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GENERAL THEORY OF ∇-MODELS

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Syntactic models of the set theory in the set theory constructed in [1]-[6] (so called ∇-models) depend on several parameters. By a specification of the parameters, models in which the continuum hypothesis (or other statements) does not hold may be constructed. It was proved in [7]-[8] that the number of parameters may be limited to two, namely to a complete Boolean algebra and an ultrafilter on it.

With respect to this fact it seems to be reasonable to present the whole theory of ∇-models once more in a quite simpler form. This new form enables to find a general method of finding a set-theoretical formula $\bar{\varphi}(B)$ with one free variable to every closed normal set theoretical formula φ in such a way that the following holds true: If the statement "There is a complete Boolean algebra B such that $\bar{\varphi}(B)$ " is provable in the set theory (or, even, is only consistent with it) then φ is consistent with the set theory (including the axiom of choice).

At the end of the present paper, we prove that all the consistency results (of some very general kind) which can be obtained by the method of ∇-models can be obtained also by the method of standard models of the set theory in some extension of the set theory whose consistency relative to the set theory is provable by means of some particular ∇-models.

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1. Classic parametric model

All the definitions are done in (or with respect to) the set theory Σ^* with the axiom of choice.

Metadefinition 1. Let φ be an elementary formula. The formula $\varphi^{P,R}$ (called the translation by the variables P,R) is defined in the following recursive way:

1) if φ is atomic, say, $x \in y$, then $\varphi^{P,R}$ is $\langle x, y \rangle \in R$,

2) if φ is φ_1 & φ_2, $\neg \varphi_1$ respectively, then $\varphi^{P,R}$ is $\varphi_1^{P,R}$ & $\varphi_2^{P,R}$, $\neg \varphi_1^{P,R}$ respectively,

3) if φ is $(\exists x)\psi$, then $\varphi^{P,R}$ is $(\exists x)(x \in P \,\&\, \psi^{P,R})$.

If ψ is a formula equivalent to an elementary formula φ in the set theory then $\psi^{P,R}$ is defined as $\varphi^{P,R}$.
We shall write φ^P instead of $\varphi^{P,E}$.

Remark. The formula $x = y$ is understood as an abbreviation of $\neg(\exists z)(z \in y \,\&\, \neg(z \in x)) \,\&\, \neg(\exists z)(z \in x \,\&\, \neg(z \in y))$.

Definition 1. Let P be a class, R a relation. For $x \in P$, we define $\Phi_{PR}(x) = \{y; y \in P \,\&\, \langle y, x \rangle \in R\}$. For $X \subseteq P$, we define $\Psi_{PR}(X) = \{y; y \in P \,\&\, (\exists z)(z \in X \,\&\, (z = y)^{P,R})\}$.

R is said to be a model-relation on P (shortly, $Mr(P, R)$) iff

1) $(\forall x \in P)(\exists y \subseteq P)(\Phi_{P,R}(x) = \Psi_{P,R}(y))$

2) $(\forall x \subseteq P)(\exists y \in P)(x \subseteq \Phi_{P,R}(y))$

3) $(\forall x, y \in P)(\exists z \in P)((z = \mathcal{F}_i(x,y))^{P,R})$ for $i = 1, \dots, 8$

($\mathcal{F}_i$ are the Gödel's operations)

4) $(\forall x \in P)((x \neq 0 \rightarrow (\exists y)(y \in x \,\&\, x \cap y = 0))^{P,R})$

5) $((\exists x)(x = \omega_0))^{P,R}$

6) $((\forall x, y, z)(x \in z \,\&\, x = y . \rightarrow y \in z))^{P,R}$

<u>Definition 2</u>. Let $x^*, y^*, \ldots$ be variables denoting elements of P. Define

$$Cls^*(X) \equiv .. X \in P . \vee . X \subseteq P \,\&\, X = \Psi_{P,R}(X) \,\&\, \neg(\exists x^*)(X = \Phi_{P,R}(x^*)) \,\&$$
$$\& (\forall x^*)(\exists y^*)(\Phi_{P,R}(y^*) = X \cap \Phi_{P,R}(x^*))$$

$$X \in^* Y \equiv .. X \in P \,\&\, Y \in P \,\&\, \langle X, Y \rangle \in R . \vee . X \in P \,\&\, Y \notin P \,\&$$
$$\& \, Cls^*(Y) \,\&\, X \in Y .$$

<u>Metatheorem 1</u>. The predicates Cls^*, $\in^*$ form a parametric model of the set theory with the range of parameters $Mr\,(P, R)$.

<u>Proof</u>. See [9],[10]. (We have only to suppose that a secondary condition $(\forall x \subseteq P)(\neg(\exists y \in P)(x = \Phi(y)) \rightarrow x \notin P)$ is added to the formula $Mr\,(P, R)$.)

The parametric model in Def. 3 with the range of parameters $Mr\,(P, R)$ will be called the classic parametric model.

<u>Metadefinition 2</u>. Absolute formulas are defined inductively:

1) Atomic formulas of the form $x \in y$ are absolute,
2) if φ_1, φ_2 are absolute then $\varphi_1 \,\&\, \varphi_2$, $\neg \varphi_1$ are,
3) if $\varphi(x, y, \ldots)$ is absolute then $(\exists x)(x \in y \,\&$ $\& \,\varphi(x, y, \ldots))$ is.

<u>Lemma 1</u>. The following formulas are equivalent, in Σ, <u>to some</u> absolute formulas x): $x = y$, $x \in \mathcal{F}_i(y, z)\,(i = 1, \ldots, 8)$, $Ord(x)$ etc.

x) In fact, to prove equivalences of these formulas to some absolute formulas only axioms of the groups A,B are needed.

Definition 3. $Comp(P) \equiv (\forall x)(x \in P \to x \subseteq P)$.

Metatheorem 2. Let $\varphi(x, \dots, y)$ be an absolute formula. Then the following is provable in the set theory:
$Comp(P) \to (\forall x \dots y \in P)(\varphi^P(x, \dots, y) \equiv \varphi(x, \dots, y))$.

The inductive proof is easy.

Definition 4. A class P is said to be a model-class (shortly, $Mcl(P)$) iff
1) $Comp(P)$, 2) $(\forall x \in P)(\exists y \in P)(x \subseteq y)$, 3) $(\forall x, y \in P)(\mathcal{F}_i(x, y) \in P)$ $(i = 1, \dots, 8)$.

Theorem 1. $Mcl(P) \to Mr(P, E)$.

Lemma 2. $Mcl(L) \& (\forall X)(Mcl(X) \to L \subseteq X)$ (where L is the class of all constructible sets).

The parametric model which is a specification of the classic parametric model by the conditions $Mcl(P)$, $R = E$, is called the complete parametric model. If P_0 is a constant such that $Mcl(P_0)$ is provable then the specification of the complete parametric model by $P = P_0$ is called the complete model determined by P_0 and it is denoted by $\Delta(P_0)$. The model $\Delta(L)$ is denoted shortly by Δ .

Lemma 3. Every cardinal number is a cardinal number in the sense of the parametric complete model.

Definition 5. The sets r_α are defined inductively for $\alpha \in On$. $r_0 = \{0\}$, $r_\alpha = \mathcal{P}(\bigcup_{\beta<\alpha} r_\beta)$. For $x \in \bigcup_{\alpha \in On} r_\alpha$, $\tau(x)$ is the least $\alpha \in On$ such that $x \in r_\alpha$. The number $\tau(x)$ is called the type of x .

Lemma 4. The axiom D is equivalent to $V = \bigcup_{\alpha \in On} r_\alpha$.

Lemma 5. Let $Mcl(P)$. Denote the r'_α s in the sense of $\Delta(P)$ by r_α^P . Then $r_\alpha^P = P \cap r_\alpha$.

