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# APPROXIMATE SOLUTIONS OF EQUATIONS IN BANACH SPACES BY THE NEWTON ITERATIVE METHOD. PART 2. HAMMERSTEIN INTEGRAL EQUATIONS

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This paper is the continuation of the paper [5], the knowledge of which is assumed.

§ 3. We shall study the solution of the Hammerstein integral equation

(1) $$\Phi u \stackrel{df}{=} u(x) - \int_{-1}^{1} \mathcal{K}(x,t)\, \mathcal{F}(t, u(t))\, dt - g(x) = 0$$

by the method of collocation.

Let the function $g(x)$ be continuous on the interval $\mathcal{J} \equiv \{x \mid |x| \le 1\}$, the kernel $\mathcal{K}(x,t)$ be continuous on the square $Q \equiv \{(x,t) \mid |x| \le 1,\ |t| \le 1\}$, the function $\mathcal{F}(t,z)$ be continuous as well as its first and second derivatives with respect to $z$ on the set $\mathcal{M} \equiv \{(t,z) \mid |t| \le 1, -\infty < z < \infty\}$. Some other assumptions will be given later.

Let us denote by A the matrix of the interpolating points $x_k^{(m)}$, $k = 0, 1, \dots, m$ ; $m = 1, 2, \dots$, where we have

(2) $$-1 \le x_m^{(m)} < x_{m-1}^{(m)} < \dots < x_1^{(m)} < x_0^{(m)} \le 1$$

for the points of order m .

If f is a function continuous on $\mathcal{J}$, let us denote by $\mathcal{P}_m(f, x)$ the Lagrangian interpolating polynomial for the function f with the interpolating points $x_0^{(m)}$, $x_1^{(m)}, \dots, x_m^{(m)}$; $l_k^{(m)}(x)$ be the Lagrangian interpolating coefficients. For given m , there is

$$l_k^{(m)}(x_i^{(m)}) = \delta_{ik} \ , \quad i, k = 0, 1, \dots, m \quad ,$$

$$\mathcal{P}_m(f, x) = \sum_{k=0}^{m} l_k^{(m)}(x) f(x_k^{(m)}) \ .$$

Furthermore, let $A_1$ be the matrix of Chebyshev interpolating points

(3) $$x_{k1}^{(m)} = \cos \frac{2k+1}{2(m+1)} \pi \ , \quad k = 0, 1, \dots, m \ ; \ m = 1, 2, \dots \ ,$$

$A_2$ be the matrix of the points called the Lanczos interpolating points,

(4) $$x_{k2}^{(m)} = \cos \frac{k}{m} \pi, \quad k = 0, 1, \dots, m \ ; \quad m = 1, 2, \dots$$

(Lanczos [11],[12] ). The interpolating polynomials shall be denoted by $\mathcal{P}_m^{(1)}(f, x)$, $\mathcal{P}_m^{(2)}(f, x)$ resp., the Lagrangian interpolating coefficients by $l_{k1}^{(m)}(x)$, $l_{k2}^{(m)}(x)$ resp.

The Chebyshev interpolating points of order m are the zeros of the Chebyshev polynomial $T_{m+1}(x)$ , the Lanczos interpolating points of order $m$ are the zeros of the Chebyshev polynomial of the second kind $\mathcal{U}_{m-1}(x)$ and the points $x = -1$, $x = 1$ . We have the following

Lemma. For each function f absolutely continuous on the interval $\langle -1, 1 \rangle$ the following relations take place:

(5) $$\lim_{m\to+\infty} \max_{-1\le x\le 1} |f(x) - \mathcal{P}_m^{(i)}(f,x)| = 0, \quad i = 1, 2,$$

(6) $$\sum_{k=0}^{m} |\ell_{k1}^{(m)}(x)| \le 4\sqrt{2} + \frac{2}{\pi} \lg m, \quad -1 \le x \le 1,$$

(7) $$\sum_{k=0}^{m} |\ell_{k2}^{(m)}(x)| \le 8 + \frac{2}{\pi} \lg m, \quad -1 \le x \le 1.$$

Proof. The relations (6),(7) have been proved by Berman [1], the relation (5), for $i = 1$, has been proved by Krylov [10]. Let us now prove the relation (5) for $i = 2$. We shall do it by showing that the assumptions of the theorem of Berman [2] are valid: the relation

(x) $$\lim_{m\to\infty} \max_{-1\le x\le 1} |f(x) - \mathcal{P}_m(f,x)| = 0$$

is fulfilled for each function $f$ which is absolutely continuous on $\langle -1, 1 \rangle$ if the matrix of the interpolating points has the following properties: there exists an $m_0 \in \mathcal{N}$ such that, for $m \ge m_0$ and for each $x \in \langle -1, 1 \rangle$,

1) a) $|\ell_k^{(m)'}(x)| \ge |\ell_{k+1}^{(m)'}(x)|$ for $x_{k+1} < x_k < x$,

b) $|\ell_k^{(m)'}(x)| \le |\ell_{k+1}^{(m)'}(x)|$ for $x < x_{k+1} < x_k$,

2) $|\ell_k^{(m)'}(x)| < M$, where $M = \text{const}$ is independent on m and x.

It is easy to see that, for $0 \le x \le 1$,

$$\ell_{k2}^{(m)}(x) = \frac{\mathcal{L}_m(x)}{(x-x_k)\mathcal{L}_m'(x_k)} = \frac{(-1)^k}{m} \sin m\vartheta \sin\vartheta\, F(\vartheta_k) =$$
$$= R(\vartheta)\, F(\vartheta_k),$$

where $\mathcal{L}_m(x) = \prod_{k=0}^{m}(x - x_k)$, $x_k = x_{k2}^{(m)}$ ,

$$x = \cos\vartheta\,, \quad 0 \le \vartheta \le \pi \,,$$

$$F(\vartheta_k) = \frac{1}{\cos\vartheta - \cos\vartheta_k} \quad \text{for} \quad k = 1, 2, \dots, m-1\,,$$

$$F(\vartheta_0) = \frac{1}{2(x-1)}\,, \quad F(\vartheta_m) = \frac{1}{2(x+1)}$$

Clearly, $F(\vartheta_k) > 0$ for $\vartheta < \vartheta_k$ ,

$F(\vartheta_k) < 0$ for $\vartheta > \vartheta_k$, $k = 1, 2, \dots, m-1$,

$F(\vartheta_0) < 0$ ,

$F(\vartheta_m) > 0$ .

Let $-1 \le x < x_{k+1} < x_k < x_0 = 1$ , and therefore $\pi \ge \vartheta > \vartheta_{k+1} > \vartheta_k > \vartheta_0 = 0$ . Then

$$|\ell_{k+1,2}(x)| = |R(\vartheta)| \frac{1}{\cos\vartheta_k - \cos\vartheta} \ge |R(\vartheta)| \frac{1}{\cos\vartheta_{k+1} - \cos\vartheta} = |\ell_{k2}(x)|.$$

If $x_2 \le x < x_1 < 0$, then $\vartheta_2 \ge \vartheta > \vartheta_1 > \vartheta_0$ , and we have

$$|\ell_{12}(x)| = |R(\vartheta)| \frac{1}{\cos\vartheta_1 - \cos\vartheta} \ge |R(\vartheta)| \frac{1}{2(1-\cos\vartheta)} = |\ell_{02}(x)|.$$

This proves the assumption 1 a) of the Berman theorem. The proof of 1 b) is analogous.

