) h_n$$

and

(1.2) $$T^* f = \sum_{n=1}^{+\infty} \lambda_n (f, h_n) e_n$$

2. If $S(1)$ is the unit sphere in H then $M = T(S(1))$ is a compact set. For the sequence $(\varphi_n) \subset H$ we denote the error of the best approximation of M by $\varphi_1, \dots, \varphi_n$ as $\rho_n(M; \varphi_1, \dots, \varphi_n)$, i.e.

$$\rho_n(M; \varphi_1, \dots, \varphi_n) = \sup_{g \in M} \inf_{\alpha_1, \dots, \alpha_n} \| g - \sum_{i=1}^{n} \alpha_i \varphi_i \| . \tag{2.1}$$

We further denote by $\rho_n(M)$ the value of the error of the best n-dimensional approximation of M, i.e.

$$\rho_n(M) = \inf_{\varphi_1, \dots, \varphi_n} \rho_n(M; \varphi_1, \dots, \varphi_n) . \tag{2.2}$$

Theorem 1. Let $T : H \to H$ be a completely continous operator in the form (1.1) and $M = T(S(1))$. Then

$$\rho_n(M) = \rho_n(M; h_1, \dots, h_n) = \lambda_{n+1} . \tag{2.3}$$

Proof. 1. We have

$$\rho_n(M; h_1, \dots, h_n) = \sup_{\|f\| \le 1} [\sum_{k=n+1}^{+\infty} \lambda_k^2 |(f, e_k)|^2]^{\frac{1}{2}} \le$$
$$\le \lambda_{n+1} \sup_{\|f\| \le 1} [\sum_{k=n+1}^{+\infty} |(f, e_k)|^2]^{\frac{1}{2}} = \lambda_{n+1}$$

and, on the other hand, for $f = e_{n+1}$ it is

$$\inf_{\alpha_1, \dots, \alpha_n} \| Tf - \sum_{k=1}^{n} \alpha_k h_k \| = \| Te_{n+1} \| = \lambda_{n+1} .$$

Hence the right hand side equality is proved.

2. For any $\varphi_1, \dots, \varphi_n$ there exists $\tilde{f} = \sum_{k=1}^{n+1} a_k e_k$ such that $\sum_{k=1}^{n+1} |a_k|^2 = \| \tilde{f} \|^2 = 1$ and $\sum_{k=1}^{n+1} a_k \lambda_k (h_k, \varphi_i) = 0$ for $i = 1, \dots, n$. Then

$$\inf_{\alpha_1,\dots,\alpha_n} \| T\tilde{f} - \sum_{k=1}^{n} \alpha_k \varphi_k \| = \| T\tilde{f} \| = [\sum_{k=1}^{n+1} |a_k|^2 \lambda_k^2]^{\frac{1}{2}} \geq \lambda_{n+1}$$

and, by it,

$$\rho_n(M; \varphi_1, \dots, \varphi_n) \geq \lambda_{n+1} .$$

From this the left hand side equality follows.

The asymptotic behaviour of the minimal error $\rho_n(M)$ was examined in [2] for some classes of integral operators in L^2.

Theorem 2. If M is a compact set and (φ_n) is a complete sequence in H then

$$\lim_{n \to +\infty} \rho_n(M; \varphi_1, \dots, \varphi_n) = 0 \tag{2.4}$$

and if (2.4) holds and $\bigcup_{n=1}^{+\infty} n M$ is a dense set in H then (φ_n) is a complete sequence.

Proof. We denote by $L(\Phi)$ the linear hull of a sequence (φ_n).

1. Let (φ_n) be a complete sequence, i.e. $\overline{L(\Phi)} = H$ ($\overline{M}$ denotes the closure of M) and let P_n^{Φ} be the projection onto $L(\varphi_1, \dots, \varphi_n)$. Then

$$\lim_{n \to +\infty} \| g - P_n^{\Phi} g \| = 0 \tag{2.5}$$

for any $g \in M$. As the functions $\| g - P_n^{\Phi} g \|$ are continuous on M, there exists the sequence $(g_n) \subset M$

such that

$$\rho_n(M;\varphi_1,\dots,\varphi_n) = \|g_n - P_n^{\Phi} g_n\|, \quad n = 1,\dots .$$

We suppose

$$(2.6) \qquad \rho_n(M;\varphi_1,\dots,\varphi_n) \geq \alpha > 0$$

for all n. By the compactness of M ,there exist (g_{n_k}) and $g^* \in M$ such that $\lim\limits_{k\to+\infty} g_{n_k} = g^*$. Now, from (2.5) it follows that there exists k_0 such that for any $k \geq k_0$

$$\|g^* - P_{n_k}^{\Phi} g^*\| < \frac{\alpha}{2} \quad \text{and} \quad \|g_{n_k} - g^*\| < \frac{\alpha}{2}$$

hold. Thus,

$$\alpha \leq \|g_{n_k} - P_{n_k}^{\Phi} g_{n_k}\| \leq \|g^* - P_{n_k}^{\Phi} g^*\| + \|g_{n_k} - g^* - P_{n_k}^{\Phi}(g_{n_k} - g^*)\| < \alpha .$$

It is the contrary to the assumptions (2.6) and hence (2.4) is valid.