2. A sheaf over a complete Boolean algebra.

Definition 1. Let B be a complete Boolean algebra with the operations $\vee, \wedge, \bigvee, \bigwedge,$ - and constants 0,1, let $<$ denote the ordering of this algebra. Let P be a class, F a function associating with every ordered pair $\langle x, y \rangle \in P^2$ an element $F(x, y) \in B$. Then F is called a sheaf of binary relations (shortly, a sheaf) over B on P .

Metadefinition 1. Let $\varphi(x_1, \dots, x_n)$ be an elementary formula. We associate with φ a logical operation in n + 2 variables $B, F, x_1^1, \dots, x_n^1$ denoted $F_B \ulcorner \varphi(x_1 \dots x_n) \urcorner$ (or, shortly, $F\ulcorner \varphi(x_1 \dots x_n)\urcorner$) by the following recursion:
$F\ulcorner x \in y \urcorner = F(x, y)$, $F\ulcorner \varphi \,\&\, \psi\urcorner = F\ulcorner\varphi\urcorner \wedge F\ulcorner\psi\urcorner$, $F\ulcorner \neg\varphi\urcorner =$
$= - F\ulcorner\varphi\urcorner$, $F\ulcorner(\exists x)\varphi(x, y \dots)\urcorner = \bigvee\{F\ulcorner\varphi(x, y\dots)\urcorner, x \in P\}$.

Lemma 1. If φ is a formula provable in the predicate calculus then the following is provable in the set theory: For every sheaf over a complete Boolean algebra B, $F\ulcorner\varphi\urcorner = 1$.

Lemma 2. If $\varphi \equiv \psi$ is a formula provable in the predicate calculus then the following is provable in the set theory: For every sheaf over a complete Boolean algebra B, $F\ulcorner\varphi\urcorner = F\ulcorner\psi\urcorner$.

Metadefinition 2. Let $\psi(x \dots y)$ be a normal formula equivalent to an elementary formula $\varphi(x \dots y)$ in the set theory. Then we define $F\ulcorner\psi(x \dots y)\urcorner = F\ulcorner\varphi(x \dots y)\urcorner$.

Definition 2. Let z be an ultrafilter on B . The relation $R_z^P = Lim_z^P\, F$ is defined on P by the equivalence $\langle x, y\rangle \in R_z^P \equiv F(x, y) \in z$. If $\varphi(x \dots y)$ is a normal formula then $\varphi^*(x \dots y)$ denotes the translation $\varphi^{P,R^P_x}(x \dots y)$.

Definition 3. A sheaf F is called internal iff $F\ulcorner(\forall x,y,z)(x \in z \,\&\, x = y . \rightarrow y \in z)\urcorner = 1$. In the sequel, all sheaves are supposed to be internal.

Definition 4. A sheaf over B on P is called complete iff the following holds true: Let $\{u_x, x \in b\} \subseteq B$, $x \neq y \rightarrow u_x \wedge u_y = 0$, $\{a_x, x \in b\} \subseteq P$. Then there is an $a \in P$ such that $u_x \leq F\ulcorner a_x = a\urcorner$ for every $x \in b$ (b being an arbitrary index set).

Metatheorem 1. Let $\varphi(x \dots y)$ be normal. Then the following is provable in the set theory: Let F be a complete (and internal) sheaf over B on P, then $\varphi^*(x \dots y) \equiv$ $\equiv F\ulcorner\varphi(x \dots y)\urcorner \in z$ for every $x, \dots, y \in P$.

Proof. Suppose φ to be elementary. The proof is done by induction; suppose φ to be a formula $(\exists y)\psi(y \dots)$ (other cases are trivial). Let $((\exists y)\psi(y \dots))^*$ hold. Then there is a $y \in P$ such that $\varphi^*(y \dots)$, hence $F\ulcorner\psi(y \dots)\urcorner \in$ $\in z$, but $F\ulcorner\psi(y \dots)\urcorner \leq F\ulcorner\varphi\urcorner$.

On the other hand, let $F\ulcorner\varphi\urcorner \in z$. Put $u = F\ulcorner(\exists y)\psi(y \dots)\urcorner$. For $y \in P$, put $v_y = F\ulcorner\psi(y \dots)\urcorner$. Let, for a moment, $<$ be a well-ordering of P. Put $u_y = v_y - V\{u_x, x < y\}$. Put $b = \{y, u_y \neq 0\}$. Evidently $V\{u_y, y \in b\} = u$. Further, $x \neq y \rightarrow u_x \wedge u_y = 0$. Consequently, there is an $a \in P$ such that $u_x \leq F\ulcorner x = a\urcorner$, and thus $u \leq F\ulcorner\psi(a, \dots)\urcorner$. By the induction hypothesis, $\psi^*(a, \dots)$, which implies φ^*.

3. The sheaf of functions

<u>Definition 1</u>. Let a be a set, B a complete Boolean algebra. A function f is called a (B, a)-function iff $\mathfrak{D}(f) \subseteq a$ and $\mathcal{W}(f) \subseteq B - \{0\}$. The set of all (B, a)-functions is denoted by $C(B, a)$. For $x \notin \mathfrak{D}(f)$, we formally put $f(x) = 0$.

<u>Definition 2</u>. The sets $c_\alpha(B)$ are defined inductively. $c_0(B) = \{0\}$, $c_\alpha(B) = C(B, \bigcup_{\beta<\alpha} c_\beta(B))$. Put $C(B) = \bigcup_{\alpha \in On} c_\alpha(B)$. For $f \in C(B)$, $\rho(f)$ is the least $\alpha \in On$ such that $f \in c_\alpha(B)$.

<u>Metadefinition 1</u>. Let φ be absolute. The operation $F^{<}\ulcorner\varphi\urcorner$ is defined inductively. $F^{<}\ulcorner x \in y\urcorner = F\ulcorner x \in y\urcorner$, $F^{<}\ulcorner\varphi \,\&\, \psi\urcorner = F^{<}\ulcorner\varphi\urcorner \wedge F^{<}\ulcorner\psi\urcorner$, $F^{<}\ulcorner\neg\varphi\urcorner = -F^{<}\ulcorner\varphi\urcorner$, $F^{<}\ulcorner(\exists x \in y)\psi(x, y, \ldots)\urcorner = \bigvee\{F\ulcorner x \in y\urcorner \wedge F^{<}\ulcorner\psi(x, y, \ldots)\urcorner, \rho(x) < \rho(y)\}$.

<u>Definition 3</u>. The sheaf F of functions over B on $C(B)$ is defined inductively with respect to $max(\rho(x), \rho(y))$:

$$F(x, y) = \bigvee\{F^{<}\ulcorner x = z\urcorner \wedge y(z),\ z \in C(B)\}.$$

<u>Remark</u>. If $\alpha = max(\rho(x), \rho(y))$, $F(x, y)$ is defined for $\rho(x) < \rho(y) = \alpha$, then $F^{<}\ulcorner x = y\urcorner$ is defined for $max(\rho(x), \rho(y)) = \alpha$ and, finally, $F(x, y)$ is defined for $\rho(y) \leq \rho(x) = \alpha$.

In the sequel F (or F_B) denotes the sheaf just defined.

