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ON THE DIFFERENTIABILITY OF MAPPINGS AND CONVEX FUNCTIONALS

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1. Introduction. This paper is a continuation of our dealings [1],[2],[3],[4] concerning the differentiability of mappings in linear normed spaces. Theorems 1,2 give sufficient conditions under which a mapping F is Lipschitzian and possesses the Lipschitz Fréchet derivative on a convex open subset of a linear normed space X. Theorem 3 deals with the Darboux property of the Gâteaux differentials, Theorem 4 is the generalization of Roll's theorem. It is shown that the set of all $x \in X$ where the Gâteaux differential of a demicontinuous mapping exists is a $F_{\sigma\delta}$ -set (Theorem 5). For the recent papers concerning the differentiability of mappings cf. the papers cited in [1],[2].

Second part of this paper is devoted to the study of convex functionals.

Main result (Theorem 6) of this part is an extension and generalization of S. Mazur's result [13,§ 9] to convex functionals and simultaneously it contains some answer to an open question c) by M.Z. Nashed [5,p.75] concerning the differentiability of convex functionals. This paper concludes the study of the one-sided Gâteaux differentials $V_+ \varphi(x, h)$ of convex functional φ [Theorem 8]. For recent investigations in the theory of convex functionals see [6] and the papers cited here, [7], [8], [9].

2. <u>Notations and definitions</u>. Let X, Y be linear normed spaces, $(X \to Y)$ the space of all linear continuous mapping of X into Y. Throughout this paper by a word "space" there is meant a real space. We shall use the symbols " $\longrightarrow$ ", " $\xrightarrow{w}$ " to denote the strong and weak convergence in X, Y. A pairing between $e^* \in X^*$ (or $z^* \in Y^*$) and $x \in X$ (or $y \in Y$) is denoted by (x, e^*), (or by (y, z^*)). A mapping $F: X \to Y$ is said to be demicontinuous [10] if $x_n \to x_0$ implies $F(x_n) \xrightarrow{w} F(x_0)$. We shall use the notations of Gâteaux, Fréchet differentials and derivatives in the sense of [11, chapt. I.]. We shall say that a mapping $F: X \to Y$ possesses the Lipschitz Fréchet derivative $F'(x)$ on a subset E of X if there exists a positive constant M such that $\| F'(x_1) - F'(x_2) \| \leqq M \| x_1 - x_2 \|$ holds for every $x_1, x_2 \in E$. Recall that a mapping $F: X \to Y$ is said to be uniformly Fréchet differentiable [11, chapt. I] on a set $E \subset X$ if for any positive constant ε there exists $\sigma > 0$ such that if $0 < \| h \| < \sigma$, then $\| w(x, h) \| < \varepsilon \| h \|$ for each $x \in E$, where $w(x, h) = F(x+h) - F(x) - F'(x)h$ ($F'(x)$ denotes the Fréchet derivative of F at x).

3. <u>Theorem 1</u>. Let X be a Banach space, Y a linear normed space, $F: X \to Y$ a continuous mapping of X into Y having on a convex bounded open subset E of X the first and second Gâteaux differentials $VF(x, h)$,

$V^2 F(x, h, k)$. Suppose that $V^2 F(x, h, k)$ is demicontinuous at $h = 0$, $k = 0$ uniformly with respect to $x \in E$.

Then F is Lipschitzian on E and possesses the Lipschitz Fréchet derivative $F'(x)$ on E . Moreover, F is uniformly Fréchet differentiable on E .

<u>Proof</u>. Since $V^2 F(x, h, k)$ is demicontinuous at $(0,0)$ uniformly with respect to $x \in E$, for any $e^* \in Y^*$ and any positive constant $M > 0$ there exist m_1, $m_2 > 0$ such that if $\|h\| \leq m_1$, $\|k\| \leq m_2$, then $|(V^2 F(x, h, k), e^*)| \leq M$ for every $x \in E$. As e^* is linear and continuous and $V^2 F(x, \lambda h, \alpha k) = \lambda \alpha V^2 F(x, h, k)$ for any real λ, α, we have that for every $h, k \in X$ and $x \in E$.

