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§ 1. Introduction. There is a number of interesting fixed point theorems for multivalued mappings with applications in functional analysis and the theory of games (see [1] -[3]). The Glicksberg generalization [2] of the Kakutani theorem [3] on fixed points is as follows:

Theorem (Glicksberg). Let X be a locally convex linear topological space and C a compact convex subset of X . Then every closed multivalued mapping $f: C \to 2^C \cap \mathcal{K}(X)$ has a fixed point in C (i.e. $x \in f(x)$ for some $x \in C$). (For the notations and definitions see § 2.)

Recently Sadovskij [4] has proved the following

Theorem (Sadovskij). Every concentrative self-mapping of a convex closed bounded subset in a Banach space has at least one fixed point.

Recall that the sum of a contraction and a completely continuous mapping is concentrative.

This paper deals with some generalizations of the Glicksberg's and Sadovskij's theorems (see § 4). The method of § 4 is derived from the Sadovskij's proof of his theorem. This method can be formulated for multivalued mappings between sets (without topologies). We use a slight modification of a result of Michael [5]. Let us note that not all locally convex spaces are paracompact. [1)]

In § 5 we mention a fixed point theorem for onevalued weakly continuous mappings in weakly compact (non-convex) subsets of a Banach space and a proposition generalizing Problem 1 [7,p.262].

§ 2. Notations and definitions. Let $\mathbb{R}$, resp. $\mathbb{C}$ denote the field of real, resp. complex numbers. Let X be a linear space (over $\mathbb{R}$ or $\mathbb{C}$) and $M \subset X$. Then co M and sp M denote the convex and linear hull of M into X, resp. If X is a linear topological space and $M \subset X$, then $\overline{\text{co}}$ M and $\overline{\text{sp}}$ M denote the closed convex and closed linear hull of M in X, resp.

For every set C put $2^C = \{M \in \exp C : M \neq \emptyset\}$ (= the system of all nonempty subsets of C). Under a multivalued mapping of a set C into another set D we mean a mapping $f : C \to 2^D$.

Let X and Y be topological (Hausdorff) spaces and $f : X \to 2^Y$ a multivalued mapping of X into Y. Then f is called:

(1) Lower semi-continuous (l.s.c.) if the set $\{x \in X : f(x) \cap V \neq \emptyset\}$ is open in X for every open set V in Y.

(2) Upper semi-continuous (u.s.c.) if the set $\{x \in X : f(x) \subset V\}$ is open in X for every open set V in Y.

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1) Cf. Stone A.H., Paracompactness and product spaces, Bull. Amer. Math. Soc. 54, 1948, 977-982.

(3) Closed if the graph $Gr(f)=\{(x,y): x\in X, y\in f(x)\}$ of f is closed in $X\times Y$.

For Y a linear topological space denote:

$\mathcal{K}(Y)=\{C\in 2^Y : C$ convex $\}$,

$\mathcal{F}(Y)=\{C\in 2^Y : C$ convex closed $\}$,

$\mathcal{C}(Y)=\{C\in 2^Y : C$ convex compact $\}$.

Let (M, d) be a pseudometric space. Then a mapping $f: M\to M$ is called a contraction if there exists a constant $\alpha\in\langle 0,1)$ such that

$$d(f(x), f(y))\leq \alpha\, d(x,y) \text{ for any } x,y\in M.$$

If $C\subset M$, then we define

$$Q(C)=\{\varepsilon\in\mathbb{R} : \varepsilon>0 \text{ and there is a finite } \varepsilon\text{-net for } C\}.$$

The number $\chi(C)=\inf Q(C)$ $(\inf\emptyset=+\infty)$ is called the measure of non-compactness of C. If (M_1, d_1) is another pseudometric space, then a mapping $f: M\to M_1$ is called concentrative if f is continuous and for any bounded non-precompact subset C of M

$$\chi_1(f(C))<\chi(C)$$ (χ_1 is the measure of non-compactness in (M_1, d_1)).