<u>Lemma 1</u>. $F^{<}\ulcorner x = y\urcorner \wedge F^{<}\ulcorner y = z\urcorner \leq F^{<}\ulcorner x = z\urcorner$ (i)

$F^{<}\ulcorner x = y\urcorner \wedge F\ulcorner x \in z\urcorner \leq F\ulcorner y \in z\urcorner$ (ii)

$F^{<}\ulcorner x = y\urcorner \wedge F\ulcorner z \in x\urcorner \leq F\ulcorner z \in y\urcorner$ (iii)

<u>Proof</u>. The lemma is proved inductively with respect to the class $\mathcal{D}(\mathcal{D} \subseteq On^3 \,\&\, (\forall\alpha, \beta, \gamma)(\langle\alpha, \beta, \gamma\rangle \in \mathcal{D} \equiv \alpha \leq \beta \leq \gamma))$

ordered lexicographically. Put $ix(\alpha,\beta,\gamma) = \langle\bar{\alpha},\bar{\beta},\bar{\gamma}\rangle$, where $\{\alpha,\beta,\gamma\} = \{\bar{\alpha},\bar{\beta},\bar{\gamma}\}$, $\langle\bar{\alpha},\bar{\beta},\bar{\gamma}\rangle \in D$. Let $\langle\alpha,\beta,\gamma\rangle$ be the least triple such that there are $x, y, z \in C(B)$ for which $ix(\varphi(x),\varphi(y),\varphi(z)) = \langle\alpha,\beta,\gamma\rangle$ and the lemma does not hold. If $\varphi(z) < \varphi(x)$ then (iii) is trivial; let $\varphi(x) \leq \varphi(z)$, let (iii) do not hold. Then there is a z_0, $\varphi(z_0) < \varphi(x)$, $u = F^{<}\ulcorner x = y\urcorner \wedge$ $\wedge F\ulcorner z_0 \in x\urcorner \wedge F\ulcorner z = z_0\urcorner \wedge F\ulcorner z \notin y\urcorner \neq 0$.
But $ix(\varphi(z_0),\varphi(y),\varphi(z)) < ix(\varphi(x),\varphi(y),\varphi(z))$, i.e. $u \leq F\ulcorner z_0 \notin y\urcorner$ by the induction hypothesis. Hence $F^{<}\ulcorner x = y\urcorner \wedge F\ulcorner z_0 \in x\urcorner \wedge F\ulcorner z_0 \notin y\urcorner \neq 0$, which contradicts to $ix(\varphi(x),\varphi(y),\varphi(z_0)) < ix(\varphi(x),\varphi(y),\varphi(z))$.

Let (i) do not hold, let $F^{<}\ulcorner x = y\urcorner \wedge F^{<}\ulcorner y = z\urcorner \wedge F^{<}\ulcorner x \neq z\urcorner \neq 0$. Then there is a q, $\varphi(q) < \varphi(x)$, $u = F^{<}\ulcorner x = y\urcorner \wedge F^{<}\ulcorner y = z\urcorner \wedge F\ulcorner q \in$ $\in x\urcorner \wedge F\ulcorner q \notin z\urcorner \neq 0$. Because of $\varphi(q) < \varphi(x)$ we have $u \leq F\ulcorner q \in$ $\in y\urcorner$, as $ix(\varphi(q),\varphi(y),\varphi(z)) < ix(\varphi(x),\varphi(y),\varphi(z))$ we have $u \leq F\ulcorner q \in z\urcorner$, which is a contradiction.

Finally, we prove (ii). $F^{<}\ulcorner x = y\urcorner \wedge F\ulcorner x \in z\urcorner =$

$$= F^{<}\ulcorner x = y\urcorner \wedge \bigvee\{F^{<}\ulcorner x = q\urcorner \wedge z(q), q \ldots\} =$$

$$= \bigvee\{F^{<}\ulcorner x = y\urcorner \wedge F^{<}\ulcorner x = q\urcorner \wedge z(q), q \ldots\} \leq$$

$$\leq \bigvee\{F^{<}\ulcorner y = q\urcorner \wedge z(q), q \ldots\} = F\ulcorner y \in z\urcorner .$$

<u>**Metatheorem 1.**</u> For every elementary formula φ, $F^{<}\ulcorner x = y\urcorner \wedge F\ulcorner \varphi(x, z, \ldots)\urcorner \leq F\ulcorner \varphi(y, z, \ldots)\urcorner$ is provable in the set theory.

<u>**Proof.**</u> If the formula is atomic, see Lemma 1. Further, the lemma is proved by the customary induction.

<u>**Metatheorem 2.**</u> Let φ be absolute. Then $F^{<}\ulcorner \varphi\urcorner = F\ulcorner \varphi\urcorner$ is provable in the set theory.

Proof. If φ is atomic, see the definition. Only the case $(\exists y \in x)\, \psi(y, x, \dots)$ is nontrivial. By the induction hypothesis, it follows $F^{<}\ulcorner\varphi\urcorner \le F\ulcorner\varphi\urcorner$. Let $u = F\ulcorner\varphi\urcorner - F^{<}\ulcorner\varphi\urcorner \ne 0$. Then there is a $y \in C(B)$ such that $u \wedge F\ulcorner y \in x\urcorner \wedge F\ulcorner\psi(y, x, \dots)\urcorner \ne 0$. Hence there is a $q \in C(B)$ such that $0 \ne u \wedge F^{<}\ulcorner q = y\urcorner \wedge x(q) \wedge$ $\wedge\, F\ulcorner\psi(q, x, \dots)\urcorner \le u \wedge x(q) \wedge F\ulcorner\psi(q, x, \dots)\urcorner$. As $\rho(q) < \rho(x)$, $F\ulcorner\psi(q, x, \dots)\urcorner \le F^{<}\ulcorner\varphi(x\dots)\urcorner$, hence $u \wedge F^{<}\ulcorner\varphi(x\dots)\urcorner \ne 0$, which is a contradiction.

Lemma 2. Let $\{u_i, i \in b\} \subseteq B$, $\{x_i, i \in b\} \subseteq C(B)$. Then there is an $f \in C(B)$ such that $F\ulcorner y \in f\urcorner = \bigvee\{F\ulcorner y = x_i\urcorner \wedge$ $\wedge\, u_i, i \in b\}$ for every $y \in C(B)$ (b is an arbitrary index set).

Proof. f is defined on $\{x_i, i \in b\}$ by $f(x_i) = u_i$.

Theorem 1. The sheaf F is an internal complete sheaf over B on $C(B)$.

Proof. The internality follows by Lemma 1 and Metatheorem 2. Let $\{u_i, i \in b\} \subseteq B$, $\{a_i, i \in b\} \subseteq C(B)$, $i \ne j \rightarrow u_i \wedge u_j = 0$. Put $\alpha = \sup(\rho(a_i), i \in b)$, $v_f = \bigvee\{a_i(f) \wedge u_i, i \in b\}$ for $f \in c_{\alpha+1}(B)$. By Lemma 2, there is an $a \in C(B)$ such that $F\ulcorner y \in a\urcorner = \bigvee\{F\ulcorner y = f\urcorner \wedge v_f, f \in c_{\alpha+1}(B)\}$. It easily follows that $u_i \le F\ulcorner a_i = a\urcorner$.

Theorem 2. For every $y_1, y_2 \in C(B)$ there is a $z \in C(B)$ such that $F\ulcorner z = \mathcal{F}_i(y_1, y_2)\urcorner = 1$ ($\mathcal{F}_i(y_1, y_2)$ being the Gödel's operations, $i = 1, \dots, 8$).

Proof. Put $u_x = F\ulcorner x \in \mathcal{F}_i(y_1, y_2)\urcorner$, $\alpha = \max(\rho(y_1), \rho(y_2))$. By Lemma 2, there is a $z \in c_{\alpha+1}(B)$ such that

$F\ulcorner y \in z\urcorner = \bigvee\{F\ulcorner y = x\urcorner \wedge F\ulcorner x \in \mathcal{F}_i(y_1, y_2)\urcorner, x \in c_\alpha(B)\} =$

$= \bigvee\{F\ulcorner y = x\urcorner \wedge F\ulcorner x \in \mathcal{F}_i(y_1, y_2)\urcorner\} \leq F\ulcorner y \in \mathcal{F}_i(y_1, y_2)\urcorner$. The converse inequality is clear.