The proof of 2) for the Lanczos polynomials has been given by Stehlík [15], the constant M being equal to 2 .

<u>Remark</u>. Jegorova [6] presents the proof of (x) for the matrix given by the zeros of Chebyshev polynomials of the second kind $\mathcal{U}_m(x)$ using the Berman theorem as well.

The Lanczos interpolating points are obtained by adjoining the points $-1, 1$ to the zeros of $\mathcal{U}_{m-1}(x)$. In spite of this, the validity of the relation (x) is not obvious. For example, adjoining the points $-1, 1$ to the zeros of the Chebyshev polynomials $T_m(x)$, an analogous assertion will not take place; for example, for the function $|x|$ on $\langle -1, 1 \rangle$ the sequence of the interpolating polynomials diverges at the point 0 (Berman [3]).

Let us now present a special choice of the spaces and operators given in § 2 [5]. X be the space $C\langle -1, 1 \rangle$ with the Chebyshev norm, $\tilde{X}_m$ the space of polynomials of order m - 1, $\bar{X}_m$ the space of vectors $\bar{u} = (\bar{u}_0, \bar{u}_1, \dots, \bar{u}_{m-1})$ with the norm

$$\| \bar{u} \| = \max_{0 \le k \le m-1} | \bar{u}_k | .$$

The operator $P_m$ associates to each function $f \in X$ its interpolating polynomial $\mathcal{P}_{m-1}(f, x)$ with the Chebyshev or Lanczos interpolating points. The operators $\psi_{m+1}$ or $\varphi_{m+1}$ associate to each polynomial or continuous function f resp. the vector $(f(x_0^{(m)}), \dots, f(x_m^{(m)}))$ of the space $\bar{X}_m$. Clearly,

$$\| \varphi_m \| = 1 \quad \text{for} \quad m = 1, 2, \dots . \tag{8}$$

The operator $\psi_{m+1}^{-1}$ associates to the vector $\bar{v}^{(m)} = (\bar{v}_0^{(m)}, \dots, \bar{v}_m^{(m)})$ the polynomial $\sum_{k=0}^{m} \ell_k^{(m)}(x)\, \bar{v}_k^{(m)}$ ( $\ell_k^{(m)}$ denotes here and later on $\ell_{ki}^{(m)}(x)$, $i = 1, 2$ ).

Thus,

$$\| \psi_{m+1}^{-1} \| \le \max_{-1 \le x \le 1} \sum_{k=0}^{m} | \ell_k^{(m)}(x) | , \quad m = 1, 2, \dots ,$$

and the same inequality takes place for $\| \mathcal{P}_{m+1} \|$.

According to (6),(7) for $m \to \infty$, there is

(9) $\| \psi_{m+1}^{-1} \| = O(\lg m), \quad \| \mathcal{P}_{m+1} \| = O(\lg m)$ .

Defining the operators K and F by the relations

$$u \in C\langle -1, 1\rangle, \; v = Ku \Longleftrightarrow v(x) = \int_{-1}^{1} \mathcal{K}(x,t)u(t)\,dt,$$

$$f \in C\langle -1, 1\rangle, \; g = Ff \Longleftrightarrow g(x) = \mathcal{F}(x, f(x)),$$

the equation I of § 2 [5] gets the form (1), the equations III,IV being of the forms

(10) $\bar{u}_i^{(m)} - \int_{-1}^{1} \mathcal{K}(x_i^{(m)}, t)\, \mathcal{F}(t, \sum_{s=0}^{m} l_s^{(m)}(t)\, \bar{u}_s^{(m)})\,dt = g(x_i),$

$i = 0, 1, \dots, m$ ,

(11) $\bar{v}_i^{(m)} - \int_{-1}^{1} \mathcal{K}(x_i^{(m)}, t) \sum_{s=0}^{m} l_s^{(m)}(t)\, \mathcal{F}(t_s, \bar{v}_s^{(m)})\,dt = g(x_i),$

$i = 0, 1, \dots, m$ ,

respectively.

Now, it is possible to prove

<u>Theorem 1</u>. Suppose the validity of the assumptions given above, as well as:

A) The function $g(x)$ is the equation (1) be absolutely continuous on $\langle -1, 1\rangle$. Let $\mathcal{K}(x, t)$ be an absolutely continuous function of x on $\langle -1, 1\rangle$ for each $t \in \langle -1, 1\rangle$.

B) Let the linear integral equation

$$v(x) - \int_{-1}^{1} \mathcal{K}(x, t)\, \mathcal{F}_z'(t, u_0(t))\, v(t)\,dt = f(x),$$

$u_0$ being a polynomial of degree s and $f \in C\langle -1, 1\rangle$ being arbitrary, have the resolvent $\mathcal{G}(x, t)$, and, furthermore,

$$(12) \quad \int_{-1}^{1} |\mathcal{G}(x,t)|\,dt \leq \alpha, \quad |x| \leq 1,$$

$$(13) \quad \left|\int_{-1}^{1} \mathcal{K}(x,t)\,\mathcal{F}(t, u_0(t))\,dt - u_0(x) + g(x)\right| \leq$$
$$\leq \beta, \quad |x| \leq 1,$$

$$(14) \quad \int_{-1}^{1} |\mathcal{K}(x,t)\,\mathcal{F}''_{zz}(t, u(t))|\,dt \leq \varkappa$$

on the set $S \overset{df}{=} \{(x,z) \,|\, |x| \leq 1,\ |z - u_0(x)| \leq r\}$,

$$(15) \qquad r > r_0 \overset{df}{=} \frac{1 - \sqrt{1 - 2h}}{h}(1+\alpha)\beta,$$

$$(16) \qquad h \overset{df}{=} (1+\alpha)^2 \beta \varkappa < \frac{1}{2}.$$

Then the systems of equations (10) have, starting from a certain $m \geq s$, solutions $\bar{u}_*^{(m)} = (\bar{u}_{0*}^{(m)}, \dots, \bar{u}_{m*}^{(m)})$ to which there converge the Newton iterative processes (ordinary or modified) with the initial approximations

$$(17) \quad \bar{u}_0^{(m)} = (u_0(x_0^{(m)}), \dots, u_0(x_m^{(m)})).$$

Furthermore, there is

$$(18) \quad \lim_{m\to\infty} \max_{-1 \leq x \leq 1} \left|u^*(x) - \sum_{k=0}^{m} l_k^{(m)}(x)\,\bar{u}_{k*}^{(m)}\right| = 0,$$

$u^*$ being the solution of (1) given by the Newton iterative process with the initial approximation $u_0$.

If

c) $\quad \varkappa < \varkappa_1 \overset{df}{=} \dfrac{1+\sqrt{1-2h}}{h}\,(1+\alpha)\beta\,,$

then, starting from a sufficiently large m , the solutions of (10) are unique on the set

$$\bar{S} = \{\bar{u} \mid \max_{0 \le k \le m} |\bar{u}_k - \bar{u}_{k0}| \le \|\psi_m^{-1}\|^{-1}\varkappa\}\,.$$

Proof. It is clear that the assertion of the theorem is identique with the assertion of Theorem 1, § 2 [5] for the special spaces and operators chosen above. Therefore it suffices to show that the assumptions A) - C), resp. D) of that theorem take place. The assumptions C,D) follow immediately from our assumptions B),C). It remains to show that A), B) of Theorem 1 § 2 take place.