2. From (2,4) it follows (2.5) for any $g \in M$. It means that $M \subset \overline{L(\Phi)}$ and therefore $\bigcup\limits_{n=1}^{+\infty} n\,M \subset \overline{L(\Phi)}$. By the density $\bigcup\limits_{n=1}^{+\infty} n\,M$ in H , the completeness of (φ_n) is proved.

But the convergence theorem does not say too much on the suitability of choice of an approximating sequence (φ_n). Therefore, we define

<u>Definition 1</u>. A sequence $(\varphi_n) \subset H$ is called to be

an almost optimal approximation of M if there exists a constant C such that

(2.7) $$\rho_n(M;\varphi_1,\dots,\varphi_n)\le C\rho_n(M)$$

holds for any n.

If (ε_n) is an orthonormal base in H then for $M=T(S(1))$

$$\rho_n(M;\varepsilon_1,\dots,\varepsilon_n)=\sup_{\|f\|\le 1}[\sum_{k=n+1}^{+\infty}|(f,T^*\varepsilon_k)|^2]^{\frac{1}{2}}$$

is valid and it is clear that we need some further information of $(T^*\varepsilon_n)$ to determine the quality of the approximation. The following example shows that.

<u>Example 1</u>. Let T be in the form (1.1) and $\lim_{n\to+\infty}\frac{\lambda_{n-1}}{\lambda_n}=+\infty$. We put $\varepsilon_{2n-1}=h_{2n}$, $\varepsilon_{2n}=h_{2n-1}$, $n=1,\dots$. Then (ε_n) is an orthonormal base and

$$\lim_{n\to+\infty}\frac{\rho_{2n-1}(M;\varepsilon_1,\dots,\varepsilon_{2n-1})}{\rho_{2n-1}(M)}\ge\lim_{n\to+\infty}\frac{\lambda_{2n-1}}{\lambda_{2n}}=+\infty .$$

3. <u>Definition 2</u>. A sequence $(\varphi_n)\subset H$ is called to be strong minimal (see [3],[4]) or strong maximal if there exists a positive constant c_1 or c_2 such that for the eigenvalues $(\mu_k^{(n)})$, $k=1,\dots,n$; $n=1,\dots$ of the Gramms' matrices $((\varphi_i,\varphi_j))_{i,j=1,\dots,n}$ the inequality

(3.1) $$c_1\le\mu_k^{(n)}$$

or

(3.2) $$\mu_k^{(n)}\le c_2$$

holds.

It is proved in [3] that a strong minimal sequence

(φ_n) has the uniquely determined the biorthogonal sequence $(\omega_n) \subset \overline{L(\Phi)}$.

<u>Theorem 3</u>. Let a sequence $(\varphi_n) \subset H$ have the biorthogonal sequence (ω_n) and let (ε_n) be an orthonormal base in H . Then the following statements are equivalent.

(A) (φ_n) is strong minimal.

(B) (ω_n) is strong maximal.

(C) The operator $U_1 : H \longrightarrow \ell^2$ which is defined by

(3.3) $$U_1 f = ((f, \omega_n))$$

is linear bounded.

(D) For any $f \in H$ it is $\sum_{n=1}^{+\infty} |(f, \omega_n)|^2 < +\infty$.

(E) The set $E = \{ f \in H ; \sum_{n=1}^{+\infty} |(f, \omega_n)|^2 < +\infty \}$ is the set of the second category of H .

(F) The linear operator U_2 which is defined on $L(\Phi)$ by

(3.4) $$U_2 \varphi_n = \varepsilon_n$$

has a bounded extension on H .

(G) The operator U_3 which is defined on $L(\varepsilon)$ by

(3.5) $$U_3 \varepsilon_n = \omega_n$$

has a bounded extension on H .

(H) The operator $U_4 : \ell^2 \longrightarrow H$ which is defined by

(3.6) $$U_4((\alpha_n)) = \sum_{n=1}^{+\infty} \alpha_n \omega_n$$

is linear bounded.

(I) There exists a constant K such that for every natural number n and complex numbers $\alpha_1, \dots, \alpha_n$;

$\beta_1, \dots, \beta_n$ the inequality

(3.7) $$\left|\sum_{k=1}^{n} \alpha_k \beta_k\right| \le K \left[\sum_{k=1}^{n} |\alpha_k|^2\right]^{\frac{1}{2}} \cdot \left\|\sum_{k=1}^{n} \beta_k \varphi_k\right\|$$

holds.