<u>Remark</u>. Put $z(y_1) = z(y_2) = 1$, $\mathcal{D}(z) = \{y_1, y_2\}$; clearly z fulfils the assertion of the preceding Metatheorem with respect to $\mathcal{F}_1(y_1, y_2) = \{y_1, y_2\}$. The operation associating with every y_1, y_2 the function z just defined will be denoted by $\{y_1, y_2\}_B$. Further, we define $\langle y_1, y_2\rangle_B = \{\{y_1, y_1\}_B, \{y_1, y_2\}_B\}_B$. Clearly, it holds $F\ulcorner w = \langle y_1, y_2\rangle\urcorner = 1$ for $w = \langle y_1, y_2\rangle_B$.

<u>Lemma 3</u>. $F\ulcorner(\forall y \neq 0)(\exists x \in y)(x \cap y = 0)\urcorner = 1$.

<u>Proof</u>. Let $y \in C(B)$. Put $u = F\ulcorner y \neq 0\urcorner$. It suffices to prove that $u \leq \bigvee\{F\ulcorner x \in y \,\&\, x \cap y = 0\urcorner, x \in C(B)\}$. Suppose that there is a v, $0 \neq v \leq u$, such that $v \wedge F\ulcorner x \in y \,\&\, x \cap y = 0\urcorner = 0$ for every $x \in C(B)$. Let x_0 be an element of the least type such that $w = v \wedge \wedge F\ulcorner x_0 \in y\urcorner \neq 0$. We prove $w \wedge F\ulcorner x_0 \in y\urcorner \neq 0\urcorner = 0$. Let $w \wedge F\ulcorner(\exists z)(z \in x_0 \,\&\, z \in y)\urcorner \neq 0$. Then there is a $z_0 \in C(B)$ such that $\varphi(z_0) < \varphi(x_0)$ and $w \wedge \wedge F\ulcorner z_0 \in x_0 \,\&\, z_0 \in y\urcorner \neq 0$. Hence $v \wedge F\ulcorner z_0 \in y\urcorner \neq 0$, which is a contradiction.

<u>Definition 4</u>. The function q_α is defined on $c_\alpha(B)$ by $q_\alpha(x) = 1$ for every $x \in c_\alpha(B)$.

<u>Lemma 4</u>. $q_\alpha \in c_{\alpha+1}(B)$.

<u>Lemma 5</u>. If $a \subseteq C(B)$ then there is an $\alpha \in On$ such that $(\forall x \in a)(F\ulcorner x \in q_\alpha\urcorner = 1)$.

<u>Proof</u>. Take $\alpha = \sup(\varphi(x), x \in a) + 1$.

4. The sheaf of functions on a subalgebra.

<u>Definition 1</u>. Let $B' \subseteq B$, $0 \in B'$, $1 \in B'$, let

B' be a complete Boolean algebra with respect to the same operations as B is. Then B' is said to be a complete subalgebra of B. $L(B)$ is the set of all complete subalgebras of B.

Lemma 1. $L(B)$ is a complete lattice with respect to the ordering by set-inclusion. The algebra B is the greatest element of $L(B)$, the algebra $B_0 = \{0, 1\}$ is the least element of it.

Lemma 2. If $B' \in L(B)$ then $C(B') \subseteq C(B)$. If $f \in C(B')$ then $\rho(f)$ is the least α such that $f \in c_\alpha(B')$.

Metatheorem 1. Let φ be absolute. Then the following is provable in the set theory: Assume $F_{B'}(x,y) = F_B(x,y)$ for every $x, y \in c_\alpha(B')$. Then $F_{B'} \ulcorner \varphi(x \ldots y) \urcorner = F_B \ulcorner \varphi(x \ldots y) \urcorner$ for every $x, \ldots, y \in c_\alpha(B')$.

Proof. By induction; the only nontrivial case is that φ is $(\exists y)(y \in x \mathbin{\&} \psi(y, x \ldots))$. Evidently, $F_{B'} \ulcorner \varphi(x \ldots) \urcorner \leq F_B \ulcorner \varphi(x \ldots) \urcorner$. Put $u = F_B \ulcorner \varphi(x \ldots) \urcorner - F_{B'} \ulcorner \varphi(x \ldots) \urcorner$ and assume $u \neq 0$. Then there are q, $y \in C(B)$ such that $0 \neq u \wedge F_B \ulcorner y = q \urcorner \wedge x(q) \wedge F_B \ulcorner \psi(y, x \ldots) \urcorner$. $x(q) \neq 0$ and $x \in C(B')$ implies $q \in C(B')$. Consequently, $u \wedge F_B \ulcorner q \in x \mathbin{\&} \psi(q, x \ldots) \urcorner \neq 0$, hence $u \wedge F_{B'} \ulcorner \psi(x \ldots) \mathbin{\&} q \in x \urcorner \neq 0$, and $u \wedge F_{B'} \ulcorner \varphi(x \ldots) \urcorner \neq 0$, which is a contradiction.

Theorem 1. $F_B(x, y) = F_{B'}(x, y)$ for every x, $y \in C(B')$.

Proof. By transfinite induction with respect to

$max(\rho(x), \rho(y))$.

$F_B(x,y) = \bigvee\{F_B\ulcorner x = z\urcorner \wedge y(z), z \in C(B)\} = \bigvee\{F_B\ulcorner x = z\urcorner \wedge$
$\wedge y(z), z \in C(B')\} = \bigvee\{F_{B'}\ulcorner x = z\urcorner \wedge y(z), z \in C(B')\} = F_{B'}(x,y)$.

The preceding metatheorem has been used to the formula $x = x$.

<u>Definition 2</u>. Let $X \subseteq C(B)$. We define

$\overline{X} = \{y \in C(B), \bigvee\{F\ulcorner y = x\urcorner, x \in X\} = 1$.

$F_{BB'}$ is the restriction of F_B onto $(\overline{C(B')})^2$.

<u>Lemma 2</u>. $F_{BB'}(f,g) = \bigvee\{F_B\ulcorner f = x \,\&\, g = y\urcorner \wedge F_{B'}\ulcorner x \in y\urcorner,$
$x, y \in C(B')\}$ for every $f, g \in \overline{C(B')}$.

<u>Proof</u>. Let $f, g \in \overline{C(B')}$. Then $F_B\ulcorner f \in g\urcorner =$

$F_B\ulcorner f \in g\urcorner \wedge \bigvee\{F_B\ulcorner f = x \,\&\, g = y\urcorner, x, y \in C(B')\} =$

$= \bigvee\{F_B\ulcorner f \in g \,\&\, f = x \,\&\, g = y\urcorner, x, y \in C(B')\} =$

$= \bigvee\{F_B\ulcorner x \in y \,\&\, f = x \,\&\, g = y\urcorner, x, y \in C(B')\} =$

$= \bigvee\{F_B\ulcorner f = x \,\&\, g = y\urcorner \wedge F_{B'}\ulcorner x \in y\urcorner, x, y \in C(B')\}$.

<u>Metatheorem 2</u>. Let φ be elementary. Then the following is provable in the set theory: For every $f \ldots g \in \overline{C(B')}$,
$F_{BB'}\ulcorner\varphi(f \ldots g)\urcorner = \bigvee\{F_B\ulcorner f = x \,\&\, \ldots \,\&\, g = y\urcorner \wedge F_{B'}\ulcorner\varphi(x \ldots y)\urcorner,$
$x \ldots y \in C(B')\}$.

<u>Proof.</u> By induction; for φ atomic see the preceding lemma. Let φ be $(\exists g)\psi(g,f)$. $F_{BB'}\ulcorner(\exists g)\psi(g,f)\urcorner =$

$= \bigvee\{F_B\ulcorner f = x \,\&\, g = y\urcorner \wedge F_{B'}\ulcorner\psi(y,x)\urcorner, g \in \overline{C(B')}, x, y \in C(B')\} \leq$

$\leq \bigvee\{F_B\ulcorner f = x\urcorner \wedge \bigvee\{F_B\ulcorner g = y\urcorner, y \in C(B'), g \in \overline{C(B')}\} \wedge$

$\wedge \bigvee\{F_{B'}\ulcorner\psi(y,x)\urcorner, y \in C(B')\}, x \in C(B')\} =$

$= \bigvee \{ F_B \ulcorner f = x \urcorner \wedge F_{B'} \ulcorner (\exists y)\, \psi(y, x) \urcorner,\ x \in C(B') \}$, as

$\bigvee \{ F_B \ulcorner g = y \urcorner,\ y \in C(B') \} = 1$ for every $g \in \overline{C(B')}$.