Let X be a locally convex linear topological space and P a defining system of pseudonorms for X (i.e. $\{p^{-1}(\langle 0,\varepsilon)) : p\in P, \varepsilon\in(0,1)\}$ is a base for neigborhoods of o in X).

Then a multivalued mapping f of a subset C of X into X is said to satisfy the condition (C) if for any bounded subset M of C and for every $p \in P$ such that M is non - p - precompact there is

$$\chi_p(f(M)) < \chi_p(M) .$$

(χ_p is the measure of non-compactness of the pseudonormed space (X, p)). If f is, in addition, one-valued and continuous, then it is called <u>concentrative</u> (P -concentrative).

If X and Y are topological spaces and $f : X \to 2^Y$ a multivalued mapping of X into Y then a continuous mapping $\varphi : X \to Y$ is called a <u>continuous selection</u> of f if $\varphi(x) \in f(x)$ for each $x \in X$. If Y is a linear topological space then f is said to have the <u>almost continuous selection property</u> if for every neighborhood V of o in Y there exists a continuous mapping $\varphi_V : X \to Y$ such that $\varphi_V(x) \in (f(x) + V) \cap co\, f(X)$ for any $x \in X$.

§ 3. <u>Remarks</u>. Let X, Y be topological spaces and $f : X \to 2^Y$. The multivalued mapping f is l.s.c if and only if for each convergent net $x_\alpha \to x$ in X and any $y \in f(x)$ there are $y_\alpha \in f(x_\alpha)$ such that $y_\alpha \to y$ in Y . The mapping f need not be closed (for example, let $X = \langle 0,1 \rangle$, $Y = X \times X$, $f(x) = \{(x,y) : y \in \langle 0,1 \rangle\}$ for $x \in \langle 0,1)$ and $f(1) = \{0\}$). If f is closed, then f(x) is closed for any $x \in X$. If Y is regular and f is u.s.c. and f(x) is closed in Y for any $x \in X$ then f is closed. If f is closed and f(X) is relatively compact in Y (i.e. $\overline{f(X)}$ is compact in Y) then f is u.s.c.

The following Proposition 1 is a slight modification of Michael's result [5].

Proposition 1. If X is a paracompact space, Y a linear topological space, $f : X \longrightarrow \mathcal{K}(Y)$ a l.s.c. multivalued mapping of X into Y, V a convex neighborhood of o in Y, then there exists a continuous mapping $\varphi_V : X \rightarrow Y$ such that $\varphi_V(x) \in (f(x) + V) \cap co\, f(X)$ for each $x \in X$.

Therefore if Y is locally convex, then f has the almost continuous selection property.

Suppose (M, d) is a pseudometric space and χ its measure of non-compactness, it is easy to prove the following assertions:

(i) $C \subset M$ is bounded iff $\chi(C) < +\infty$,

(ii) $C \subset M$ is precompact (i.e. totally bounded) iff $\chi(C) = 0$.

Let (X, p) be a pseudonormed space and χ its measure of non-compactness. Then $\chi(\overline{co}\ C) = \chi(C)$ for every subset C of X.

If f and g are two multivalued mappings from some subsets of a locally convex space X into X which satisfy the condition (C) (with respect to a same defining system P of pseudonorms for X), then its composition $f \circ g$ also satisfies the condition (C). Every precompact [2] multivalued mapping in X satisfies the condition (C).

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2) i.e. it maps bounded sets into precompact sets.

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§ 4. <u>Theorem 1</u>. Let X be a locally convex (Hausdorff) linear topological space (over $\mathbb{R}$ or $\mathbb{C}$) and C a non-empty convex closed subset of X . Further, let f be a multivalued mapping of C into itself such that the following conditions are satisfied:

(i) there exists a non-empty subset K of C such that $\overline{co}\, f(K) \supset K$;

(ii) if Ω is a convex closed subset of C such that $\overline{co}\, f(\Omega) = \Omega$, then Ω is compact;

(iii) f admits a continuous selection on any convex compact subset of C .