<u>Proof</u>. It will be done by the following schema $(B) \Longleftrightarrow (A) \Rightarrow (C) \Rightarrow (D) \Rightarrow (E) \Rightarrow (F) \Rightarrow (G) \Rightarrow (H) \Rightarrow (I) \Rightarrow (A)$.

1. The equivalence of statements (A) and (B) was proved in [4].

2. (A) $\Longrightarrow$ (C). For $f \in H$ we denote by g the projection of f onto $\overline{L(\Phi)}$. Let $P_n^{\Phi} f = \sum_{k=1}^{n} a_k^{(n)} \varphi_k$. Then $s\text{-}\lim P_n^{\Phi} f = g$ and hence $w\text{-}\lim P_n^{\Phi} f = g$. Especially, it means $\lim_{n \to +\infty} a_k^{(n)} = (g, \omega_k)$ for all k. By the strong minimality of (φ_n) we obtain

$$\sum_{k=1}^{n} |a_k^{(n)}|^2 \le \frac{1}{c_1} \|P_n^{\Phi} f\|^2 \le \frac{1}{c_1} \|g\|^2 \le \frac{1}{c_1} \|f\|^2$$

and therefore

$$\sum_{k=1}^{m} |a_k^{(n)}|^2 \le \frac{1}{c_1} \|f\|^2$$

is valid for all $n \ge m$. By the limit process for $n \to +\infty$ and then for $m \to +\infty$, we have

$$\sum_{k=1}^{+\infty} |(g, \omega_k)|^2 \le \frac{1}{c_1} \|f\|^2 .$$

According to the choice of the biorthogonal sequence (ω_n), the equalities

$$(f, \omega_n) = (g, \omega_n)$$

hold for all n . It means

(3.8) $$\sum_{k=1}^{+\infty} |(f, \omega_k)|^2 \le \frac{1}{c_1} \|f\|^2 ,$$

i.e. the operator U_1 is linear bounded on H .

3. (C) $\Longrightarrow$ (D) $\Longrightarrow$ (E). It is quite clear from the fact

that the complete normed linear space is the second category of itself.

4. (E) $\Longrightarrow$ (F). We define finite dimensional (and hence bounded) operators

$$A_n f = \sum_{k=1}^{n} (f, \omega_k)\, \varepsilon_k \,.$$

Then

$$\| A_n f \| = \Big[\sum_{k=1}^{n} |(f, \omega_k)|^2\Big]^{\frac{1}{2}}$$

and

$$\limsup_{n \to +\infty} \| A_n f \| = \Big[\sum_{k=1}^{+\infty} |(f, \omega_k)|^2\Big]^{\frac{1}{2}} \,.$$

The set $\{ f \in H \,;\, \limsup_{n \to +\infty} \| A_n f \| < +\infty \} = E$ coincides, by the Banach-Steinhaus principle of condensation of singularities (see [5],p.73) either with H or it is a set of the first category of H. By the assumption, $E = H$. Since $\| A_n f \|$ are convex continous functionals on H, we can use the Gelfand lemma on such functionals (see [1],pp.68-70) to obtain that the functional

$$\sup_n \| A_n f \| = \Big[\sum_{k=1}^{+\infty} |(f, \omega_k)|^2\Big]^{\frac{1}{2}}$$

is also continous on H, i.e. there exists a constant K such that the inequality

$$\Big[\sum_{k=1}^{+\infty} |(f, \omega_k)|^2\Big]^{\frac{1}{2}} \le K \| f \|$$

holds for every $f \in H$. Therefore the operator U_2 which is defined by (3.4) is bounded on $L(\Phi)$ and, to be one, it has a bounded extension on H.

5. (F) $\Longrightarrow$ (G). Since U_2 is a linear bounded operator on H then the operator $\widetilde{U}_2$ which is defined by $\widetilde{U}_2 f = U_2 f$ for $f \in \overline{L(\Phi)}$ and $\widetilde{U}_2 f = 0$ for $f \in H \dot{-} \overline{L(\Phi)}$ is the same. The adjoint operator $\widetilde{U}_2^*$ is also linear and bounded on H. We have

$$(\widetilde{U}_2 f, \varepsilon_n) = (f, \widetilde{U}_2^* \varepsilon_n) \tag{3.9}$$

for every $f \in H$ and thus $(f, \widetilde{U}_2^* \varepsilon_n) = 0$ for all $f \in H \dot{-} \overline{L(\Phi)}$. Then $\widetilde{U}_2^* \varepsilon_n \in \overline{L(\Phi)}$, and, by putting $f = \varphi_k$ in (3.9), we can see that $\widetilde{U}_2^* \varepsilon_n = \omega_n$. Setting $\widetilde{U}_2^* = U_3$, we obtain the linear bounded operator on H that satisfies (3.5).