On the other hand, $F_B \ulcorner f = x \urcorner \wedge F_{B'} \ulcorner (\exists y)\, \psi(y, x) \urcorner =$

$= F_B \ulcorner f = x \urcorner \wedge \bigvee \{ F_{B'} \ulcorner \psi(y, x) \urcorner,\ y \in C(B') \} =$

$= \bigvee \{ F_B \ulcorner f = x \urcorner \wedge F_{B'} \ulcorner \psi(y, x) \urcorner,\ y \in C(B') \} \leq$

$\leq \bigvee \{ F_B \ulcorner f = x \urcorner \wedge F_B \ulcorner y = y \urcorner \wedge F_{B'} \ulcorner \psi(y, x) \urcorner,\ x, y \in C(B') \} \leq$

$\leq \bigvee \{ F_B \ulcorner f = x \ \&\ g = y \urcorner \wedge F_{B'} \ulcorner \psi(y, x) \urcorner,\ x, y \in C(B'),$

$g \in \overline{C(B')} \} = \bigvee \{ F_{BB'} \ulcorner \psi(g, f) \urcorner,\ g \in \overline{C(B')} \} = F_{BB'} \ulcorner (\exists g)\, \psi(f, g) \urcorner$.

<u>Corollary</u>. If φ is closed then $F_{BB'} \ulcorner \varphi \urcorner = F_{B'} \ulcorner \varphi \urcorner$ is provable in the set theory.

<u>Definition 3</u>. The functions $k_x \in c_{\tau(x)}(B)$ are defined for every set x by induction with respect to $\tau(x)$:

$k_0 = 0,\ k_x(f) = 1 \equiv .(\exists y)(f = k_y \ \&\ y \in x);\ \mathcal{D}(f) = \{ k_y, y \in x \}$

<u>Lemma 3</u>. $C(B_0) = \{ k_x,\ x \in V \}$.

<u>Lemma 4</u>. $F_{B_0} \ulcorner k_x \in k_y \urcorner = 1 \equiv x \in y$.

<u>Metatheorem 3</u>. Let φ be normal. Then $F_{B_0} \ulcorner \varphi(k_x, \dots$ $\dots, k_y) \urcorner = 1 \equiv \varphi(x, \dots, y)$.

5. The model $\nabla(B, z)$

<u>Definition 1</u>. Put $R_{B,z} = \mathcal{L}im_z^{C(B)} F_B$, z being an ultrafilter on B . In this and the next section, φ^* denotes the translation of φ by $C(B)$, $R_{B,z}$. The symbols Φ , Ψ will be written without indices, too.

Theorem 1. $M\kappa(C(B), R_{B,z})$.

Proof. We use Metatheorem 1 § 2 (F_B is complete by Th. 1 § 3). 1) If $x \in C(B)$ put $y = \Phi(x) \cap c_{\rho(x)}(B)$. 2) It follows from lemma 5 in § 3. 3) follows from metatheorem 4 in § 3. 4) follows from lemma 3 in § 3. The function k_{ω_0} fulfils the assertion on 5) (see also below). 6) follows from Lemma 6 in § 3 and Metatheorem 2 in § 3.

Definition 2. $\nabla(B, z)$ denotes the specification of the classic parametrical model by $P = C(B)$, $R = R_{B,z}$, B being a variable for a complete Boolean algebra, z a variable for an ultrafilter on B .

Lemma 1. Let $X \subseteq C(B)$, then $\Psi(\bar{X}) = \{y ,\ V\{F\ulcorner x = y\urcorner, x \in X\} \in z\}$.

Lemma 2. Let $X \subseteq C(B)$. Then $Cls^*(\Psi(\bar{X})) \vee \vee (\exists x^*)(\Psi(\bar{X}) = \Phi(x^*))$.

Proof. The case $X = 0$ is trivial, suppose $X \neq 0$. It suffices to prove $(\forall x^*)(\exists y^*)(\Psi(\bar{X}) \cap \Phi(x^*) = \Phi(y^*))$. Put $s_\alpha = X \cap c_\alpha$. Let h_α be the function whose domain is s_α and such that $h_\alpha(x) = 1$ for every $x \in s_\alpha$. Evidently $\Psi(\bar{X}) = \bigcup_{\alpha \in On} \Phi(h_\alpha)$. Let $g \in C(B)$. Let a be a subset of $\Phi(g)$ such that the condition 1 in Def.1 § 1 holds. There is an $\alpha \in On$ such that $a \cap \Psi(\bar{X}) \subseteq \Phi(h_\alpha)$. Put $f = g \cap^* h$. Obviously, $\Phi(f) = \Psi(\bar{X}) \cap \Phi(g)$.

Theorem 2. Let $B' \in L(B)$. Then $Mcl^*(\Psi(\overline{C(B')}))$.

Proof. First, we prove $Comp^*(\Psi(\overline{C(B')}))$. Let $f \in \Psi(\overline{C(B')})$. We may suppose $f \in \overline{C(B')}$. Let $g \in^* f$, i.e. $F\ulcorner g \in f\urcorner \in z$. We have $V\{F\ulcorner f = h\urcorner, h \in C(B')\} \in z$ and consequently

$V\{F\ulcorner g \in f\urcorner \wedge F\ulcorner f = h\urcorner,\ h \in C(B')\} \leq V\{F\ulcorner g \in h\urcorner,$
$h \in C(B')\} \leq V\{F\ulcorner g = k \,\&\, k \in h\urcorner,\ h \in C(B'),$
$k \in \mathcal{D}(h)\} \leq V\{F\ulcorner g = k\urcorner,\ k \in C(B')\} \in x$,

i.e. $g \in \Psi(\overline{C(B')})$. The condition 2 is fulfilled by the sets $\mathcal{Q}_\alpha^{B'}$ constructed over B' . 3) follows by the Metatheorem in § 4.

Theorem 3. $Ord^*(f) \equiv V\{F\ulcorner f = k_\alpha\urcorner,\ \alpha \in On\} \in x$.

Proof. By the preceding theorem, $Mcl^*(\Psi(\overline{C(B_0)})$, and $Ord(x)$ is an absolute formula.

Lemma 3. k_{ω_0} is the number ω_0^* of $\nabla(B, x)$.

Theorem 4. The axiom of choice holds true in the complete submodel of $\nabla(B, x)$ determined by the class $\Psi(\overline{C(B')})$ for every $B' \in L(B)$.