The assumption A) on K is clearly fulfilled. Further, $F''(u)$ is bounded as its norm

$\max_{-1 \le x \le 1} |\mathcal{F}''_{zz}(x, u(x))|$ is bounded on each bounded set of functions $f \in C\langle -1, 1\rangle$, this being true because the function $\mathcal{F}''_{zz}(x,z)$ is continuous on $\mathcal{M}$.

B) takes place according to (5). Further,

$$\|K - P_{m-1}K\| \le \max_{-1 \le x \le 1} \int_{-1}^{1} |\mathcal{K}(x,t) - \mathcal{P}_m(\mathcal{K}(x,t),x)|\,dt\,.$$

From our assumption A) and from (5) it follows that, for each $t \in \langle -1, 1\rangle$, the sequence of non-negative functions $\{\varepsilon_m(t)\}$, $\varepsilon_m(t) = \max_{-1 \le x \le 1} |\mathcal{K}(x,t) - \mathcal{P}_m(\mathcal{K}(x,t),x)|$ converges to zero. From this it follows (as there exists a constant M such that $|\varepsilon_m(t)| < M$ for all m and all $t \in \langle -1, 1\rangle$ ) that $\lim_{m\to\infty} \int_{-1}^{1} \varepsilon_m(t)\,dt =$
$= \int_{-1}^{1} \lim_{m\to\infty} \varepsilon_m(t)\,dt = 0$. Thus, $\|K - P_m K\| \xrightarrow[m\to\infty]{} 0$.

The proof of Theorem 1 is complete.

<u>Theorem 2</u>. A) Let the assumptions B) of Theorem 1 take place.

B) Let there exist positive numbers $\alpha$, $M$ such that $g(x) \in Lip_M \alpha$ on $\langle -1, 1 \rangle$ and $\mathcal{K}(x,t)$ as a function of x belongs to the class $Lip_M \alpha$ on $\langle -1, 1 \rangle$ for each $t \in \langle -1, 1 \rangle$. Let there exist continuous partial derivatives $\mathcal{F}'_x(x,z)$, $\mathcal{F}''_{zx}(x,z)$ on $\mathcal{M}$.

C) Let there, starting from some $m = m_0$, exist the matrices $\{ a_{ij}^{(m)} \}$ inverse to the matrices $\{ b_{ij}^{(m)} \}$ where

$$(19)\quad b_{ij}^{(m)} = \delta_{ij} - \int_{-1}^{1} \mathcal{K}(x_i^{(m)}, t) \frac{\partial \mathcal{F}}{\partial z}(t, \sum_{s=0}^{m} l_s^{(m)}(t) \bar{u}_{s*}^{(m)}) l_j(t)\, dt$$

$$i, j = 0, 1, \dots, m,$$

$\bar{u}_*^{(m)}$ being the solutions of the systems (10) from the assertion of Theorem 1.

Let there exist a constant $c > 0$ such that, for all $m \geq m_0$, there is

$$(20)\qquad \max_{0 \leq i \leq m} \sum_{j=0}^{m} | a_{ij}^{(m)} | < c .$$

Then the systems (11) have, for $m \geq m_0$, solutions $\bar{v}_*^{(m)}$ such that

$$(21)\qquad \lim_{m \to \infty} \max_{0 \leq k \leq m} | \bar{u}_{k*}^{(m)} - \bar{v}_{k*}^{(m)} | = 0 .$$

<u>Proof</u>. As in Theorem 1, it is clear that, for our choice of spaces and operators, our assumptions imply the assumptions A),B) of theorem 2 § 2 [5], and the assertion

of our theorem is identical to that one of Theorem 2,§ 2. It remains to show that the assumption C) of that theorem, i.e., the relations (19) - (22), § 2 [5] take place. According to (8),(9), it suffices to study the sequences

$$\|g - P_m g\|, \|K - P_m K\|, \|(I - P_m)Fu^*\|, \|(I - P_m)F'(u^*)\| .$$

It is easy to show that, if $f \in Lip_M \alpha$ on $\langle -1, 1 \rangle$, there is

$$(22) \qquad \|(I - P_m)f\| = O\left(\frac{lg m}{m^{\alpha}}\right) .$$

Indeed, $q_m$ being the polynomial of the best approximation for f on $\langle -1, 1 \rangle$ there is

$$\|f - P_m f\| = \|(I - P_m)(f - q_m)\| \leq (1 + \|P_m\|)\|f - q_m\| \leq$$

$$\leq (1 + \|P_m\|) \max_{-1 \leq x \leq 1} |f(x) - q_m(x)| \leq (1 + \|P_m\|)\, \varepsilon_m .$$

Following the Jackson theorem [14] there is $\varepsilon_m \leq C \frac{M}{m^{\alpha}}$, C depending only on the length of the interval and on the number $\alpha$. According to (9), this implies (22). Specially, for there is

$$(23) \qquad \|g - P_m g\| = O\left(\frac{lg m}{m^{\alpha}}\right) .$$

Further, there is

$$\|K - P_m K\| \leq \max_{-1 \leq x \leq 1} \int_{-1}^{1} |\mathcal{K}(x,t) - \mathcal{P}_m(\mathcal{K}(x,t), x)|\, dt .$$

From the assumption about $\mathcal{K}(x, t)$, it follows that, for each $t \in \langle -1, 1 \rangle$, there is $|\mathcal{K}(x,t) - q_m(x,t)| \leq$ $\leq C \frac{M}{m^{\alpha}}$ where C does not depend on t . Thus we have, for $m \to \infty$,

$$(24) \qquad \| K - P_m K \| = \mathcal{O}\left( \frac{lg\, m}{m^{\alpha}} \right) .$$

The function $\int_{-1}^{1} \mathcal{K}(x,t)\, \mathcal{F}(t, v(t))\, dt$, for $v \in C\langle -1, 1 \rangle$ belongs to the class $Lip_{M'_v}\, \alpha$, $M'_v = M \int_{-1}^{1} | \mathcal{F}(t, v(t)) |\, dt$.

As $g \in Lip_M\, \alpha$, it follows from the equation (1) that each its continuous solution u (namely $u^*$) belongs to the class $Lip_{M_{1u}}\, \alpha$, $M_{1u} = min\,(M, M'_u)$. The assumptions about the derivatives $\mathcal{F}'_x$, $\mathcal{F}''_{xx}$ imply the existence of numbers $M_2$, $M_3$ such that

$$\mathcal{F}(x, u^*(x)) \in Lip_{M_2}\, \alpha, \quad \mathcal{F}'_z(x, u^*(x)) \in Lip_{M_3}\, \alpha .$$

Using the same procedure as that for proving (23), we get the relations

$$(25) \quad \| (I - P_{m+1}) F u^* \| = \mathcal{O}\left( \frac{lg\, m}{m^{\alpha}} \right) \quad \text{for } m \to \infty,$$

$$(26) \quad \| (I - P_{m+1}) F'(u^*) \| = \mathcal{O}\left( \frac{lg\, m}{m^{\alpha}} \right) \quad \text{for } m \to \infty .$$

It follows from (8),(9),(23),(24),(25),(26) that the left-hand sides of (19),(20),(21),(22) of § 2 [5] are, for the given $(p, \kappa, s, t)$, $\mathcal{O}\left( \frac{(lg\, m)^4}{m^{\alpha}} \right)$, $\alpha > 0$, and they have, for $m \to \infty$, the limit equal to zero. This completes the proof of Theorem 2.