6. (G) $\Longrightarrow$ (H). For an operator U_3 which satisfies (3.5) we have

$$U_3 \left(\sum_{n=1}^{+\infty} \alpha_n \varepsilon_n\right) = \sum_{n=1}^{+\infty} \alpha_n \omega_n$$

and

$$\left\| \sum_{n=1}^{+\infty} \alpha_n \omega_n \right\| \leq K \left[\sum_{n=1}^{+\infty} |\alpha_n|^2\right]^{\frac{1}{2}} \tag{3.10}$$

for all $(\alpha_n) \in \ell^2$. It is quite clear now that the operator U_4 from (3.6) is linear bounded on ℓ^2.

7. (H) $\Longrightarrow$ (I). By (H), the inequality (3.10) holds for every $(\alpha_n) \in \ell^2$. For any natural number n and complex numbers $\beta_1, \dots, \beta_n$ we have

$$\left|\sum_{k=1}^{n} \alpha_k \beta_k\right| = \left|\left(\sum_{k=1}^{n} \beta_k \varphi_k, \sum_{k=1}^{n} \bar{\alpha}_k \omega_k\right)\right| \leq K \left[\sum_{k=1}^{n} |\alpha_k|^2\right]^{\frac{1}{2}} \cdot \left\|\sum_{k=1}^{n} \beta_k \varphi_k\right\| .$$

8. (I) $\Longrightarrow$ (A). For fixed chosen natural n, $\sum_{k=1}^{n} \alpha_k \beta_k$ is the linear continous functional on the space of n-tuples $(\alpha_1, \dots, \alpha_n)$ with the norm equaling to $\left[\sum_{k=1}^{n} |\beta_k|^2\right]^{\frac{1}{2}}$.

By the assumption (3.7) the inequality

$$\left[\sum_{k=1}^{n} |\beta_k|^2\right]^{\frac{1}{2}} \leq K \left\|\sum_{k=1}^{n} \beta_k \varphi_k\right\|$$

is valid. With respect to the following determination of the minimal eigenvalue of the Gramms' matrix $((\varphi_i, \varphi_j))_{i,j=1,\dots,n}$

$$\mu_1^{(n)} = \min \frac{\|\sum_{k=1}^{n} \beta_k \varphi_k\|}{[\sum_{k=1}^{n} |\beta_k|^2]^{\frac{1}{2}}}$$

we obtain

$$\mu_1^{(n)} \geq \frac{1}{K} > 0 ,$$

what means that (φ_n) is the strong minimal sequence.

Remark. The condition (E) under the assumption (φ_n) is a complete sequence in H can be replaced by the following condition

(E') E is a G_δ-set in H .

Proof. E is dense in H as $L(\Phi) \subset E$ and (φ_n) is complete. E being a dense G_δ-set in the complete space H, it cannot be a set of the first category of H (see Kuratowski: Topologie I).

We denote by H_Φ the completeness $L(\Phi)$ with respect to the scalar product

(3.11) $$(\varphi_k, \varphi_n)_\Phi = \delta_{k,n} .$$

So we have

(3.12) $$\|\sum_{k=1}^{n} (f, \omega_k)\varphi_k\|_\Phi = [\sum_{k=1}^{n} |(f, \omega_k)|^2]^{\frac{1}{2}} .$$

Corollary 1. Let (φ_n) be strong minimal and (ω_n) be complete in H . Then there exists the embedding of H into H_Φ that is continous.

Proof. By (3.12) and the part D of theorem 3, we have

$$\|f\|_\Phi = [\sum_{n=1}^{+\infty} |(f, \omega_n)|^2]^{\frac{1}{2}} .$$

This definition of the norm is correct as (ω_n) is complete. Using now the part C of theorem 3 we obtain a constant $K_1 > 0$ such that

(3.13) $$\|f\|_{\Phi} \leq K_1 \|f\|_H .$$

<u>Corollary 2</u>. Let (φ_n) be strong maximal and let exist a biorthogonal sequence (ω_n) to (φ_n). Then there exists the embedding of H_{Φ} into H that is continous.

<u>Proof</u>. The sequence (φ_n) is an orthonormal base in H_{Φ} and therefore for any $f \in H_{\Phi}$ there exists $(\alpha_n) \in \ell^2$ *such that*

$$f = \sum_{n=1}^{+\infty} \alpha_n \varphi_n \quad \text{and} \quad \|f\|_{\Phi} = [\sum_{n=1}^{+\infty} |\alpha_n|^2]^{\frac{1}{2}} .$$

By the part H of theorem 3, the series $\sum_{n=1}^{+\infty} \alpha_n \varphi_n$ is also convergent in H and

(3.14) $$\|f\|_H = \|\sum_{n=1}^{+\infty} \alpha_n \varphi_n\|_H \leq K_2 [\sum_{n=1}^{+\infty} |\alpha_n|^2]^{\frac{1}{2}} = K_2 \|f\|_{\Phi} .$$

<u>Corollary 3.</u> Let (φ_n) be strong minimal and strong maximal and complete in H. Then (φ_n) and its biorthogonal (ω_n) are bases in H and the spaces H_{Φ} and H_{Ω} are topologically equivalent to H.