(1) $|(V^2 F(x, h, k), e^*)| = \dfrac{\|h\|}{m_1} \dfrac{\|k\|}{m_2} |(V^2 F(x, \dfrac{m_1 h}{\|h\|}, \dfrac{m_2 k}{\|k\|}), e^*)| \leq N \|h\| \|k\|$,

where $N = M(m_1 \cdot m_2)^{-1}$. Suppose now that x_0, x_1 are two arbitrary points of E, h is an arbitrary (but fixed) element of X . Set

$\varphi(t) = (VF(x_0 + tk, h), e^*)$, $k = x_1 - x_0$, $t \in \langle 0, 1 \rangle$.

Then $\varphi(0) = (VF(x_0, h), e^*)$, $\varphi(1) = (VF(x_1, h), e^*)$.

There exists the derivative $\varphi'(t)$ and

$$\varphi'(t) = (V^2 F(x_0 + tk, h, k), e^*) .$$

According to the mean-value theorem

(2) $(VF(x_1, h) - VF(x_0, h), e^*) = (V^2 F(x_0 + \tau k, h, k), e^*)$,

$0 < \tau < 1$. From (1),(2) it follows that

$$|(VF(x_1,h) - VF(x_0,h), e^*)| \leqq N \|x_1 - x_0\| \|h\| .$$

Using the Hahn-Banach theorem we obtain

$$(3) \qquad \|VF(x_1,h) - VF(x_0,h)\| \leqq N \|x_1 - x_0\| \|h\| .$$

Thus $VF(x,h)$ is uniformly continuous in $x \in E$. According to [11, § 3] $VF(x,h) = DF(x,h)$. Since X is a Banach space, F a continuous mapping of X into Y, using the Baire's theorems we have that $DF(x,h) = F'(x)h$, where $F'(x)$ denotes the Gâteaux derivative of F at $x \in E$. Hence

$$\|F'(x_1)h - F'(x_0)h\| \leqq N \|x_1 - x_0\| \|h\|$$

for every $x_1, x_0 \in E$. Since $h \in X$, $h \neq 0$ is an arbitrary element of X, we can choose it such that

$$\|F'(x_1)h - F'(x_0)h\| \geqq \frac{1}{2} \|F'(x_1) - F'(x_0)\| .$$

Thus

$$(4) \qquad \|F'(x_1) - F'(x_0)\| \leqq 2N \|x_1 - x_0\|$$

for every $x_1, x_2 \in E$. Therefore the Gâteaux differential is uniformly continuous in the sense of the uniform convergence of the transformations in $(X \to Y)$. Thus $F'(x)$ is the Fréchet derivative for every $x \in E$. Fixing $x^* \in E$, we have that

$$\|F'(x)\| \leqq \|F'(x^*)\| + 2NK$$

for every $x \in E$, where $K = \sup_{x,y \in E} \|x - y\| < \infty$. Thus $F'(x)$ is uniformly bounded on E. In view of the mean-value theorem F is Lipschitzian on E. Using the first part of Theorem 4.2 [11] we see that F is uniformly Fréchet differentiable on E. This completes the proof.

Remark 1. From theorem 1 it follows that F is bounded on each convex closed subset P of E and that $\| F'(x) \| \leqq \leqq k$ uniformly with respect to $x \in E$. The assumption that $V^2 F(x, h, k)$ is demicontinuous at $(0,0)$ uniformly with respect to $x \in E$ can be replaced by the stronger one: $\| V^2 F(x, h, k) \| \leqq M \| h \| \| k \|$ holds for every $x \in E$ and $h, k \in X$; $(M > 0)$.

Corollary 1. Let X, Y be linear normed spaces, $F: X \to \to Y$ a mapping of X into Y having on $D_R = \{ x \in X : \| x \| < R \}$ the first and second Gâteaux differentials $VF(x, h)$, $V^2 F(x, h, k)$. Suppose that $VF(0, h) = 0$, $\| V^2 F(x, h, k) \| \leqq M \| h \| \| k \|$ for every $h, k \in X$ and $x \in E$.

Then F is Lipschitzian on E and possesses the Lipschitz Fréchet derivative $F'(x)$ on E. Furthermore, F is uniformly Fréchet differentiable on E.

Theorem 2. Let X, Y be linear normed spaces, $F: X \to Y$ a mapping of X into Y having on a convex bounded open subset $E \subset X$ the first and second Gâteaux differentials $VF(x, h)$, $V^2 F(x, h, k)$. Suppose that for an arbitrary (but fixed) element $h \in X$ $VF(x, h)$ is demicontinuous in $x \in E$ and for every $x \in E$ $VF(x, h)$ is demicontinuous at $h = 0$. Assume that $V^2 F(x, h, k)$ is demicontinuous at $(0,0)$ uniformly with respect to $x \in E$.