Then f has a fixed point in C , i.e. there is a point $x_0 \in C$ such that $x_0 \in f(x_0)$.

<u>Proof</u>. Let

$$\mathfrak{G} = \{\Omega \subset C : \Omega = \overline{co}\,\Omega,\ K \subset \Omega,\ f(\Omega) \subset \Omega\} .$$

This system $\mathfrak{G}$ has the following property:

(P) $\qquad \Omega \in \mathfrak{G} \Longrightarrow \overline{co}\, f(\Omega) \in \mathfrak{G}$.

Indeed, let $\Omega \in \mathfrak{G}$ and $\Omega_1 = \overline{co}\, f(\Omega)$. Certainly, $\Omega_1 = \overline{co}\,\Omega_1$. By (i) we have $K \subset \overline{co}\, f(K) \subset \subset \overline{co}\, f(\Omega) = \Omega_1$. Since $\Omega_1 = \overline{co}\, f(\Omega) \subset \Omega$, we have $f(\Omega_1) \subset \Omega_1$. Thus (P) is proved.

Let $C_0 = \bigcap \mathfrak{G}$. Then $\emptyset \neq C_0 \in \mathfrak{G}$ since $K \subset C_0 = \overline{co}\, C_0$ and $f(C_0) = f(\bigcap \mathfrak{G}) \subset \bigcap f(\mathfrak{G}) \subset \bigcap \mathfrak{G} = C_0$. But (P) implies $\overline{co}\, f(C_0) \in \mathfrak{G}$. Therefore $C_0 = \overline{co}\, f(C_0)$.

From (ii) it follows that C_0 is compact. Hence by (iii) f admits a continuous selection φ on C_0 . Then φ is a continuous self-mapping of the compact convex subset C_0 of locally convex space X , and by Tychonoff Fixed-Point Theorem there exists a fixed point $x_o \in C_o \subset C$ of φ, i.e. $x_o = \varphi(x_o)$. Hence $x_o = \varphi(x_o) \in f(x_o)$. This completes the proof. Q.E.D.

Remark. It is evident that the condition (i) in Theorem 1 is equivalent to the following (formally stronger) condition:

(i′) there exists a non-empty convex closed subset K of C such that $\overline{co}\, f(K) \supset K$.

From the proof of Theorem 1 it is clear that the set K in the condition (i) is relatively compact.

Lemma 1. Let X be a locally convex (Hausdorff) space and C a compact convex subset of X. Then each closed multivalued mapping of C into itself which has the almost continuous selection property has a fixed point in C.

Proof. Let $f : C \to 2^C$ be closed with the almost continuous selection property. Then for any convex symmetric neighborhood V of o in X there exists a continuous mapping $\varphi_V : C \to C$ such that $\varphi_V(x) \in f(x) + V$ for aby $x \in C$. By Tychonoff Fixed Point Theorem φ_V has a fixed point $x_V \in C$. Then $x_V \in f(x_V) + V$. Since C is compact, the net $\{x_V\}$ has a convergent subnet $\{x_W\}$. Let be $x_W \to x_o$. The closedness of f implies that $x_o \in f(x_o)$. Indeed, since $x_W \in f(x_W) + W$ there are $y_W \in f(x_W)$ such that $x_W - y_W \in W$. Since $x_W \to x_o$, there is $y_W \to x_o$. From the closedness of f we have $x_o \in f(x_o)$. The lemma is proved. Q.E.D.

Theorem 2. Let C be a convex closed subset of a locally convex (Hausdorff) space X (over $\mathbb{R}$ or $\mathbb{C}$). Let $f : C \to 2^C \cap \mathcal{K}(X)$ be a closed multivalued self-mapping of C which satisfies the conditions (i) and (ii) of Theorem 1. Then f has a fixed point in C.