<u>Remark</u>. In concrete cases, the verification of the assumption about the matrices $\{ a_{ij}^{(m)} \}$ may be difficult.

In spite of this, the theorem gives some information about the possibility of solving (11) instead of (10). For example, for the Gauss interpolating points (the zeros of Legendre polynomials), the theorem cannot be proved. In fact, in this case there is $\| P_m \| = O(\sqrt{m})$, and the relations (19) - (22) of § 2 [5] do not take place.

It is very advantageous to use the system (11) instead of (10) namely in the case when the function $\mathcal{F}(t, z)$ depends on z only. Then the equation (11) gets the simple form

$$\bar{v}_i^{(m)} = \sum_{k=0}^{m} A_{ik}\, \mathcal{F}(\bar{v}_k^{(m)}) + g(x_i^{(m)}), \quad i = 0, 1, \dots, m,$$

$$A_{ik} = \int_{-1}^{1} \mathcal{K}(x_i^{(m)}, t)\, l_k(t)\, dt .$$

Example 1. Let the equation (1) have the form

$$u(x) = \int_0^1 \mathcal{K}(x, t)\, e^{u(t)} dt + \tfrac{1}{2}(x - x^2), \; 0 \le x \le 1 ,$$

$$\mathcal{K}(x, t) = x(1 - t) \quad \text{for} \quad x \le t ,$$

$$= t(1 - x) \quad \text{for} \quad t \le x .$$

To solve this equation is equivalent to the boundary problem

$$u''(x) + e^{u(x)} + 1 = 0, \quad 0 < x < 1 ,$$

$$u(0) = u(1) = 0 .$$

The initial approximation was obtained using the theorem of Bohl [4].

We get the approximate solution in the form

$$\tilde{u}_0(x) = 1{,}7234\, (\cos \alpha_1 x + \alpha_2 \sin \alpha_1 x - 1) ,$$

$$\alpha_1 = 1,07522, \quad \alpha_2 = 0,59617 .$$

For the initial approximation there was taken the polynomial $u_0(x)$ of degree four obtained from $\tilde{u}_0(x)$ by the interpolation with the Lanczos interpolating points $x_k^{(m)}$ (for the interval $\langle 0, 1 \rangle$ ). It is easy to show that the conditions of Theorems 1 and 2 are satisfied with the exception of the condition C) of Theorem 2 (see Remark following Theorem 2). We are going to verify them for $\tilde{u}_0$ and then to show that they will take place for $u_0$ , too.

In the first place, the functions $\mathcal{K}(x, t)$, $\mathcal{F}(t, z) = e^z$, $g(x) = \frac{1}{2}(x - x^2)$ have clearly all the needed properties and so the assumptions A) of Theorem 1 and B) of Theorem 2 are fulfilled. It remains to verify B) and C) of Theorem 1.

There is

$$\max_{0 \le x \le 1} \int_0^1 \mathcal{K}(x, t)\, dt = 0,125 ,$$

$$\max_{0 \le x \le 1} |\tilde{u}_0(x)| \le 0,28293 ,$$

$\| K F'(\tilde{u}_0) \| < 0,16588 < 1$ , so

$$\| [I - K F'(\tilde{u}_0)]^{-1} \| \le \frac{1}{1 - \| K F'(\tilde{u}_0) \|} < 1,19887 = 1 + \alpha ;$$

$$\| \tilde{u}_0 - K F \tilde{u}_0 - g \| < 0,57381 = \beta ;$$

$\| K F''(u) \| < 0,5 = \varkappa$, if $\max_{0 \le x \le 1} |u(x)| \le 1,3863$ ,

i.e. for $\| u - \tilde{u}_0 \| < 1,1034 = r$ ;

$$(1 + \alpha)^2 \beta \varkappa < 0,4124 = h < \frac{1}{2} ;$$

$$\frac{1}{2}[1-\sqrt{1-2h}](1+\alpha)\beta < 0,96982 = r_0 < r ;$$

$$\frac{1}{2}[1+\sqrt{1-2h}](1+\alpha)\beta > 2,36631 = r_1 > r .$$

The result for the polynomial $u_0(x)$ will be also valid as the error of the approximation is not larger than

$$\frac{1}{5!}\max_{0\le x\le 1}|\tilde{u}_0^{(5)}(x)|\frac{1}{64}\max_{0\le x\le 1}|\tilde{\mathcal{L}}_5(x)| < 1,2 \cdot 10^{-4} .$$

From the validity of the verified conditions there follows that the equation (1) has a unique solution $u^*(x)$ on the ball $\bar{S}(u_0, r)$ of the space $C\langle 0,1\rangle$.

For the discretization, we have used the collocation with Lanczos interpolating points $\{x_k^{(m)}\}$, $k = 0,1,\dots,m$ for m = 4,6,8 , i.e., with 5, 7, 9 points, resp.

Denoting the Lanczos polynomial of order m for $\langle -1, 1\rangle$ by $\mathcal{L}_m(x)$ there will be, for the polynomial $\tilde{\mathcal{L}}_m(x)$ transformed to the interval $\langle 0, 1\rangle$,

$$\tilde{\mathcal{L}}_m(x) = \mathcal{L}_m(2x-1) .$$

The interpolating points $x_k^{(m)}$ are the zeros of $\tilde{\mathcal{L}}_m(x)$:

$$x_k^{(m)} = \frac{1}{2}\left(1+\cos\frac{k\pi}{m}\right), \quad k = 0,1,\dots,m .$$

The Lagrangian interpolating coefficients $l_k^{(m)}(x) = \dfrac{\tilde{\mathcal{L}}_m(x)}{(x-x_k^{(m)})\tilde{\mathcal{L}}_m'(x_k^{(m)})}$ were obtained from the Lanczos expression ( [11],chap.4)

$$l_k^{(m)}(x) = \frac{2}{m} \sum_{s=0}^{m}{}'' \cos k \frac{2\pi}{m} \tilde{T}_s(x) ,$$

where $\sum_{s=0}^{m}{}'' = \frac{1}{2} a_0 + a_1 + \dots + a_{m-1} + \frac{1}{2} a_m$ and $\tilde{T}_s(x)$

are Chebyshev polynomials transformed to $\langle 0, 1 \rangle$.

The systems (10) have the form

$$\bar{u}_i^{(m)} = \int_0^1 \mathcal{K}(x_i^{(m)}, t)\, e^{\sum_{k=0}^{m} l_k^{(m)}(t) \bar{u}_k^{(m)}} dt - \frac{1}{2}(x_i^{(m)} - [x_i^{(m)}]^2),$$

$i = 0, 1, \dots, m$ .

The systems (11) become much more simple:

$$\bar{v}_i^{(m)} = \sum_{k=0}^{m} A_{ik}^{(m)} e^{\bar{v}_k^{(m)}} - \frac{1}{2}(x_i^{(m)} - [x_i^{(m)}]^2), \quad i = 0, 1, \dots, m ;$$

where $A_{ik}^{(m)} = \int_0^1 \mathcal{K}(x_i^{(m)}, t)\, l_k^{(m)}(t)\, dt$ .

These systems were solved on the computer Ural II. This method, in the form just presented, is not suitable for a great number of interpolating points because the coefficients of the polynomials $l_k^{(m)}(x)$ become too large (for m = 4 the greatest coefficient is about $8.10^1$ , for m = 10 about $9.10^5$ ).