Then F is Lipschitzian on E and possesses the Lipschitz Fréchet derivative $F'(x)$ on E. Moreover, F is uniformly Fréchet differentiable on E.

Proof. The proof depends on Proposition 3 [2] and the arguments similar to that of proof of Theorem 1.

Let x_1, x_2 be two arbitrary elements of X. We use the following significations for the following subsets of X : $\{x_t : x_t = x_1 + t(x_2 - x_1),\ t \in \langle 0, 1 \rangle\} = \langle x_1, x_2 \rangle$, $\{x_t : x_t = x_1 + t(x_2 - x_1),\ t \in (0, 1)\} = (x_1, x_2)$.

Theorem 3. Let X, Y be linear normed spaces, $F: X \to Y$ a mapping of X into Y, having the Gâteaux differential $VF(x, h)$ on $\langle x_1, x_2 \rangle \subset X$. If for some $e^* \in Y^*$

(5) $\quad (VF(x_1, h), e^*) < a < (VF(x_2, h), e^*)$,

where $h = x_2 - x_1$, then there exists $x_0 \in (x_1, x_2)$ such that $a = (VF(x_0, h), e^*)$.

Proof. Set $\varphi(t) = (F(x_1 + th), e^*)$, $t \in \langle 0, 1 \rangle$. Then there exists $\varphi'(t)$ and $\varphi'(t) = (VF(x_1 + th, h), e^*)$. According to (5) $\varphi'(0) < a < \varphi'(1)$. Since φ possesses the derivative $\varphi'(t)$ on $\langle 0,1 \rangle$, $\varphi'(t)$ assumes all values between $\varphi'(0)$ and $\varphi'(1)$. Therefore there exists $\theta \in (0, 1)$ such that $a = \varphi'(\theta) = (VF(x_0, h), e^*)$, $x_0 = x_1 + \theta h$.

Theorem 4. Let X, Y be linear normed spaces, $F: X \to Y$ a mapping of X into Y. Suppose that F is a continuous mapping on $\langle x_1, x_2 \rangle \subset X$ having n-th Gâteaux differential $V^n F(x, h, \ldots h)$ on (x_1, x_2). Assume that there exist the points $x^{(i)} = \langle x_1, x_2 \rangle$, $(i = 1, 2, \ldots n+1)$, $x^{(i)} = x_1 + t_i(x_2 - x_1)$, where $0 \leqq t_1 < t_2 < \ldots < t_{n+1} \leqq 1$, such that $F(x^{(i)}) = 0$, $(i = 1, 2, \ldots n+1)$.

Then for every $e^* \in Y^*$ there exists

$\xi(e^*) \in (x_1, x_2)$ such that $(V^n F(\xi, h, h, \ldots \ldots h), e^*) = 0$, where $h = x_2 - x_1$.

Proof. If $n = 1$, then our theorem is valid according to ordinary Roll's theorem. Suppose its validity for $n - 1$. According to Roll's theorem there exist $\theta_1, \theta_2, \ldots, \theta_n$, $t_1 < \theta_1 < t_2 < \theta_2 < t_3 < \ldots t_n < \theta_n < t_{n+1}$ such that $\varphi'(\theta_j) = 0$, $(j = 1, 2, \ldots n)$, where $\varphi'(t) = (VF(x_1 + th, h), e^*)$. Apply Theorem 4 to $g(t) = \varphi'(t)$ and the interval $\langle \theta_1, \theta_n \rangle$. Then there exists $\theta \in (\theta_1, \theta_n)$ such that $g^{(n-1)}(\theta) = 0$. But this denotes that $(V^n F(x_1 + \theta h, h, h, \ldots h), e^*) = 0$.

Lemma 1. Let X, Y be linear normed spaces, F: X $\longrightarrow$ Y a demicontinuous mapping of X into Y. Then for any positive number c the set $E(c) = \{x \in X : \| F(x) \| \leqq c\}$ is closed in X.

Proof. Suppose that $x_n \to x_0$, $x_n \in E(c)$, $x_0 \in X$. Since F is demicontinuous, $F(x_n) \xrightarrow{w} F(x_0)$. As $\| F(x_0) \| \leqq \underline{\lim}_{n \to \infty} \| F(x_n) \| \leqq c$, we have that $x_0 \in E(c)$.