<u>Proof</u>. According to theorem 3 the biorthogonal (ω_n) is also strong minimal and strong maximal in H. By the part D of this theorem, $\sum_{n=1}^{+\infty} |(f, \varphi_n)|^2$ is convergent for all $f \in H$ and, by the part G, there exists a linear bounded operator U_3 such that

$$U_3 ((\sum_{n=1}^{+\infty} (f, \varphi_n) e_n) = \sum_{n=1}^{+\infty} (f, \varphi_n) \omega_n .$$

Next, by the completeness of (φ_n), we can see

(3.15) $$f = \sum_{n=1}^{+\infty} (f, \varphi_n) \omega_n .$$

So it is proved that (ω_n) is a base in H. As a base, (ω_n) is complete. In the same way we can prove that (φ_n) is also a base in H. The topological equivalence of H_Φ and H_Ω to H follows now directly from corollary 1 and 2.

If a sequence (φ_n) fulfils the assumptions of corollary 3 then it is called to be Riesz base in H. (See [6]).

<u>Corollary 4</u>.[6] Let (ε_n) be an orthonormal base. A sequence (φ_n) constitutes Riesz base if and only if there exists an operator $\mathcal{U}$ which is defined by (3.4) and has the following properties

(i) $\mathcal{U}$ has a bounded extension on H.

(ii) There exists the inverse $\mathcal{U}^{-1}$ that is bounded and defined on H.

<u>Proof</u>. 1. Let (φ_n) be Riesz base. The property (i) follows immediately from the part F of theorem 3 and, by the part H, it is $\mathcal{R}(\mathcal{U}) = H$. Let $\mathcal{U}f = 0$ for $f = \sum (f, \omega_n)\varphi_n$. Then $(f, \omega_n) = 0$ for all n, and, by the completeness of (ω_n), $f = 0$. Hence $\mathcal{U}^{-1}$ exists. Using now (3.13), we obtain

$$\|\sum_{n=1}^{+\infty} (f, \omega_n)\varepsilon_n\| = \|\mathcal{U}f\| \leq K_1 \|f\| .$$

It means that $\mathcal{U}^{-1}$ is bounded.

2. Let $\mathcal{U}$ have the properties (i), (ii). The sequence (φ_n) is strong minimal, by the part F of theorem 3. As $\varphi_n = \mathcal{U}^{-1}\varepsilon_n$, we can use (ii) and the part G to obtain (φ_n) is also strong maximal. If $(f, \varphi_n) = 0$ for all n then $((\mathcal{U}^{-1})^* f, \varepsilon_n) = 0$, i.e. $(\mathcal{U}^{-1})^* f = 0$ and it

is finally $f = 0$. It proves that (φ_n) is a complete sequence and therefore (φ_n) constitutes Riesz base.

4. After the preceding section we can now return to the problem of almost optimal approximations.

Theorem 4. Let T be a completely continuous operator in the form (1.1). Let $(\varphi_n) \subset \mathcal{R}(T)$ constitute Riesz base in H and let (ω_n) be the biorthogonal sequence to (φ_n). Let $\left(\frac{T^*\omega_n}{\lambda_n}\right)$ be strong maximal in H. Then (φ_n) is an almost optimal approximation for $M = T(S(1))$.

Proof. Let $g = Tf \in M$. Then

$$g = \sum_{k=1}^{+\infty} (g, \omega_k)\varphi_k = \sum_{k=1}^{+\infty} (f, T^*\omega_k)\varphi_k$$

and

$$\inf_{\alpha_1,\dots,\alpha_n} \| g - \sum_{k=1}^{n} \alpha_k \varphi_k \| \le \| g - \sum_{k=1}^{n} (f, T^*\omega_k)\varphi_k \| = \| \sum_{k=n+1}^{+\infty} (f, T^*\omega_k)\varphi_k \| .$$

By (3.14), we have

$$\| \sum_{k=n+1}^{+\infty} (f, T^*\omega_k)\varphi_k \| \le K_2 \left[\sum_{k=n+1}^{+\infty} |(f, T^*\omega_k)|^2 \right]^{\frac{1}{2}} \le K_2 \lambda_{n+1} \left[\sum_{k=1}^{+\infty} |(f, \frac{T^*\omega_k}{\lambda_k})|^2 \right]^{\frac{1}{2}}.$$

We use now the parts C and D of theorem 3 to obtain

$$\rho_n(M; \varphi_1, \dots, \varphi_n) \le c K_2 \lambda_{n+1} .$$

The theorem is proved.