Proof. Let A_0 be a one-to-one mapping of the class $\{k_\alpha,\ \alpha \in On\}$ onto $C(B')$. Let $g \in A \equiv (\exists\alpha, f)(\langle f, k_\alpha\rangle \in A_0 \,\&\, g = \langle f, k_\alpha\rangle_{B'})$. We prove that $\Psi(\overline{A})$ is a mapping of the class of ordinals onto $\Psi(\overline{C(B')})$ in sense of $\nabla(B,x)$. Let $\langle f_1, g\rangle^* \in^* \Psi(\overline{A}), \langle f_2, g\rangle^* \in^* \Psi(\overline{A})$, let $f_1 \neq^* f_2$. Then $0 \neq F\ulcorner f_1 \neq f_2 \,\&\, \langle f_1, g\rangle = \langle h_1, k\rangle \,\&\, \langle f_2, g\rangle = \langle h_2, k\rangle\urcorner$ for some h_1, h_2, k such that $k = k_\alpha$, $\langle h_1, k\rangle \in A_0, \langle h_2, k\rangle \in A_0$, i.e. $F\ulcorner f_1 \neq f_2 \,\&\, f_1 = f_2\urcorner \neq 0$, which is a contradiction. Hence $Un^*(\Psi(\overline{A}))$. Let $f \in \Psi(\overline{C(B')})$, i.e. $V\{F\ulcorner f = h\urcorner,\ h \in C(B')\} \in x$. Then there is a set $\{u_i,\ i \in b\}$ of elements of B and a set $\{h_i, i \in b\} \subseteq C(B')$ such that $i \neq i_1 \rightarrow u_i \wedge u_{i_1} = 0$, $u_i \leq F\ulcorner f = h_i\urcorner$ and $V\{u_i,\ i \in b\} \in x$.

Using the fact that F_B is complete it follows that there is a k such that $u_i = F\ulcorner k = d_i \urcorner$, where $d_i = \langle A' h_i , h_i \rangle_{B'}$. Evidently, $k \in \overline{C(B')}$. Hence, $\Psi(\overline{A})$ is a $\nabla(B, z)$-mapping onto $\overline{C(B')}$. It remains to prove that $\Psi(\overline{A})$ is a class in the sense of $\Psi(\overline{C(B')})$. We prove $(\forall x \in \Psi(\overline{C(B')}))(\exists y \in \Psi(\overline{C(B)}))(\Psi(\overline{A}) \cap \Phi_B(x) = \Phi_B(y))$ similarly as we have proved Lemma 2.

6. Cardinal numbers of the model $\nabla(B, z)$.

Definition 1. $f \in C(B)$ is said to be a relatively cardinal number of $\nabla(B, z)$ iff $V\{F\ulcorner f = k_\alpha \urcorner, \alpha \in N\} \in z$ (N being the class of all cardinal numbers).

Lemma 1. $f \in C(B)$ is a relatively cardinal number of $\nabla(B, z)$ if and only if f is a cardinal number in sense of the $\nabla(B, z)$-model-class $\Psi(\overline{C(B_0)})$.

Proof. See the Metatheorem 3 in § 4.

Lemma 2. Every cardinal number of $\nabla(B, z)$ is a relatively cardinal number of $\nabla(B, z)$.

Proof. See § 1.

Definition 2. A set $\{w(\gamma, \delta), \gamma < \omega_\alpha \ \& \ \delta < \omega_\beta\}$ is said to be an (α, β)-system in z iff there is a $u \in \in z$ such that

(i) $\gamma_1 \neq \gamma_2 \rightarrow w(\gamma_1, \delta) \wedge w(\gamma_2, \delta) = 0$,

(i') $\delta_1 \neq \delta_2 \rightarrow w(\gamma, \delta_1) \wedge w(\gamma, \delta_2) = 0$,

(ii) $V\{w(\gamma, \delta), \gamma < \omega_\alpha\} = u$, for every $\delta < \omega_\beta$,

(ii') $V\{w(\gamma, \delta), \delta < \omega_\beta\} = u$. for every $\gamma < \omega_\alpha$.

Theorem 1. Relatively cardinal numbers k_{ω_α}, k_{ω_β} have in $\nabla(B, z)$ the same cardinality if and only if there is an (α, β)-system in z .

Proof. Let k_{ω_α}, k_{ω_β} have the same cardinality in sense of $\nabla(B, z)$. Let $\varphi(x, \omega_\alpha, \omega_\beta)$ be an elementary formula saying " x is a one-to-one mapping of ω_α onto ω_β". Then there is a $f \in C(B)$ such that $\varphi^*(f, k_{\omega_\alpha}, k_{\omega_\beta})$. That means that $u = F\ulcorner\varphi(f, k_{\omega_\alpha}, k_{\omega_\beta})\urcorner \in z$. Put, for $\gamma < \omega_\alpha$, $\delta < \omega_\beta$, $w(\gamma, \delta) = u \wedge F\ulcorner k_{\langle\gamma,\delta\rangle} \in f\urcorner$. Evidently, $w(\gamma, \delta)$ form an (α, β)-system in z .

Let, conversely, $\{w(\gamma,\delta), \gamma < \omega_\alpha \,\&\, \delta < \omega_\beta\}$ be an (α, β)-system in z , let u be such that (ii) and (ii$'$) hold true. The function f is defined in the following way: $f(k_{\langle\gamma,\delta\rangle}) = w(\gamma, \delta)$, f undefined otherwise. We have $F\ulcorner\varphi(f, k_{\omega_\alpha}, k_{\omega_\beta})\urcorner = u \in z$, hence $\varphi^*(f, k_{\omega_\alpha}, k_{\omega_\beta})$.

Lemma 3. If there is an (α, β)-system in z and $\alpha \leq \alpha_1 \leq \beta_1 \leq \beta$, then there is an (α_1, β_1)-system in z .

Proof. As k_{ω_α} and k_{ω_β} have in $\nabla(B, z)$ the same cardinality, the same holds true for $k_{\omega_{\alpha_1}}$, $k_{\omega_{\beta_1}}$.

Definition 3. Let $a \subseteq B$. We define $Ex_B(a) \equiv (\forall x, y \in a)(x \neq y \rightarrow x \wedge y = 0)$.

$\mu(B) = \min\{\omega_\alpha, \neg(\exists a)(\mathrm{card}\, a = \aleph_\alpha \,\&\, Ex_B(a) \,\&\, a \subseteq B)\}$.

Lemma 4. $\mu(B)$ is a regular cardinal number.

Proof. See [3]; the proof given there for topological spaces may be easily modified for complete Boolean algebras.

Lemma 5. Let $\mu(B) \leq \omega_\alpha$. Then there is no (α, β)-system for any $\beta \neq \alpha$.

Proof. Let $\{w(\gamma, \delta), \gamma < \omega_\alpha \,\&\, \delta < \omega_\beta\}$ be such an (α, β)-system. First, let $\alpha < \beta$. Put $a_\gamma = \{\delta, w(\gamma, \delta) \neq 0 \,\&\, \delta < \omega_\beta\}$. It follows $\mathrm{card}\, a_\gamma < \mu(B)$. Hence $\mathrm{card}(\bigcup\{a_\gamma, \gamma < \omega_\alpha\}) \leq \omega_\alpha$. Consequently, there is a $\delta < \omega_\beta$ such that $\delta \notin a_\gamma$ for every $\gamma < \omega_\alpha$. That means that $w(\gamma, \delta) = 0$ for every $\gamma < \omega_\alpha$, which is a contradiction with (ii).

Let $\beta < \alpha$. By the preceding part of the present proof, we may suppose $\omega_\beta < \mu(B)$. Put $a_\delta = \{\gamma, w(\gamma, \delta) \neq 0 \,\&\, \gamma < \omega_\alpha\}$. Evidently, $card\, a_\delta < \mu(B)$. As $\mu(B)$ is regular, $card(\bigcup\{a_\delta, \delta < \omega_\beta\}) < \mu(B) \leq \omega_\alpha$ and therefore is a $\gamma < \omega_\alpha$ such that $\gamma \notin a_\delta$ for every $\delta < \omega_\beta$, which is a contradiction with (ii′).

On the end of [4], an example of topological spaces depending on two cardinal numbers $\aleph_\alpha$, $\aleph_\beta$ is given (the continuum-hypothesis assumed) such that the following holds true for the algebra B of all open regular sets:

Let $\aleph_\alpha$ be regular, $\aleph_\alpha \leq cf(\aleph_\beta)$. Then

(i) $\mu(B) < \aleph_{\beta+1}$, hence there is no (γ, δ)-system in B for $\aleph_\gamma \geq \aleph_{\beta+1}$, $\delta \neq \gamma$,

(ii) further, there is no (γ, δ)-system for $\aleph_\gamma < \aleph_\alpha$, $\delta \neq \gamma$,

(iii) but there is an (α, β)-system in B.