Proof. Let C_0 be as in the proof of Theorem 1. Since C_0 is compact convex and $f_0 = f \mid C_0$ is a closed self-mapping of C_0 and all the sets $f_0(x)$ are convex the Glicksberg's Theorem can be applied. Hence there exists a point $x_0 \in C_0$ such that $x_0 \in f_0(x_0) = f(x_0)$. The proof is complete. Q.E.D.

Remark. If the mapping f in Theorem 2 is in addition l.s.c. with almost continuous selection property then the Theorem 2 can be proved from the Lemma 1.

It is clear that the mapping f in Theorem 2 is from C to $2^C \cap \mathcal{F}(X)$, in fact. Also $f_0 : C_0 \to 2^{C_0} \cap \mathcal{C}(X)$ (since C_0 is compact).

Lemma 2. Let C be a convex closed subset of a linear topological space X. Let f be a multivalued mapping of C into itself. If there exists a point $x_0 \in C$ such that $x_0 \in \overline{co} \bigcup_{n=1}^{\infty} f^n(x_0)$, then f satisfies the condition (i) of Theorem 1.

Proof. Let $K = \bigcup_{n=0}^{\infty} f^n x_0$. Then $f(K) \cup \{x_0\} = K$. Since $x_0 \in \overline{co} f(K)$, there is $K \subset \overline{co} f(K)$. Q.E.D.

Lemma 3. Let C be a convex complete bounded subset of a locally convex space X and f a multivalued self-mapping of C which satisfies the condition (C). Then f satisfies the condition (ii) of Theorem 1.

Proof. Let $\Omega \subset C$ be a set such that $\Omega = \overline{co} f(\Omega)$. Then Ω is bounded, and $\chi_p(f(\Omega)) = \chi_p(\overline{co} f(\Omega)) = \chi_p(\Omega)$ for each $p \in P$ (P is a defining system of pseudonorms for X with respect to which f satisfies the condition (C)).

Hence Ω is precompact. Since C is complete (and Ω closed), the set Ω is compact. Q.E.D.

Proposition 2. Let X be a locally convex linear topological (Hausdorff) space and C a complete bounded convex subset of X. If f is a multivalued precompact closed self-mapping of C with the almost continuous selection property on any compact convex subset of C, then it has a fixed point in C.

Proof. From the precompactness of f and the completeness and boundedness of C it follows that the convex set $C_0 = \overline{co}\, f(C)$ is compact. Since $f_0 = f \mid C_0 : C_0 \rightarrow 2^{C_0}$ satisfies the conditions of Lemma 1 the proposition follows.

Q.E.D.

Remark. The mapping f in Prop. 2 is compact (i.e. it maps bounded sets into relatively compact sets), in fact. Also the mapping f in Prop.2 has the almost continuous selection property on any compact subset of C iff it has this property on any compact convex subset of C.

Proposition 3. Let X be a locally convex linear topological space and C a complete bounded convex subset of X. If $f : C \rightarrow 2^C$ is a precompact multivalued mapping which has a continuous selection on any compact (convex) subset of C then it has a fixed point in C.

Proof. Let $C_0 = \overline{co}\, f(C)$. Then C_0 is compact, $f(C_0) \subset C_0$ and f has a continuous selection φ on C_0. From the Tychonoff Fixed Point Theorem it follows the existence of a fixed point for φ. This fixed point is a fixed point of f, too. Q.E.D.

Proposition 4. Let X be a locally convex linear topological (Hausdorff) space and C a convex closed subset of X. Let f be a closed multivalued mapping from C to $2^C \cap \mathcal{K}(X)$ such that $f(C)$ is relatively compact. Then f has a fixed point in C.

Proof. Let $C_0 = \overline{co}\, f(C)$ and apply the Glicksberg Theorem to C_0. Q.E.D.