For the computation of $A_{ik}^{(m)}$ there was used the double precision in the subprogram for the computation of the values of a polynomial in interpolating points. For m = 4,6, we had also the exact values of $A_{ik}^{(m)}$ , and the error of computation was on the last eigth place. The systems were solved by the modified Newton method with the initial approximations $(\bar{v}^{(m)})_0 = (u_0(x_0^{(m)}), \dots, u_0(x_m^{(m)}))$ . The vectors of solutions $(\bar{v}^{(m)})_n$ , for all m ,did not change starting from the second iteration. The residuum-vectors were of order $10^{-11}$ maximally (the order of solutions is $10^{-2}$ ) - see Table 1.

The proof of existence and the error estimate were done using the Theorem of Kantorovich [5] and the formula for the error estimate of the modified Newton method (see [7], p. 633). In all the three cases, the theoretical error was smaller than $10^{-11}$ . This theoretical estimate does not entirely describe the reality because of the rounding-off errors (see Table 1 - a slight disturbance of the symmetry).

If we denote by $\bar{v}^{(m)}$ the solutions obtained by the procedure just described, we get the polynomials approximating the solution of (1) in the form

$$v^{(m)}(x) = \sum_{k=0}^{m} l_k^{(m)}(x)\, \bar{v}_k^{(m)}$$

The values of these polynomials were tabulated in equidistant points with the step 0,1 - see Table 2.

The values $\max_{0 \le x \le 1} | v^{(m)}(x) - u^*(x) |$ can be estimated in the following manner(similarly as it is made in the paper of Mysovskich [13] for the solution obtained by the method of mechanic quadrature): the function $v^{(m)}(x)$ may be taken as the initial approximation of $u^*(x)$ in the Theorem of Kantorovich [5], and we have:

If

1) $\| \bar{\Gamma}_m \| \le c_m$ ,

where $\bar{\Gamma}_m = (I - A_m)^{-1}$, $A_m h = \int_0^1 \mathcal{K}(x,t)\, e^{v^{(m)}(t)}\, h(t)\, dt$ ,

2) $\max_{0 \le x \le 1} \int_0^1 \mathcal{K}(x,t)\, e^{u(t)} dt \le K_m$ for $\max_{0 \le x \le 1} | u(x) - v^{(m)}(x) | \le \delta_m$ ,

3) $\max_{0\le x\le 1} |v^{(m)}(x) - \int_0^1 \mathcal{K}(x,t) e^{v^{(m)}(t)} dt - \frac{1}{2}(x-x^2)| \le \eta_m$ ,

4) $\sigma_m \ge \sigma_{0m} \overset{df}{=} \frac{1-\sqrt{1-2h_m}}{h_m} c_m \eta_m$ ,

$h_m \overset{df}{=} c_m^2 K_m \eta_m \le \frac{1}{2}$ ,

then

$$\max_{0\le x\le 1} |u^*(x) - v^{(m)}(x)| \le \sigma_{0m} .$$

This estimate has been done for $m = 4$ (analogously it could be done for other $m$'s):

1) There is $\|A_4\| \le 0{,}125 . \max_{0\le x\le 1} e^{v^{(4)}(x)} < 0{,}153463 < 1$ ,

so

$\|\Gamma_4\| \le 1{,}1813$ .

2) Let us choose $\sigma_4 = 0{,}00055$ . There is

$\max_{0\le x\le 1} \int_0^1 \mathcal{K}(x,t)dt \max_{\Omega_4} e^{u(x)} < 0{,}1534 = K_4$ ,

$\Omega_4 = \{u \mid \max |u(x) - v^{(4)}(x)| \le \sigma_4\}$ .

3) The verification of the condition 3) is somewhat complicated as the integral

$$\mathcal{J}_4(x) = \int_0^1 \mathcal{K}(x,t) e^{v^{(4)}(t)} dt$$

cannot be written as a combination of elementary functions.

We shall estimate it in the following manner:

There is $0 \le v^{(4)}(x) \le 0{,}3$ , $x \in \langle 0,1 \rangle$ .

On the interval $\langle 0,1 \rangle$ let us approximate the function $f(x) = e^{0{,}3x}$ by the polynomial

$P_4(x) = \sum_{k=0}^{4} l_k^{(4)}(x) f(x_k^{(4)})$ .

Now, there is

$$e^{v^{(4)}(x)} = P_4\left(\frac{10}{3} v^{(4)}(x)\right) + Q(x) = L_4(x) + Q(x).$$

Let us write

$$\tilde{\mathcal{T}}_4(x) = \int_0^1 \mathcal{K}(x,t) L_4(t)\,dt .$$

There is $\max\limits_{0 \le x \le 1} |Q(x)| \le \frac{1}{2^3 5!}\left(\frac{3}{10}\right)^5 \cdot 1{,}35 < 0{,}00000342$,

and so

(x) $$\max_{0 \le x \le 1} |\mathcal{T}_4(x) - \tilde{\mathcal{T}}_4(x)| < 0{,}00000043 .$$

With regard to (x), there has been found

$$\eta_4 = 0{,}000037 .$$

4) There is $h_4 = 0{,}0000007928$ ,

$$\delta_{04} = 0{,}0000437 < \delta_4 ,$$

so that

$$\max_{0 \le x \le 1} |u^*(x) - v^{(4)}(x)| \le 0{,}0000437 .$$

<u>Example 2</u>. In the same way as in Example 1, there has been solved the system of two nonlinear equations obtained when solving the problem of nonlinear bending of a thin circular plate clamped on the boundary subjected to a uniform lateral pressure.

In this example, all the conditions of the Theorems 1 and 2 are not verified and the error estimate is not given. In fact, the computations for doing this are rather complicated and not well available in the practice. We wanted only to show that the method of collocation can be used also in a non-trivial case using rather few interpolating points.

The system of two differential equations of the second order is in this case given by (see Keller-Reiss [8])

$$\mathcal{L}\varphi(x) = -\varphi(x)\psi(x) - Px^2 ,$$

$$\mathcal{L}\psi(x) = \frac{1}{2}\varphi^2(x) , \qquad 0 < x < 1 ,$$

$$\mathcal{L} \overset{df}{=} x\frac{d}{dx}\frac{1}{x}\frac{d}{dx}\, x ,$$

$$\varphi(0) = \psi(0) = 0, \ \varphi(1) = 0, \ \psi'(1) = \nu\cdot\psi(1) ,$$

where P is a constant proportional to the pressure, $\nu$ is a constant depending on the material.

Introducing new unknown functions $u = \frac{8}{P}\varphi$, $v = \frac{8}{P}\psi$, the absolute term will be the same for all P , and we get the system

$$\mathcal{L}u(x) = -\frac{P}{8}u(x)v(x) - 8x^2 ,$$

$$\mathcal{L}v(x) = \frac{P}{16}u^2(x), \qquad 0 < x < 1 ,$$

$$u(0) = v(0) = 0, \ u(1) = 0, \ v'(1) = \nu v(1) .$$

Using Green´s functions, we get the equivalent system of two integral equations

$$(1') \begin{cases} \Psi_1[u,v] \equiv u(x) + \frac{P}{8}\int_0^1 G_1(x,t)u(t)v(t)dt - f(x) = 0, \\ \Psi_2[u,v] \equiv v(x) - \frac{P}{16}\int_0^1 G_2(x,t)u^2(t)dt = 0 , \end{cases}$$

where $f(x) = x - x^3$ ,

$$G_1(x,t) = \frac{1}{2}(x - \frac{1}{x})t \quad \text{for } t \le x ,$$

$$= \frac{1}{2}(t - \frac{1}{t})x \quad \text{for } x \le t,$$

$$\mathcal{G}_2(x,t) = -\frac{1}{2}(kx+\frac{1}{x})t \quad \text{for } t \leq x ,$$
$$= -\frac{1}{2}(kt+\frac{1}{t})x \quad \text{for } x \leq t ,$$

$k = \dfrac{1+\nu}{1-\nu}$ .