7. Cardinalities of power-sets in $\nabla(B, x)$.

__Definition 1.__ Let $x \subseteq B$, $v \in B$; we define $x \wedge\wedge v = \{w \wedge v, w \in x\}$.

__Definition 2.__ $R(x, u) \equiv . u \in B \,\&\, x \subseteq B \,\&$
$\& (\forall v)(0 \neq v \leq u \rightarrow x \wedge\wedge v \neq \{0, v\})$
(read: x decomposes u).

__Theorem 1.__ Let a be a set of cardinality $\aleph_\alpha$. Then
$\mathcal{P}^*(k_a) =^* k_{\mathcal{P}(a)} \equiv \neg(\exists b)(\exists u \in x)(R(b, u) \,\&\, card\, b \leq \aleph_\alpha)$.

Proof. Let there exist $u \in x$ and b such that $card\ b \leq \aleph_\alpha$ and b decomposes u. Let f be a function mapping the set $\{k_x, x \in a \& x \neq 0\}$ onto b. Evidently, $f \in C(B)$, $f \subseteq^* k_a$.
Suppose $f \in^* k_{\mathcal{P}(a)}$. Then there is a v, $0 \neq v \leq u$ and $y \subseteq a$ such that $v \leq F\ulcorner f = k_y \urcorner$. $x \in y$ implies $F\ulcorner k_x \in f \urcorner \geq v$, i.e. $f(k_x) \wedge v = v$; $x \notin y$ implies $f(k_x) = 0$ hence $b \wedge\wedge v = \{0, v\}$ which is a contradiction with $R(b, u)$.

Let, conversely, $\mathcal{P}(k_a) \neq k_{\mathcal{P}(a)}$. In any case $k_{\mathcal{P}(a)} \subseteq^* \mathcal{P}^*(k_a)$; hence, there is a $f \in^* \mathcal{P}^*(k_a)$ such that $f \notin^* k_{\mathcal{P}(a)}$. Put $u = F\ulcorner f \notin k_{\mathcal{P}(a)}\urcorner$. Evidently $u \in x$. Put $b = W(f)$. If $R(b, u)$ did not hold then there would exist a v, $0 \neq v \leq u$ and $y \subseteq a$ such that $v \leq F\ulcorner f = k_y \urcorner$. This would be a contradiction with the definition of u.

Definition 2. Let h_{ω_α} be the cardinal number of $\nabla(B, x)$ which is the cardinal of $\mathcal{P}^*(k_{\omega_\alpha})$ in $\nabla(B, x)$.

Lemma 1. $card^*(k_{2^{\aleph_\alpha}}) \leq^* h_{\omega_\alpha}$.

Definition 3. $P_\alpha(B) = \{x, x \subseteq B \& card\ x \leq \aleph_\alpha\}$;

$Rs(a, u, \alpha) \equiv . a \subseteq P_\alpha(B) \& (\forall x \in a) R(x, u) \&$

$\& (\forall x, y \in a)(\forall v)(0 \neq v \leq u \& x \neq y . \rightarrow x \wedge\wedge v \neq y \wedge\wedge v)$

(say, a is a sytem of substantially different α-systems decomposing u).

Theorem 2. $(\exists a)(\exists u)(Rs(a, u, \alpha) \& card\ a = \aleph_\beta) \rightarrow$ $\rightarrow card^*(k_{\omega_\beta}) \leq^* h_{\omega_\alpha}$.

Proof. Let, for $y \in a$, f_y be a function mapping the set $\{k_\gamma, \gamma < \omega_\alpha\}$ onto y. Let r be a one-to-one mapping of ω_β onto a . The function g is defined by the equivalence $g(\langle f_y, k_\gamma \rangle_B) = 1 \equiv y = \kappa(\gamma)$, otherwise undefined. Evidently, $g \in C(B)$. Let $f = \mathcal{P}^*(k_{\omega_\alpha})$. Let $\varphi(x, y, z)$ be an elementary formula saying " x is a one-to-one mapping of y into z ". Evidently, $u \leq$ $\leq F\ulcorner \varphi(g, k_{\omega_\beta}, f)\urcorner$, hence $\varphi^*(g, k_{\omega_\beta}, f)$, i.e. $k_{\omega_\beta} \leq^* k_{\omega_\alpha}$. (We prove e.g. that $u \leq F\ulcorner g$ is a mapping$\urcorner$. Let $u \wedge F\ulcorner(\exists h, k_1, k_2)(k_1 \neq k_2 \;\&\; \langle h, k_1\rangle \in g \;\&\; \langle h, k_2\rangle \in g\urcorner \neq 0$. Then there are y_1, y_2 , γ_1, γ_2 such that $\gamma_1 \neq \gamma_2$ and $v = F\ulcorner\langle y_1, k_{\gamma_1}\rangle \in g \;\&\; \langle y_2, k_{\gamma_2}\rangle \in g \;\&\; y_1 \neq y_2\urcorner \wedge u \neq 0$. Let $q \in y_1$; let γ be such that $q = f_{y_1}(k_\gamma) = F\ulcorner k_\gamma \in f_{y_1}\urcorner$. We have $F\ulcorner k_\gamma \in f_{y_1}\urcorner \wedge v = F\ulcorner k_\gamma \in f_{y_2}\urcorner \wedge v$, hence $q \wedge v \in$ $\in y_2 \wedge \wedge v$. I.e., $y_1 \wedge \wedge v = y_2 \wedge \wedge v$, which is a contradiction.)

Lemma 2. Let $\mathrm{Mcl}(M)$, let the axiom of choice hold in the sense of M . Let $\mathrm{card}^M \mathcal{P}^M(\alpha) < \omega_\beta = \mathrm{card}\ \mathcal{P}(\alpha)$. Then there is an $a \subseteq (\mathcal{P}(\alpha) - M)$ such that $\mathrm{card}\ a = \omega_\beta$ and, for every $y_1, y_2 \in a$, $y_1 \neq y_2$, there is no relation $\kappa \in M$ such that $\kappa'' y_1 = y_2$, say there is a system of M-substantially different M-nonconstructible subsets of α .

Proof. Let $\mathrm{card}^M \mathcal{P}^M(\alpha) = \omega_\gamma^M$. Then there are $\omega_\gamma^M < \omega_\beta$ relations $\kappa \in \mathcal{P}(\alpha \times \alpha) \cap M$. Let us have subsets $y_\sigma\ (\sigma < \varepsilon < \omega_\beta)$ of α such that $y_\sigma \notin M$, $\sigma_1 \neq \sigma_2 \rightarrow$ $\rightarrow \neg(\exists \kappa \in M)(y_{\sigma_1} = \kappa'' y_{\sigma_2})$. Put $Y = (\mathcal{P}(\alpha) - \mathcal{P}^M(\alpha)) -$ $- \{y, (\exists \kappa \in M)(\exists \sigma < \varepsilon)(y = \kappa'' y_\sigma)\}$. Surely, $Y \neq 0$; choose $y_\varepsilon \in Y$. Take $a = \{y_\sigma, \sigma < \omega_\beta\}$.

Remark. Note that the property $\varphi_0(a, \omega_\beta, \alpha)$ saying " a is a system of ω_β M-substantially different M-nonconstructive subsets of α is equivalent to an elementary formula (we may take $\mathcal{P}_{\tau(\alpha)+3} \cap M$ instead of M).