Theorem 3. Let X be a locally convex linear topological space and C a convex complete bounded subset of X such that any precompact countably subset of C is relatively sequentially compact. Then any concentrative self-mapping f of C has a fixed point in C.

Proof. Let $x \in C$ and K = the set of all (sequentially) limit points of the sequence $\{f^n(x) : n = 1, 2, \dots\}$. Since $f(\{f^n(x) : n = 1, 2, \dots\}) \cup \{f(x)\} = \{f^n(x) : n = 1, 2, \dots\}$, there is $\chi_p(\{f^n(x) : n = 1, 2, \dots\}) = 0$ for all $p \in P$ (P is a defining system of pseudonorms for X with respect to which f is concentrative). Hence this sequence is precompact. Then $K \neq \emptyset$ owing to the relative sequentially compactness of the sequence. We shall show that $f(K) = K$. It is clear that $f(K) \subset K$. Let $z \in K$. Then $z = \lim_{k \to \infty} f^{n_k}(x)$.

Since $\{f^{n_k - 1}(x) : k = 1, 2, \dots\}$ is relatively sequentially compact, there is its convergent subsequence $f^{n_{k_s} - 1}(x) \to y \in C$. It follows from the continuity of f that $z = f(y)$. Therefore $f(K) = K$ and the set K satisfies the condition (i) of the Theorem 1.

From Lemma 3 it follows that f satisfies the condition (ii) of Theorem 1.

The condition (iii) of Theorem 1 is satisfied trivially.

Hence we can apply Theorem 1. It follows that f has a fixed point in C . Q.E.D.

§ 5. Theorem 4. Let X be a normed linear space, C a weakly compact subset of X and f a self-mapping of C such that

(K) $\|f(x) - f(y)\| < \|x - y\|$ for all $x, y \in C, x \neq y$.

Suppose that one from the following two conditions is satisfied:

(1) f is weakly continuous (resp. sequentially weakly continuous) on C ;

(2) the set C and the functional $\|x - f(x)\|$ are convex.

Then f has a unique fixed point in C .

Proof. Let $\varphi(x) = \|x - f(x)\|$ for $x \in C$.

Let be satisfied the condition (1). Since $\|\cdot\|$ is weakly lower-semicontinuous, $(I - f)$ is weakly continuous and $\varphi = \|\cdot\| \cdot (I - f)$ the functional φ is weakly lower-semicontinuous.

Let be satisfied the condition (2). From the convexity and continuity (cf. the condition (2)) of the functional $\varphi(x) = \|x - f(x)\|$ it follows the weak lower-semicontinuity of φ .

Also in each case φ is weakly lower-semicontinuous on the weakly compact set C . Therefore there exists a point

$x_0 \in C$ such that $\varphi(x_0) = \min\{\varphi(x) : x \in C\}$. From the condition (K) it follows that

$\varphi(x_0) = 0$, i.e. $x_0 = f(x_0)$.

(If in the case (1) the mapping f is sequentially weakly continuous only it suffices to note that any weakly compact subset of a normed space is sequentially weakly compact.) Q.E.D.

Proposition 5. Let X be a compact topological space and d a non-negative lower-semicontinuous function on $X \times X$ such that $d(x,y) = 0$ iff $x = y$ for $x, y \in X$. Let be $f : X \to X$ continuous and such that

(K) $d(f(x), f(y)) < d(x,y)$ for $x, y \in X$, $x \neq y$.

Then the mapping f has exactly one fixed point in X .

Proof. Let be $\varphi(x) = d(x, f(x))$ for all $x \in X$. Then the function φ is lower-semicontinuous on the compact space X . Hence there is a point $x_0 \in X$ such that

$$\varphi(x_0) = \min\{\varphi(x) : x \in X\} .$$

From (K) it follows that $\varphi(x_0) = 0$, i.e. $x_0 = f(x_0)$

Q.E.D.

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