Theorem 5. Let X be a linear normed space, Y a Banach space, F: X $\longrightarrow$ Y a demicontinuous mapping of X into Y.

Then the set Z of all x $\in$ X where the Gâteaux differential $VF(x, h)$ exists is a $F_{\sigma\delta}$-set.

Proof. Let h be any (but fixed) element of X. Set $f(t, x, h) = \frac{1}{t}(F(x + th) - F(x))$, $Z_{mn} = \{x \in X : 0 < |t| \leqq \frac{1}{m}, 0 < |t'| \leqq \frac{1}{m} \Rightarrow \| f(t, x, h) - f(t', x, h) \| \leqq \frac{1}{n}\}$.

According to lemma 1 Z_{mn} are closed sets in X. Set

$Z_n = \bigcup_{m=1}^{\infty} Z_{mn}$, $Z = \bigcap_{n=1}^{\infty} Z_n$. Then Z is a $F_{\sigma\delta}$-set. Suppose that $VF(x_0, h)$ exists. Then for any integer n there exists $\delta_n > 0$ such that if $0 < |t| < \delta_n$, then $\| f(t, x_0, h) - VF(x_0, h)\| \leq \frac{1}{2^n}$.

If $m^{-1} < \delta_n$ and $0 < |t| \leq \frac{1}{m}$, $0 < |t'| \leq \frac{1}{m}$, then

$$\| f(t, x_0, h) - f(t', x_0, h)\| \leq \frac{1}{n} .$$

Hence $x_0 \in Z_n$ for every n $(n = 1, 2, \dots)$ and thus $x_0 \in Z$. Suppose that $x_0 \in Z$. Since $x_0 \in Z_n$ for every n $(n = 1, 2, \dots)$, there exists m_0 such that if $m \geq m_0$ $(m^{-1} \leq m_0^{-1})$, then

$$\| f(\tfrac{1}{m}, x_0, h) - f(\tfrac{1}{m_0}, x_0, h)\| \leq \frac{1}{n} .$$

Since Y is complete, there exists $\lim_{m \to \infty} f(\frac{1}{m}, x_0, h) = f^*(x_0, h)$. If $0 < |t| \leq \frac{1}{m_0}$, then $\| f(t, x_0, h) - f(\frac{1}{m_0}, x_0, h)\| \leq \frac{1}{n}$.

Hence

$$\| f(t, x_0, h) - f^*(x_0, h)\| \leq \| f(t, x_0, h) - f(\tfrac{1}{m_0}, x_0, h)\| + \| f(\tfrac{1}{m_0}, x_0, h) - f(\tfrac{1}{m}, x_0, h)\| + \| f(\tfrac{1}{m}, x_0, h) - f^*(x_0, h)\| \longrightarrow 0 \quad \text{as} \quad t \to 0 .$$

Therefore $VF(x_0, h) = f^*(x_0, h)$.

<u>Corollary 2</u>. Let X be a linear normed space, Y a Banach space, $F: X \longrightarrow Y$ a demicontinuous mapping of X into Y.

Then the set P of all $x \in X$ where the Gâteaux differential does not exist is a $G_{\delta\sigma}$-set.

4. Convex functionals. We shall deal only with real functionals defined on real linear normed spaces X. We shall say that $\varphi : F \to E_1$ (E_1 denotes the set of all real numbers) is a convex functional on X if $\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$ for every $x, y \in X$ and $t \in \langle 0, 1 \rangle$. If φ is a convex functional defined on a convex open subset D of a linear topological space L, then there exists $V_+\varphi(x, h)$ for every $x \in D$, where

$$V_+\varphi(x, h) = \lim_{t \to 0_+} \frac{\varphi(x+th) - \varphi(x)}{t}, \quad (t > 0).$$

Moreover, the one-sided Gâteaux differential $V_+\varphi(x, h)$ is positively homogeneous and subadditive in $h \in X$ (cf.[12], chapt.10).

Set $V_-\varphi(x, h) = -V_+\varphi(x, -h)$. We shall prove the following:

Lemma 2. Let φ be a convex functional defined on a linear normed space X. Then

(6) $\varphi(x) - \varphi(x-h) \leq V_-\varphi(x, h) \leq V_+\varphi(x, h) \leq \varphi(x+h) - \varphi(x)$.

for every $x \in X$, $h \in X$.