Remark. It is obvious that the strong maximality of $\left(\frac{T^*\omega_n}{\mu_n}\right)$, where $\mu_n = O(\lambda_n)$, is sufficient for the validity of theorem 4.

We shall need the following lemma for the proof of the converse theorem.

Lemma 1. ([7], p.325 .) Let (a_n) be a sequence of positive numbers such that $\sum_{n=1}^{+\infty} a_n$ is convergent. If we denote $\sigma_n = \sum_{k=n}^{+\infty} a_k$ then for any $\alpha < 1$ the series $\sum_{n=1}^{+\infty} \frac{a_n}{\sigma_n^\alpha}$ is also convergent.

Theorem 5. Let T be a completely continuous operator in the form (1.1) and let $(\varphi_n) \subset \mathcal{R}(T)$ be strong minimal and an almost optimal approximation for $M = T(S(1))$. Then $\left(\frac{T^* \omega_n}{\lambda_n^\alpha}\right)$, where (ω_n) is the biorthogonal sequence to (φ_n), is strong maximal for any $\alpha < 1$.

Proof. We denote $P_n^\Phi g = \sum_{k=1}^{n} a_k^{(n)} \varphi_k$ for $g \in M$. In the same way as in the part 2 of the proof of theorem 3 we obtain

$$\sum_{k=1}^{p} |a_k^{(n)} - a_k^{(m)}|^2 \leq \frac{1}{c_1} \| P_n^\Phi g - P_m^\Phi g \|^2$$

for all natural p in the case that we define $a_k^{(n)} = 0$ for $k > n$. Since (φ_n) must be complete (see theorem 2) it is $\lim_{n \to +\infty} P_n^\Phi g = g$. Thus (see the preceding noted proof)

$$(4.1) \qquad \sum_{k=1}^{+\infty} |(g, \omega_k) - a_k^{(n)}|^2 \leq \frac{1}{c_1} \| g - P_n^\Phi g \|^2 .$$

Particularly, the inequality

$$\sum_{k=n+1}^{+\infty} |(f, T^* \omega_k)|^2 \leq \frac{1}{c_1} \| Tf - P_n^\Phi Tf \|^2 \leq \frac{c^2}{c_1} \lambda_{n+1}^2$$

holds for all $f \in S(1)$. By that and lemma 1, we can see that $\sum_{k=1}^{+\infty} \left|\left(f, \frac{T^* \omega_k}{\lambda_k^\alpha}\right)\right|^2$ is convergent for any $\alpha < 1$. Using the parts B, E of theorem 3 we finish the proof.

Remark. Let (φ_n) be strong minimal and complete in H. Then, by (4.1), it follows that

$$\sum_{k=1}^{n} |(g, \omega_k) - a_k^{(n)}|^2 \leq \frac{1}{c_1} \| g - P_n^\Phi g \|^2 .$$

If (φ_n) is, moreover, strong maximal, i.e. (φ_n) is Riesz base, we have the following important result in practice

$$\|g-\sum_{k=1}^{n}(g,\omega_k)\varphi_k\| \leq K\|g-P_n^{\Phi}g\| .$$

These inequalities can be described as follows. If the Riesz base (φ_n) is an almost optimal approximation for M then the finite dimensional approximations $\sum_{k=1}^{n}(g,\omega_k)\varphi_k$ of an element $g \in M$ give also an almost optimal approximation.

Proof. We have

$$\|g-\sum_{k=1}^{n}(g,\omega_k)\varphi_k\| \leq \|g-P_n^{\Phi}g\|+\|\sum_{k=1}^{n}[(g,\omega_k)-a_k^{(n)}]\varphi_k\| \leq$$

$$\leq \|g-P_n^{\Phi}g\|+c\,[\sum_{k=1}^{n}|(g,\omega_k)-a_k^{(n)}|^2]^{\frac{1}{2}} \leq (1+\frac{c}{\sqrt{c_1}})\|g-P_n^{\Phi}g\| .$$

5. In this section we shall show the further condition for the almost optimal approximation that will be suitable for use in practice.

If T, U are completely continous operators on H and $\mathcal{R}(T) \subset \mathcal{R}(U)$ then we say that U is a majorant operator to T.

Lemma 2. An operator U is a majorant operator to T if and only if there exists a linear bounded operator $A: H \to H$ such that $T = UA$.

Proof. 1. If $T = UA$, then it is clear that $\mathcal{R}(T) \subset \mathcal{R}(U)$.