This problem has been solved for $\nu = 0,3$ and for the following values of P : P = 50, P = 300, P = 1400 , P = 3000, P = 5000 .

The choice of the spaces and operators is as follows:

X is the space of couples of continuous functions on the interval $\langle 0,1 \rangle$, $w = [u ; v]$ , with the norm

$$\| w \|_X = max \{ \| u \|_{C\langle 0,1 \rangle} , \| v \|_{C\langle 0,1 \rangle} \} .$$

$\tilde{X}_{m+1}$ is the space of couples of polynomials of order m ,

$\bar{X}_{m+1}$ the space of couples of $(m+1)$-dimensional vectors $\bar{w}^{(m)} = [\bar{u}^{(m)}; \bar{v}^{(m)}]$ with the norm

$$\| \bar{w}^{(m)} \|_{\bar{X}} = max \{ \bar{u}_0^{(m)}, \ldots, \bar{u}_m^{(m)} ; \bar{v}_0^{(m)}, \ldots, \bar{v}_m^{(m)} \} .$$

The operator $\Psi$ mapping X into X is given by

$$\Psi [u ; v] = [\Psi_1 [u ; v] ; \Psi_2 [u ; v]] ,$$

the operators F, K by

$$F[u ; v] = [-uv ; u^2] ,$$

$$K[y ; z] = [-\frac{P}{8}\int_0^1 \mathcal{G}_1(x,t)y(t)dt ; \frac{P}{16}\int_0^1 \mathcal{G}_2(x,t)z(t)dt] .$$

Analogously, the operators $P_m^{(2)}, \varphi^{(2)}, \psi^{(2)}$ are

$$P_m^{(2)}[u ; v] = [P_m u ; P_m v] ,$$

$$\varphi^{(2)}[u ; v] = [\varphi u ; \varphi v] ,$$

$$\psi^{(2)}[u;v] = [\psi u;\psi v] .$$

The solution has been obtained by the method of collocation with the Lanczos polynomial $\mathcal{L}_m(x)$ for m = 6 , i.e., for 7 interpolating points $x_i = x_i^{(6)}$ on the interval $\langle 0,1\rangle$

The system (11) is now a system of 2(m + 1) ( = 14 ) algebraic equations for 2(m + 1) unknowns.

$$(11') \quad \begin{cases} \bar{u}_i + \frac{P}{8}\sum_{k=0}^{6} A_{ik}\,\bar{u}_k\,\bar{v}_k - \bar{f}_i = 0 , \\ \bar{v}_i - \frac{P}{16}\sum_{k=0}^{6} B_{ik}\,\bar{u}_k^2 = 0 , \end{cases}$$

where

$$A_{ik} = A_k(x_i),\quad A_k(x) = \int_0^1 G_1(x,t)\,l_k^{(6)}(t)\,dt ,$$

$$B_{ik} = B_k(x_i),\quad B_k(x) = \int_0^1 G_2(x,t)\,l_k^{(6)}(t)\,dt .$$

The coefficients $A_{ik}$ , $B_{ik}$ were calculated exactly as well as by means of the computer Ural II (again using the double precision as in Example 1). The obtained results were exact on 6 - 7 places in the case of $A_{ik}$ , and on 5 places in the case of $B_{ik}$ , this being not as good as in the case of the simple Green´s function in Example 1.

The initial approximation has been obtained in the following way: we look for a function $u_0^{(P)}(x)$ of the form $\lambda_P \cdot f(x)$, $\lambda_P = const$ (depending on P ). If we denote by $\kappa_0^{(P)}$ the residuum of (1´) for $u = u_0$ ,

$$\kappa_0^{(P)}(x) = -\frac{P}{8}\int_0^1 G_1(x,t)\,u_0(t)\,v_0(t)\,dt + (x-x^3)(1-\lambda_P) =$$

$$= -\frac{1}{2}\frac{P^2}{64}\int_0^1\int_0^1 \mathcal{G}_1(x,s)\,\mathcal{G}_2(s,t)\,u_0(s)\,u_0^2(t)\,dt\,ds +$$

$$+ (1-\lambda_p)(x-x^3) \; ,$$

and we determine $\lambda_p$ from the condition

$$\int_0^1 K_0^{(P)}(x)\,dx = 0 \; .$$

This leads to the equation of order three for $\lambda_p$ ,

$$\lambda_p^3 + a_p\,\lambda_p - a_p = 0, \quad a_p = \frac{A}{P^2} \; , \quad A = 20493,66 \; ,$$

with the unique positive solution

$$\lambda_p = \sqrt{\frac{4a_p}{3}}\; sh\, \vartheta_p \; , \quad sh\, 3\vartheta_p = \frac{3}{\sqrt{\frac{4a_p}{3}}} \; .$$

The functions $v_0^{(P)}(x)$ can then be obtained by solving the second equation in (1´) for $u = u_0^{(P)}$ .

In this way, the following approximated values for $\lambda_p$ have been obtained:

$$\lambda_{50} = 0,9085, \quad \lambda_{300} = 0,48839, \quad \lambda_{1400} = 0,2028 \; ,$$

$$\lambda_{3000} = 0,1258 \; , \quad \lambda_{5000} = 0,09067 \; .$$

The systems (11´) were again solved by the Newton modified method with the initial approximations

$$\bar{u}_{0_i}^{(P)} = u_0^{(P)}(x_i), \quad v_{0_i}^{(P)} = v_0^{(P)}(x_i), \quad i = 0, 1, \dots, 6 \; .$$

The vectors of solution $\bar{u}$ , $\bar{v}$ do not change for P = 50, starting from the 3[rd] , for P = 300 from the 7[th] , for P = 1400 from the 10[th] ; for P = 3000 and P = 5000 from the 11[th] iteration. For P = 50 and P = 300 , the initial approximation was so good that there could be done the proof of convergence following Theorem of Kantorovich.

For the other $P's$, in spite of the fact that the vectors of solution did not change, the sufficient conditions of Theorem of Kantorovich were not fulfilled. Therefore, for these $P's$ (1400, 3000, 5000 ), the iteration-cycle was repeated with the initial approximations equal to the vectors of solution from the first cycle.

The vectors of solution did not change any more, and now the proof of convergence could be done.

The residuum-vectors of (11′), for all the values of P, were, for the initial approximations $\bar{u}_0^{(P)}$, $\bar{v}_0^{(P)}$, of order $10^{-2}$ - $10^{-3}$ , and realizing the iterations they decreased to the maximal values of order $10^{-11}$ . The order of solution is $10^{-1}$ - $10^{-3}$ . The error estimate shows again that the systems (11′) were solved exactly within the limits of the given computer (but without having regard to the rounding-off errors as in Example 1).