Proof. From (10) [12, chapt. 10] it follows that

(7) $\varphi(x) - \varphi(x-h) \leq V_+\varphi(x, h) \leq \varphi(x+h) - \varphi(x)$.

We shall prove that $\varphi(x) - \varphi(x-h) \leq V_-\varphi(x, h)$. Since $(0 < t < 1)$

$$\varphi(x - th) = \varphi((1-t)x + t(x-h)) \leq (1-t)\varphi(x) + t\varphi(x-h),$$

$$\frac{1}{t}(\varphi(x) - \varphi(x-th)) \geq \frac{1}{t}(\varphi(x) - (1-t)\varphi(x) - t\varphi(x-h)) =$$

$$= \varphi(x) - \varphi(x-h).$$

Hence

(8) $\quad V_{-}\varphi(x,h) \geqq \varphi(x)-\varphi(x-h)$.

Replacing in $\varphi(x)-\varphi(x-h) \leqq \varphi(x+h)-\varphi(x)$ the element h by th and divide by t we have

(9) $$\frac{\varphi(x)-\varphi(x-th)}{t} \leqq \frac{\varphi(x+th)-\varphi(x)}{t}$$

Hence $V_{-}\varphi(x,h) \leqq V_{+}\varphi(x,h)$. This relation together with (7),(8) gives (6). This concludes the proof.

Remark 2. Let φ be a convex functional defined on a linear normed space X . Then $V_{-}\varphi(x,h)$ is positively homogeneous and concave functional in $h \in X$ for every $x \in X$. Indeed $V_{-}\varphi(x, h+k) = - V_{+}\varphi(x,-(h+k))$. As $V_{+}\varphi(x,-h-k) \leqq V_{+}\varphi(x,-h)+V_{+}\varphi(x,-k)$, we have that

(10) $\quad V_{-}\varphi(x,h+k) \geqq V_{-}\varphi(x,h) + V_{-}\varphi(x,k)$.

Since $(t > 0, \ \alpha > 0)$

$$\frac{1}{t}(\varphi(x)-\varphi(x-\alpha th)) = \alpha \ \frac{\varphi(x)-\varphi(x-\alpha th)}{t\alpha}$$

$V_{-}\varphi(x,h)$ is positively homogeneous in $h \in X$. This fact and (10) imply that $V_{-}\varphi(x,h)$ is concave in $h \in X$.

In next we use the following argument: If φ is a convex Lipschitzian functional with constant $M > 0$ and $V_{+}\varphi(x_0, h)$ denotes the one-sided Gâteaux differential of φ at $x_0 \in X$, then

(11) $\quad |V_{+}\varphi(x_0,h_1)-V_{+}\varphi(x_0,h_2)| \leqq M\,\|h_1-h_2\|$

for every $h_1, h_2 \in X$. This fact follows at once from the definition of $V_+ \varphi(x_0, h)$ and the properties of φ.

Theorem 6. Let X be a separable linear normed space, $\varphi : X \to E_1$ a convex Lipschitzian functional on X. Then the set Z of all $x \in X$ where the Gâteaux derivative $\varphi'(x)$ of φ exists is a G_δ-set of the second category. Moreover, if X is a Banach space, then Z contains a G_δ-set which is dense in X.

Proof. The proof of this theorem depends on some arguments of S. Mazur [13,§ 9]. Let $h_1, h_2, \ldots$ be a countable and dense subset in X, $Z_n = \{x \in X : V_+\varphi(x, h_n) = -V_+\varphi(x, -h_n), n = 1, 2, \ldots\}$. Since φ is convex, $V_+\varphi(x_0, h)$ exists and (11) is valid for every $h_1, h_2 \in X$. Thus $V\varphi(x_0, h)$ exists if and only if $V_+\varphi(x_0, h) = -V_+\varphi(x_0, -h)$, $h \in X$. Hence $x_0 \in Z \iff V_+\varphi(x_0, h_n) = -V_+\varphi(x_0, -h_n)$ in view of continuity of $V_+\varphi(x_0, h)$ in h and separability of X. Therefore $x_0 \in Z_n$ $(n = 1, 2, \ldots)$ and $Z = \bigcap_{n=1}^{\infty} Z_n$. According to lemma 2 and (9) set