2. Let $\mathcal{R}(T) \subset \mathcal{R}(U)$. If we denote $N(U) = \{f \in H;\ Uf = 0\}$, then $U_1 = U/_{H \ominus N(U)}$ is linear

and bounded. $\mathcal{U}_1^{-1}$ exists and $\mathcal{R}(\mathcal{U}) = \mathcal{D}(\mathcal{U}_1^{-1})$.
We put $A = \mathcal{U}_1^{-1} T$, i.e. $T = \mathcal{U} A$. We have only to show that A is bounded. As $\mathcal{D}(A) = H$, the operator A will be bounded if and only if it will be closed (see [1], p.150). Let $s - \lim_{n \to +\infty} f_n = f$ and $s - \lim_{n \to +\infty} A f_n = g$. Then for $T f_n = h_n$ it is $s - \lim_{n \to +\infty} h_n = T f = h$ and for $g_n = \mathcal{U}_1^{-1} h_n$, we have $s - \lim_{n \to +\infty} g_n = g$. But $h_n = \mathcal{U}_1 g_n$ and hence $s - \lim_{n \to +\infty} h_n = \mathcal{U}_1 g = h$, i.e. $g = \mathcal{U}_1^{-1} h = \mathcal{U}_1^{-1} T f = A f$. Therefore A is closed.

<u>Lemma 3</u>. Let $A : H \to H$ be linear and bounded and let $T : H \to H$ be completely continous. Let $\mathcal{U}_1 = A T$ or $\mathcal{U}_2 = T A$. Then the eigenvalues (μ_n) of $[\mathcal{U}_1^* \mathcal{U}_1]^{\frac{1}{2}}$ or $[\mathcal{U}_2^* \mathcal{U}_2]^{\frac{1}{2}}$ have the following asymptotic behaviour

$$\mu_n = O(\lambda_n). \tag{5.1}$$

<u>Proof</u>. It can be easy obtained from the mini-maximal principle of eigenvalues of completely continous self-adjoint operators (see [8],XI,§ 9).

<u>Corollary</u>. Let $\mathcal{U}$ be a majorant operator to T. Then for eigenvalues (λ_n), (μ_n) of $[T^* T]^{\frac{1}{2}}$, $[\mathcal{U}^* \mathcal{U}]^{\frac{1}{2}}$ the asymptotic behaviour

$$\lambda_n = O(\mu_n) \tag{5.2}$$

is true.

<u>Proof</u>. It is quite clear from lemma 2 and lemma 3.

<u>Theorem 6</u>. Let $\mathcal{R}(T) = \mathcal{R}(\mathcal{U})$ and let T and $\mathcal{U}$ be completely continous operators. Let (φ_n) be an almost optimal approximation for $M_{\mathcal{U}} = \mathcal{U}(S(1))$. Then (φ_n) is also an optimal approximation for $M_T = T(S(1))$.

Proof. By lemma 2, there exists the linear bounded operator A such that $T = \mathcal{U}A$ and hence

$$M_T \subset \mathcal{U}(S(\|A\|)) = \|A\| M_{\mathcal{U}} .$$

Then

(5.3) $\rho_n(M_T; \varphi_1, \dots, \varphi_n) \leq \rho_n(\|A\| M_{\mathcal{U}}; \varphi_1, \dots, \varphi_n) = \|A\| \rho_n(M_{\mathcal{U}}; \varphi_1, \dots, \varphi_n).$

Now, using the assumption and lemma 3, we obtain

$$\rho_n(M_{\mathcal{U}}; \varphi_1, \dots, \varphi_n) \leq c_1 \mu_{n+1} \leq c_1 c_2 \lambda_{n+1} .$$

These inequalities and (5.3) show that (φ_n) is an almost optimal approximation for M_T .

Remark. Let $\mathcal{R}(T) = \mathcal{R}(\mathcal{U})$ be dense in H and let T be a completely continous operator. Let $\mathcal{U}$ be also completely continous and therefore for $f \in H$ we have

(5.4) $$\mathcal{U}f = \sum_{n=1}^{+\infty} \mu_n (f, \tilde{e}_n) \tilde{h}_n .$$

The sequence $(\tilde{h}_n)$ fulfils the properties of theorem 4.

Proof. By (5.4), $(\tilde{h}_n)$ is an orthonormal base in H and hence it is Riesz base. We have only to show that $\frac{T^* \tilde{h}_n}{\lambda_n}$ is strong maximal. But according to lemma 2 there exists the linear bounded operator A on H such that $T = \mathcal{U}A$. From that it follows that $T^* = A^* \mathcal{U}^*$ and $\frac{T^* \tilde{h}_n}{\lambda_n} = \frac{\mu_n}{\lambda_n} A^* \tilde{e}_n$. With respect to lemma 2 and lemma 3, $(\frac{\mu_n}{\lambda_n} \tilde{e}_n)$ constitutes Riesz base. Using that and the part D of theorem 3 we obtain that $(\frac{T^* \tilde{h}_n}{\lambda_n})$ is strong maximal.