The functions $\tilde{u}(x)$, $\tilde{v}(x)$ giving the approximate solutions of (1′), are given by

$$\tilde{u}(x) = \sum_{k=0}^{6} l_k^{(6)}(x)\, \bar{u}_k \quad ,$$

$$\tilde{v}(x) = \sum_{k=0}^{6} l_k^{(6)}(x)\, \bar{v}_k \quad .$$

The functions $\tilde{\varphi}(x) = \frac{P}{8}\tilde{u}(x)$ , $\tilde{\psi}(x) = \frac{P}{8}\tilde{v}(x)$ were calculated in eleven equidistant points - see Table 3, graph 1, 2 .

Remark 1. The same problem has been solved by Keller and Reiss in [8] by another method. In their paper, there are no numerical results but the graph of the function $\varphi$ is the same as that presented in this paper.

Remark 2. In 1966, there appeared the paper [16] devoted to the method of collocation for nonlinear differential boundary value problems. The results of Vajnikko are based on other assumptions and techniques than are those used in this paper, and there is no regard to the approximations of the type (11). Such types of approximation are studied in a paper of Kolodner [9] . The considerations of this paper are based on other principles.

TABLE 1.

(Example 1.)

The values: $(\bar{v}^{(4)})_n = (\bar{v}_{0n}^{(4)}, \bar{v}_{1n}^{(4)}, \ldots, \bar{v}_{4n}^{(4)})$, $(\bar{\varkappa}^{(4)})_n = (\bar{\varkappa}_{0n}^{(4)}, \bar{\varkappa}_{1n}^{(4)}, \ldots, \bar{\varkappa}_{4n}^{(4)})$.

| | $\bar{v}_{in}^{(4)}$ | $\bar{\varkappa}_{in}^{(4)}$ |
|---|---|---|
| n=0, i = 0 | 0 | 0 |
| 1 | 0,13977676 | $-0,10613883 \; 10^{-3}$ |
| 2 | 0,28302447 | $-0,37531793 \; 10^{-3}$ |
| 3 | 0,13979381 | $-0,89532841 \; 10^{-4}$ |
| 4 | 0 | 0 |
| n=1, i = 0 | 0 | 0 |
| 1 | 0,13990479 | $-0,39726729 \; 10^{-8}$ |
| 2 | 0,28345147 | $-0,10069926 \; 10^{-9}$ |
| 3 | 0,13990482 | $-0,38999132 \; 10^{-8}$ |
| 4 | 0 | 0 |
| n=2, i = 0 | 0 | 0 |
| 1 | 0,13990480 | $-0,14551915 \; 10^{-10}$ |
| 2 | 0,28345149 | 0 |
| 3 | 0,13990482 | 0 |
| 4 | 0 | 0 |

TABLE 1. (part 2.)
(Example 1.)

The values: $(\bar{v}^{(6)})_n = (\bar{v}^{(6)}_{0n}, \bar{v}^{(6)}_{1n}, \dots, \bar{v}^{(6)}_{6n})$, $(\bar{\kappa}^{(6)})_n = (\bar{\kappa}^{(6)}_{0n}, \bar{\kappa}^{(6)}_{1n}, \dots, \bar{\kappa}^{(6)}_{6n})$ .

| | $\bar{v}^{(6)}_{in}$ | $\bar{\kappa}^{(6)}_{in}$ | |
|---|---|---|---|
| n = 0, i = 0 | 0 | 0 | |
| 1 | 0,069446171 | -0,59570186 | $10^{-4}$ |
| 2 | 0,21096851 | -0,18536916 | $10^{-3}$ |
| 3 | 0,28302447 | -0,35908231 | $10^{-3}$ |
| 4 | 0,21098071 | -0,17261709 | $10^{-3}$ |
| 5 | 0,069466617 | -0,39373932 | $10^{-4}$ |
| 6 | 0 | 0 | |
| n = 1, i = 0 | 0 | 0 | |
| 1 | 0,069515036 | -0,14115357 | $10^{-8}$ |
| 2 | 0,21118615 | -0,51513779 | $10^{-8}$ |
| 3 | 0,28343070 | -0,79744496 | $10^{-8}$ |
| 4 | 0,21118511 | -0,50640665 | $10^{-8}$ |
| 5 | 0,069515027 | -0,13533281 | $10^{-8}$ |
| 6 | 0 | 0 | |
| n = 2, i = 0 | 0 | 0 | |
| 1 | 0,069515037 | -0,29103831 | $10^{-10}$ |
| 2 | 0,21118616 | -0,29103831 | $10^{-10}$ |
| 3 | 0,28343071 | -0,58207661 | $10^{-10}$ |
| 4 | 0,21118512 | +0,58207661 | $10^{-10}$ |
| 5 | 0,069515029 | -0,14551915 | $10^{-10}$ |
| 6 | 0 | 0 | |

**TABLE 1.** (part 3.)

(Example 1.)

The values: $(\bar{v}^{(8)})_n = (\bar{v}^{(8)}_{0n}, \bar{v}^{(8)}_{1n}, \ldots, \bar{v}^{(8)}_{8n})$, $(\bar{\pi}^{(8)})_n = (\bar{\pi}^{(8)}_{0n}, \bar{\pi}^{(8)}_{1n}, \ldots, \bar{\pi}^{(8)}_{8n})$.

| | $\bar{v}^{(8)}_{in}$ | $\bar{\pi}^{(8)}_{in}$ |
|---|---|---|
| n = 0, i = 0 | 0 | 0 |
| 1 | 0,040566447 | $-0,46632616 \cdot 10^{-4}$ |
| 2 | 0,13977676 | $-0,98941498 \cdot 10^{-4}$ |
| 3 | 0,24070954 | $-0,24377918 \cdot 10^{-3}$ |
| 4 | 0,28302447 | $-0,35898626 \cdot 10^{-3}$ |
| 5 | 0,24071893 | $-0,23476858 \cdot 10^{-3}$ |
| 6 | 0,13979381 | $-0,82279556 \cdot 10^{-4}$ |
| 7 | 0,40588348 | $-0,24870096 \cdot 10^{-4}$ |
| 8 | 0 | 0 |
| n = 1, i = 0 | 0 | 0 |
| 1 | 0,040618381 | $-0,80763129 \cdot 10^{-9}$ |
| 2 | 0,13989554 | $-0,30267984 \cdot 10^{-8}$ |
| 3 | 0,24099134 | $-0,61409082 \cdot 10^{-8}$ |
| 4 | 0,28343051 | $-0,78580342 \cdot 10^{-8}$ |
| 5 | 0,24099132 | $-0,60827006 \cdot 10^{-8}$ |
| 6 | 0,13989551 | $-0,29976945 \cdot 10^{-8}$ |
| 7 | 0,040618366 | $-0,77125151 \cdot 10^{-9}$ |
| 8 | 0 | 0 |
| n = 2, i = 0 | 0 | 0 |
| 1 | 0,040618382 | $-0,72759576 \cdot 10^{-11}$ |
| 2 | 0,13989554 | 0 |
| 3 | 0,24099134 | $-0,29103831 \cdot 10^{-10}$ |
| 4 | 0,28343051 | $-0,58207661 \cdot 10^{-10}$ |
| 5 | 0,24099132 | $-0,29103831 \cdot 10^{-10}$ |
| 6 | 0,13989551 | 0 |
| 7 | 0,040618366 | $-0,72759576 \cdot 10^{-11}$ |
| 8 | 0 | 0 |

## TABLE 2.

(Example 1.)