$$Z_{npq} = \{x_0 \in X : \frac{1}{t}[\varphi(x_0 + th_n) - 2\varphi(x_0) + \varphi(x_0 - th_n)] \geqq \frac{1}{p}; \text{ for some } t \in \langle 0, q^{-1}\rangle\}$$

for any integers n, p, q. Since φ is continuous, Z_{npq} are closed. Obviously $X - Z_n = \bigcup_{p,q=1}^{\infty} Z_{npq}$ and $X - Z_n$ $(n = 1, 2, \ldots)$ are F_σ-sets. Hence Z_n are G_δ-sets. The sets Z_n $(n = 1, 2, \ldots)$ are dense in X. Suppose that $x^* \in X$, $x^* \notin \bar{Z}_n$. Then

$V_+\varphi(x, h_n) \neq - V_+\varphi(x, -h_n)$ for each $x \in U(x^*)$ of some neighbourhood $U(x^*)$ of x^*: $\|x - x^*\| \leqq r$ $(r > 0)$. Set $f(t) = \varphi(x^* + t h_n)$ for $t \in (-\infty, +\infty)$. Since φ is Lipschitzian, f(t) possesses the derivative $f'(t)$ almost everywhere. Let t_0 be such that $|t_0| \leqq \frac{r}{\|h_n\|}$ and suppose that $f'(t_0)$ exists. Set $x_0 = x^* + t_0 h_n$. Then $\|x_0 - x^*\| \leqq r$ and $x_0 \in Z_n$; i.e. $V_+\varphi(x_0, h_n) = - V_+\varphi(x_0, -h_n)$ which is a contradiction. Hence Z_n are dense in X (n = 1, 2,...) and X - Z are nondense. Since

$$X - Z = X - \bigcap_{n=1}^{\infty} Z_n = \bigcup_{n=1}^{\infty} (X - Z_n) ,$$

X - Z is a set of the first category in X, so Z is the set of the second category. Being Z an intersection of G_δ-sets, Z is also a G_δ-set.

Suppose that $x_0 \in Z$ is an arbitrary point of Z. Then $V\varphi(x_0, h) = V_+\varphi(x_0, h) = V_-\varphi(x_0, h)$. Since $V_+\varphi(x_0, h)$ is subadditive in $h \in X$ and $V_-\varphi(x_0, h)$ satisfies (10), we have that $V\varphi(x_0, h) = D\varphi(x_0, h)$ for every $x_0 \in Z$. According to (11)

$$|D\varphi(x_0, h_1) - D\varphi(x_0, h_2)| \leqq M \|h_1 - h_2\|$$

for every $x_0 \in Z$ and $h_1, h_2 \in X$. Hence $D\varphi(x_0, h)$ is continuous at h = 0 for every $x_0 \in Z$. Thus $D\varphi(x_0, h) = \varphi'(x_0)h$. If X is complete, according to well-known theorem Z contains a G_δ-set which is dense in X. This completes the proof.

<u>Remark 3</u>. S. Mazur [13, § 9] has proved the following result: Let $\varphi : X \to E_1$ be a functional on a separable Banach space X such that $\varphi(x+y) \leqq \varphi(x) + \varphi(y)$,

$\varphi(tx) = t\varphi(x)$, $\varphi(x) \leq M \|x\|$ hold for every x, $y \in X$, $t \geq 0$ $(M > 0)$. Then the set Z of all $x \in X$ where the Gâteaux differential $V\varphi(x, h)$ exists is a G_δ-set which is dense in X, $X - Z$ is the set of the first category.

The result of S. Mazur is contained in Theorem 6. In fact, subadditivity and positive homogeneity of φ imply convexity and subadditivity with $\varphi(x) \leq M\|x\|$ give a Lipschitz condition of φ. Indeed,

$$\varphi(x_1) = \varphi(x_1 - x_2 + x_2) \leq \varphi(x_1 - x_2) + \varphi(x_2)$$