Example 2. Let T be a completely continous operator in the form (1.1) such that $\mathcal{R}(T)$ is a dense set in H and let (φ_n) be Riesz base in H and (ω_n) be the biorthogonal sequence to (φ_n). According to corollary 4 of theorem 3 there exists the operator $\mathcal{U}$ such that

$$\varphi_n = \mathcal{U} h_n, \qquad h_n = \mathcal{U}^* \omega_n$$

and $\mathcal{U}$ and $\mathcal{U}^{-1}$ are bounded on H. From the proof of theorem 4 it follows

$$\begin{aligned} \| Tf - P_n^{\Phi} Tf \| &\le K_2 \Big[\sum_{k=n+1}^{+\infty} |(Tf, \omega_k)|^2 \Big]^{\frac{1}{2}} = \\ &= K_2 \Big[\sum_{k=n+1}^{+\infty} |(\mathcal{U}^{-1} Tf, h_k)|^2 \Big]^{\frac{1}{2}} . \end{aligned} \tag{5.5}$$

The operator $T_1 = \mathcal{U}^{-1} T$ is completely continous and, by virtue of lemma 3, (5.1) is valid for the non-decreasing sequence (μ_n) of the eigenvalues of $[T_1^* T_1]^{\frac{1}{2}}$. As $T = \mathcal{U} T_1$, the converse statement (5.2) is also true. If we put $M_1 = T_1(S(1))$, then $M_1 = \mathcal{U}^{-1}(M)$. Next, we shall suppose that (h_n) will be an almost optimal approximation of M_1, i.e.

$$\Big[\sum_{k=n+1}^{+\infty} |(T_1 f, h_k)|^2 \Big]^{\frac{1}{2}} \le c \mu_{n+1} .$$

By this ,(5.5) and (5.1) we have

$$\rho_n(M; \varphi_1, \dots, \varphi_n) \le c' \lambda_{n+1} . \tag{5.6}$$

Thus (φ_n) is an almost optimal approximation for M. The converse proposition is also true. Let (.5.6) be valid. Then, by (4.1), we obtain

$$\sum_{k=n+1}^{+\infty} |(T_1 f, h_k)|^2 = \sum_{k=n+1}^{+\infty} |(Tf, \omega_k)|^2 \le \frac{1}{c_1} \| Tf - P_n^{\Phi} Tf \|^2 \le \frac{c'^2}{c_1} \lambda_{n+1}^2 .$$

Using now (5.2), we get

$$\rho_n(M; h_1, \dots, h_n) \le c_1' \mu_{n+1} .$$

By connecting this example with the example 1, we can see that the optimal approximation (h_n) for M need not be an almost optimal approximation for the "similar" compact $M_1 = V(M)$ either, where V is a linear bounded operator.

However, we can prove the following theorem:

Theorem 7. Let (φ_n) be Riesz base in H and an almost optimal approximation for $M_T = T(S(1))$ where $T: H \to H$ is a completely continous operator. Let $C: H \to H$ be linear and bounded and let C^{-1} exist and be also bounded. Let $C(\mathcal{R}(T)) = \mathcal{R}(T)$. If we denote $C\varphi_n = \psi_n$ then (ψ_n) is an almost optimal approximation for $C^{-1}(M_T)$.

Proof. With respect to corollary 4 of theorem 3, (ψ_n) is Riesz base in H. Let (ω_k) and (η_k) be the biorthogonal sequence to (φ_n) and (ψ_n). We denote $L_n^{\Psi} f = \sum_{k=1}^{n} (f, \eta_k)\psi_k$. As

$$(\psi_k, \eta_n) = (C\varphi_k, \eta_n) = (\varphi_k, C^*\eta_n) ,$$

we have $\eta_n = (C^*)^{-1}\omega_n$ and $L_n^{\Psi} f = \sum_{k=1}^{n} (C^{-1}f, \omega_k) C\varphi_k$.

Hence

$$(5.7) \quad \|f - L_n^{\Psi} f\| = \|f - C(L_n^{\Phi}(C^{-1}f))\| \leq \|C\| \cdot \|C^{-1}f - L_n^{\Phi} C^{-1} f\| .$$

If we set $U = C^{-1}T$ then $U: H \to H$ is a completely continous operator such that $\mathcal{R}(U) = \mathcal{R}(T)$. According to the last theorem, (φ_n) is an almost optimal approximation for $M_U = U(S(1))$. From the last remark of the section 4 it follows that $(L_n^{\Phi} f)$ is also an almost optimal approximation for M_U. Now, using lemma 3, theorem

1 and (5.7), we obtain that $(L_n^{\psi} f)$ is an almost optimal approximation for M_u. Thus (ψ_n) is also an almost optimal approximation for M_u.

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