The values of the polynomials $v^{(m)}(x)$ in equidistant points $x_i = 0,1\,i$

| | $v^{(4)}(x_i)$ | $v^{(6)}(x_i)$ | $v^{(8)}(x_i)$ |
|---|---|---|---|
| i = 0 | 0 | 0 | 0 |
| 1 | 0,10036435 | 0,10037304 | 0,10037333 |
| 2 | 0,17973078 | 0,17970254 | 0,17970252 |
| 3 | 0,23712032 | 0,23708111 | 0,23707951 |
| 4 | 0,27183372 | 0,27180541 | 0,27180089 |
| 5 | 0,28345147 | 0,28343136 | 0,28343062 |
| 6 | 0,27183370 | 0,27180715 | 0,27178565 |
| 7 | 0,23712037 | 0,23708441 | 0,23704911 |
| 8 | 0,17973073 | 0,17970729 | 0,17965961 |
| 9 | 0,10036431 | 0,10037954 | 0,10032808 |
| 10 | $-0,45169145\ 10^{-7}$ | 0 | 0 |

## TABLE 3.

(Example 2.)

The values of the polynomials $\tilde{\varphi}^{(P)}(x)$, $\psi^{(P)}(x)$ in equidistant points $x_i = 0,1\, i$

| | $\tilde{\varphi}^{(50)}(x_i)$ | $\tilde{\varphi}^{(300)}(x_i)$ | $\tilde{\varphi}^{(1400)}(x_i)$ |
|---|---|---|---|
| i = 0 | 0 | 0 | 0 |
| 1 | 0,54582458 | 1,42030834 | 2,07944345 |
| 2 | 1,06195006 | 2,81844292 | 4,19079255 |
| 3 | 1,51784262 | 4,16235038 | 6,38915918 |
| 4 | 1,88103525 | 5,39500725 | 8,64690470 |
| 5 | 2,11675850 | 6,41662912 | 10,89814198 |
| 6 | 2,18776806 | 7,06862888 | 12,97523570 |
| 7 | 2,05436962 | 7,11933975 | 14,43730208 |
| 8 | 1,67464050 | 6,25149188 | 14,29070895 |
| 9 | 1,00484850 | 4,05144562 | 10,60157315 |
| 10 | 0 | 0 | 0 |
| | $\tilde{\psi}^{(50)}(x_i)$ | $\tilde{\psi}^{(300)}(x_i)$ | $\tilde{\psi}^{(1400)}(x_i)$ |
| i = 0 | 0 | 0 | 0 |
| 1 | -0,19363374 | -1,93102459 | -6,98411525 |
| 2 | -0,37623782 | -3,78312450 | -13,75727518 |
| 3 | -0,53824804 | -5,49201712 | -20,21555025 |
| 4 | -0,67226338 | -6,98688488 | -26,19116325 |
| 5 | -0,77464256 | -8,21133900 | -31,47261950 |
| 6 | -0,84625012 | -9,13873538 | -35,84621950 |
| 7 | -0,89235338 | -9,78185588 | -39,15897125 |
| 8 | -0,92166812 | -10,19695012 | -41,40287550 |
| 9 | -0,94455544 | -10,48213200 | -42,82061350 |
| 10 | -0,97036781 | -10,77014325 | -44,03261625 |

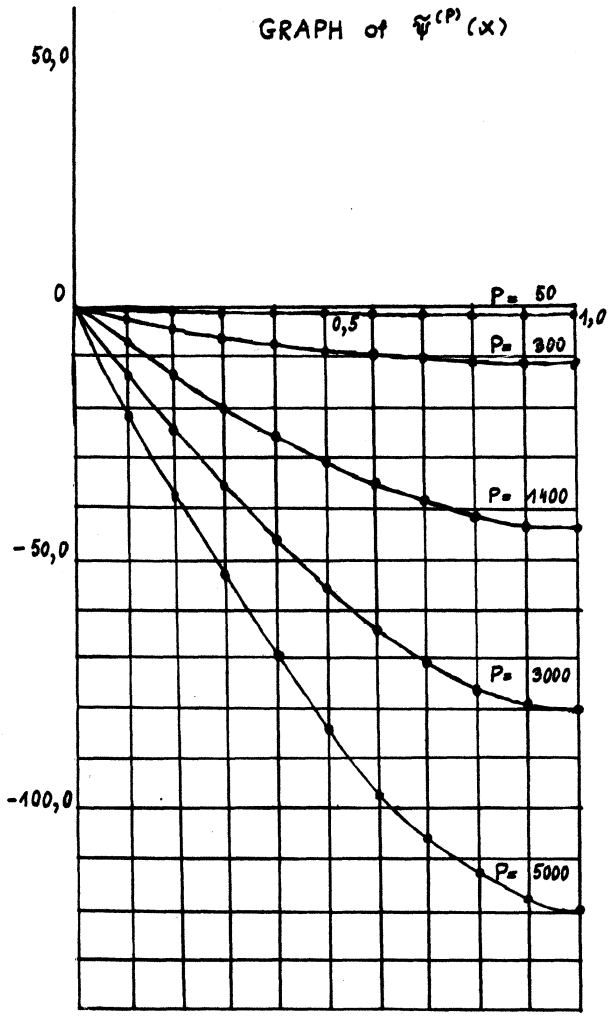
TABLE 3.(part 2.)

(Example 2.)

The values of the polynomials $\tilde{\varphi}^{(P)}(x)$, $\tilde{\psi}^{(P)}(x)$ in equidistant points $x_i = 0{,}1\, i$

| | $\tilde{\varphi}^{(3000)}(x_i)$ | $\tilde{\varphi}^{(5000)}(x_i)$ |
|---|---|---|
| i = 0 | 0 | 0 |
| 1 | 2,51775120 | 2,94737262 |
| 2 | 5,01375412 | 5,68467869 |
| 3 | 7,74926325 | 8,88324062 |
| 4 | 10,59133650 | 12,21431000 |
| 5 | 13,45863375 | 15,46958312 |
| 6 | 16,36193025 | 18,85822312 |
| 7 | 19,03935825 | 22,48038062 |
| 8 | 20,18636475 | 24,97721688 |
| 9 | 16,28039025 | 21,35742875 |
| 10 | 0 | 0 |

| | $\tilde{\psi}^{(3000)}(x_i)$ | $\tilde{\psi}^{(5000)}(x_i)$ |
|---|---|---|
| i = 0 | 0 | 0 |
| 1 | -12,27117150 | -17,70502096 |
| 2 | -24,20771850 | -34,96473562 |
| 3 | -35,67802538 | -51,60107100 |
| 4 | -46,47269625 | -67,37054812 |
| 5 | -56,26567875 | -81,85218125 |
| 6 | -64,64851500 | -94,46109869 |
| 7 | -71,23773750 | -104,58786144 |
| 8 | -75,85545000 | -111,86352800 |
| 9 | -78,78298500 | -116,55041412 |
| 10 | -81,08775750 | -120,05858125 |


GRAPH of $\tilde{\Psi}^{(P)}(x)$
50,0
0
-50,0
-100,0
0,5
1,0
P= 50
P= 300
P= 1400
P= 3000
P= 5000

R e f e r e n c e s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