implies

$$\varphi(x_1) - \varphi(x_2) \leq \varphi(x_1 - x_2) \leq M \|x_1 - x_2\| .$$

Remark 4. The following theorem is well-known [14,p.336]: If $T: D \to E_n$ is Lipschitzian on a bounded domain D of E_n (E_n denotes the euclidean n-space), then T possesses a total differential almost everywhere. The proof of this interesting assertion depends on lemma 1 [14,p.335] and Theorem of H. Rademacher [15] which proof is rather complicated and not quite easy. Since every linear normed space X with $\dim X = n$ is homeomorph to E_n, this result can be extended at once for such spaces. Combining Theorem 6 with the proof of Theorem 1 [16] and aware that every linear normed space X with $\dim X < \infty$ is a Banach space we can prove simply (without theory of measure and integrals) the following partial assertion: Let X be a linear normed space with $\dim X < < \infty$, $\varphi: X \to E_1$ a convex Lipschitzian functional on X. Then the set Z of all $x \in X$ where the Fréchet derivative $\varphi'(x)$ of φ exists is a G_δ-set of the second category and hence it contains a G_δ-set which is dense in X.

Lemma 3. Let φ be a continuous convex functional on a Banach space X . If there exists the Gâteaux differential $V\varphi(x_0, h)$ of φ at $x_0 \in X$, then $V\varphi(x_0, h) = \varphi'(x_0)h$, where $\varphi'(x_0)$ denotes the Gâteaux derivative of φ at x_0 .

Proof. Since $V_+\varphi(x_0, h)$ is subadditive in h and $V_-\varphi(x_0, h)$ satisfies (10), $V\varphi(x_0, h) = D\varphi(x_0, h)$ in view of $V_+\varphi(x_0,h) = V_-\varphi(x_0,h) = V\varphi(x_0,h)$. According to Baire's theorems $D\varphi(x_0,h) = \varphi'(x_0)h$.

Theorem 7. Let φ be a convex continuous functional defined on a Banach space X .

Then the set Z of all $x \in X$ where the Gâteaux derivative $\varphi'(x)$ of φ exists is a $F_{\sigma\delta}$-set.

Proof. This assetion follows immediately from Lemma 3 and Theorem 5.

Corollary 3. Let φ be a convex continuous functional on a Banach space X . Then the set P of all $x \in X$ where the Gâteaux derivative does not exist is a $G_{\delta\sigma}$ -set.

Now we shall investigate the properties of the one-sided Gâteaux differential $V_+\varphi(x_0, h)$ of convex functionals. These functionals have been used by M.Z. Nashed [5] to characterization of best approximation in linear normed spaces. Recall that a functional $\varphi(x)$ is said to be weakly lower-semicontinuous at $x_0 \in X$ if $x_n \xrightarrow{w} x_0$ implies $\varphi(x_0) \leq \varliminf_{n\to\infty} \varphi(x_n)$. It is easy to prove the following

Proposition 1. Suppose that $\varphi : X \to E_1$ is a functional defined on a linear normed space X . Then φ is weakly lower-semicontinuous if and only if for every real constant c the set $E(c) = \{x \in X : \varphi(x) \leqq c\}$ is weakly closed in X.

From this proposition it follows the following consequence: If $\varphi : X \to E_1$ is a quasi-convex [21] (i.e. $f(tx + (1-t)x_o) \leqq \max(f(x), f(x_o))$ for each x, $x_o \in X$, $t \in \langle 0,1 \rangle$) and lower-semicontinuous functional, then φ is weakly lower-semicontinuous. Indeed, $E(c) = \{x \in X : \varphi(x) \leqq c\}$ is a convex closed set for each real number c and hence it is weakly closed.
By Proposition 1 φ is weakly lower-semicontinuous. This result has been obtained by B.T.Poljak [21] (for convex functionals cf.[17]) by an another way. This assertion together with Th.9.2 [7] gives some conditions for extrema of such functionals in reflexive linear normed spaces (for futher results cf. [21],[6]).

<u>Theorem 8</u>. Let $\varphi : X \to E_1$ be a convex functional defined on a linear normed space X . Suppose that φ is continuous at $x_o \in X$.

Then: a) $V_+ \varphi(x_o, h)$ is a bounded functional in $h \in X$; b) $V_+ \varphi(x_o, h)$ is continuous and weakly lower-semicontinuous in h on each bounded open convex subset E of X .

<u>Proof</u>. a) Suppose that $h_n \to 0$, $h_n \in X$. According to Lemma 2

$$\varphi(x_o) - \varphi(x_o - h_n) \leqq V_+ \varphi(x_o, h_n) \leqq \varphi(x_o + h_n) - \varphi(x_o) .$$

Since φ is continuous at $x_o \in X$, $V_+ \varphi(x_o, h_n) \to 0$ as $n \to \infty$. Hence $V_+ \varphi(x_o, h)$ is continuous at $h = 0$. Thus for any positive number $M > 0$ there exists $\delta > 0$ such that if $\|h\| \leqq \delta$, then $|V_+ \varphi(x_o, h)| \leqq M$. Since $V_+ \varphi(x_o, h)$ is positively homogeneous in h , we have that

$$|V_+\varphi(x_0,h)| \leq |\frac{\|h\|}{m} V_+\varphi(x_0,\frac{mh}{\|h\|})| \leq M m^{-1}\|h\|$$

for any $h \in X$. b) According to a) $V_+\varphi(x_0, h)$ is finite on E . If $M \subset E$ is an arbitrary open bounded subset of E , then $V_+\varphi(x_0, h)$ is bounded on M . According to [18, chapt. II, § 5] $V_+\varphi(x_0, h)$ is continuous in $h \in E$. Being continuous and convex in h , $V_+\varphi(x_0, h)$ is weakly lower-semicontinuous on E . This completes the proof.

R e f e r e n c e s

[1] J.KOLOMÝ: On the differentiability of mappings in functional spaces. Comment.Math.Univ.Carolinae 8(1967), No 2, 315-330.

[2] J.KOLOMÝ, V.ZIZLER: Remarks on the differentiability of mappings in linear normed spaces. Comment.Math. Univ.Carolinae 8(1967), No 4, 691 - 704 .

[3] V.ZIZLER: On the differentiability of mappings in Banach spaces. Comment.Math.Univ.Carolinae 8(1967), No 3, 415-430.

[4] J.DURDIL: On the differentiability of Urysohn and Nemyckij operator. Comment.Math.Univ.Carolinae 8(1967), No 3, 515-553.

[5] M.Z.NASHED: Some remarks on variations and differentials. Am.Math.Monthly Vol.73(1966), No 4, 63-76.

[6] M.Z.NASHED: Supportably and weakly convex functions with application to approximation theory and nonlinear programming. Journ.Math.Anal.and Appl.18(1967), No 3, 504-521.

[7] M.M.VAJNBERG: O minimume vypuklych funkcionalov. Uspechi

mat.nauk 20(1965),No 1,239-240.

[8] R.I.KAČUROVSKIJ: O monotonnych operatorach i vypuklych funkcionalach. Uspechi mat.nauk 15(1960),No 4,213-215.

[9] E.S.LEVITAN,B.T.POLJAK: O schodimosti minimizurujuščich posledovatelnostěj v zadačach na uslovnyj ekstremum. Dokl.akad.nauk SSSR t.168(1966),No 5, 997-1000.

[10] E.F.BROWDER: Variational boundary value problems for quasi-linear elliptic equations of arbitrary order. Proc.Nat.Acad.Sci.U.S.A.Vol.50(1963), No 1,31-37.

[11] M.M.VAJNBERG: Variacionnyje metody issledovanija nelinejnych operatorov. Moskva 1956.

[12] F.A.VALENTINE: Convex sets. New York 1964.

[13] S.MAZUR: Über konvexe Mengen in linearen normierten Räumen. Studia Math.t.IV(1933),70-84.

[14] T.RADO, P.V.REICHELDERFER: Continuous transformations in analysis. Springer 1955.

[15] H.RADEMACHER: Ueber partielle und totale Differenzierbarkeit. Math.Ann.79(1919) 340-359,81(1920), 52-63.

[16] N.A.IVANOV: O differencialach Gate i Freše. Uspechi mat. nauk t.X(1955),161-166.

[17] C.L.DOLPH,G.J.MINTY: On nonlinear integral equations of the Hammerstein type.Nonlinear integral equations. Proceedings of an Advanced Seminar,1963. The Univ.of Wisconsin Press,1964,99-154.

[18] N.BURBAKI: Topologičeskije vektornyje prostranstva. Moskva 1959.

[19] R.NEVANLINNA: Bemerkung zur Funkcionalanalysis. Math. Scand.1(1953),No 1,104-112.

[20] V.JARNÍK: Diferenciální počet II.Praha 1953.

[21] B.T.POLJAK: Teoremy susčestvovanija i schodimosť minimizurujuščich posledovatelnostěj dlja zadač na ekstremum pri naličii ohraničenij. Dokl.akad. n.SSSR,166(1966)No 2,287-290.

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