Theorem 3. $k_{2^{\aleph_\alpha}} <^* k_{\omega_\beta} <^* h_{\omega_\alpha} \rightarrow (\exists a)(\exists u \in x)(Rs(a,u,\alpha) \& \operatorname{card} a = \aleph_\beta)$

Proof. $k_{2^{\aleph_\alpha}} <^* h_{\omega_\alpha}$ implies the existence of an $f \subseteq^* \subseteq^* \mathcal{P}(k_{\omega_\alpha})$ such that, in sense of the model $\nabla(B,x)$, f is a system of k_{ω_β} $\Psi(\overline{C(B_o)})$ -substantially different $\Psi(\overline{C(B_o)})$ -nonconstructive subsets of k_{ω_α}. Evidently, there is a $g \in C(B)$ such that $\varphi^*(g, k_{\omega_\beta}, f)$ (φ having the same meaning as in the proof of Theorem 2). Put $u = F\ulcorner \varphi(g, k_{\omega_\beta}, f) \& \varphi_o(f, k_{\omega_\beta}, k_{\omega_\alpha})\urcorner$. Clearly, $u \in x$. Let, for every $\gamma < \omega_\beta$, f_γ be an element of $C(B)$ such that $f_\gamma \in^* f$, $\langle f_\gamma, k_\gamma \rangle_B \in^* g$. Put $a = \{W(f_\gamma), \gamma < \omega_\beta\}$. We may suppose all the f_γ's to be defined only for some k_σ, $\sigma < \omega_\alpha$. Then $a \subseteq P_\alpha(B)$. For every $\gamma < \omega_\beta$, $Rs(W(f_\gamma), u)$ holds. If we prove $W(f_\gamma) \wedge\wedge v \neq W(f_\sigma) \wedge\wedge v$ for every $0 \neq v \leq u$ and $\gamma \neq \sigma$, then both $Rs(a,u,\alpha)$ and $\operatorname{card} a = \aleph_\beta$ will be proved. Let $W(f_\gamma) \wedge\wedge v = W(f_\sigma) \wedge\wedge v$ for some $v \leq u$, $\gamma \neq \sigma$, $\gamma, \sigma < \omega_\beta$. We define a relation r by the equivalence $\langle x,y \rangle \in r \equiv . f_\gamma(k_x) \wedge v = f_\sigma(k_y) \wedge v$. Then k_r is a relation in sense of $\Psi(\overline{C(B_o)})$ with the property contradicting to $v \leq F\ulcorner \varphi_o(f, k_{\omega_\beta}, k_{\omega_\alpha})\urcorner$.

Remark. In [4], an example of topological spaces depending on two cardinal numbers $\aleph_\alpha$, $\aleph_\beta$ is given (the continuum-hypothesis assumed) such that the following holds true for the algebra B of all open regular sets: Let $\aleph_\alpha$ be regular, $\aleph_\alpha < cf(\aleph_\beta)$. Then

(i) There is no (γ, σ)-system for any $\gamma \neq \sigma$;

(ii) If $\aleph_\sigma < \aleph_\alpha$ then, for every $u > 0$, there is no $a \subseteq P_\sigma(B)$ decomposing u;

(iii) If $\aleph_\alpha \leq \aleph_\gamma < cf(\aleph_\beta)$, then, for every $u > 0$, there are systems of $\aleph_\beta$ substantially different γ-systems decomposing u, but there are no greater systems of such systems;

(iv) If $cf(\aleph_\beta) \leq \aleph_\gamma \leq \aleph_\beta$, $\aleph_\sigma > \aleph_{\beta+1}$ then, for every $u > 0$, there are no systems of $\aleph_\sigma$ substantially different γ-systems decomposing u;

(v) If $\aleph_\beta \leq \aleph_\gamma$, $\aleph_\sigma > \aleph_\gamma$ then, for every $u > 0$, there are no systems of $\aleph_\sigma$ substantially different γ-systems decomposing u.

8. Free ultrafilters

Definition 1. Let B be a Boolean algebra (not necessarily complete). Denote the set of all ultrafilters on B by $S(B)$. If $u \in B$ then $u^0 = \{x, x \in S(B) \& u \in x\}$.

Lemma 1. The set $\{u^0, u \in B\}$ is a closed-open basis of a compact Hausdorff topology on $S(B)$.

Remark. $S(B)$ is understood as a topological space with the topology defined in Lemma 1.

Definition 2. Let $Mcl(P)$, let the axiom of choice hold in sense of P. B denotes a complete Boolean algebra in sense of P.

Lemma 2. B is a Boolean algebra in sense of the set theory.

Definition 3. The set $V(B)$ of free ultrafilters on B is defined by

$$V(B) = S(B) - \bigcup\{u^0 - \bigcup\{v^0, v \in a\}, u \in B \& u = \\ = \bigvee\{v, v \in a\} \& a \in P\}.$$

Lemma 3. The set $u^\circ - \bigcup\{v^\circ, v \in a\}$ is closed and nowhere dense in $S(B)$ for every $u \in B$, $\bigvee\{v, v \in a\} = u$.

Lemma 4. $V(B) = S(B) - \bigcup\{u^\circ - \bigcup\{v^\circ, v \in a\}, u \in B \,\&\, u = \bigvee\{v, v \in a\} \,\&\, a \in P \,\&\, Ex_B(a)\}$.

Definition 4. A set $b \subseteq B$ is said to be a pseudo-basis of B iff, for every non-zero element u of B, there is a $v \in b$, $v \neq 0$ such that $v \leq u$. $\tau(B)$ is the cardinality of the least pseudo-basis.

$d(B,b) = card\{a, a \in P \,\&\, Ex_B(a) \,\&\, a \subseteq b\}$; $d(B)$ is the minimum of all $d(B,b)$, where b is any pseudo-basis of B such that $card\ b = \tau(B)$.

Lemma 5. If $S(B)$ is not a union of $d(B)$ nowhere dense sets then $V(B) \neq 0$ (i.e., free ultrafilters exist).

Lemma 6. Let $card(\mathcal{P}^P(B)) = \aleph_0$. Then $V(B)$ is the complement of a set of the first category in $S(B)$.

Metatheorem 1. Let $\varphi_0(x)$ be a normal formula saying " B is a complete Boolean algebra". Let $\varphi(x)$ be normal, let ψ be a closed normal. Let the conjunction $\psi \,\&\, (\exists x)(\varphi(x) \,\&\, \varphi_0(x))$ be consistent with the set theory. Then the following is consistent with the set theory:

(i) There is a P such that $Mcl(P)$, ψ^P and

(ii) there is a B such that $\varphi_0^P(B) \,\&\, \varphi^P(B)$ and

(iii) $V(B)$ is the complement of a set of the first category in $S(B)$.

Proof. We deal with the set theory with the additional axiom $\psi \,\&\, (\exists x)(\varphi(x) \,\&\, \varphi_0(x))$. Let B be a complete Boolean algebra such that $\varphi(B)$, let $card\,\mathcal{P}(B) = \aleph_\alpha$. We construct a ∇-model such that k_{ω_α} is countable in sense of this model. We prove that (i) - (iii) hold in this model. Put $P = \Psi(\overline{C(B_0)})$. k_B is a complete Boolean algebra in sense of the ∇-model-class P and $\varphi(B)$ holds in sense of this class, hence, (i) and (ii) hold. The condition (iii) follows from the fact that $k_{\mathcal{P}(B)}$ is countable in sense of ∇.

9. Reduction of the sheaf of functions in a free ultrafilter.

$C^P(B)$ denotes the sheaf of functions constructed over $B \in P$ in sense of P, P being a model-class.

Definition 1. Let $x \in V(B)$, $f \in C(B)$. The sets $w(f, x)$ (the value of f in z) are defined inductively with respect to $\rho(f)$.

$w(0, x) = 0$; $w(f, x) = \{w(g, x), f(g) \in x\}$.

Further, put $W_x^B = \{w(f, x), f \in C^P(B)\}$. (We write shortly W_x instead of W_x^B.)

Lemma 1. $Comp(W_x)$.

Lemma 2. $x \in P$ implies $w(k_x, z) = x$.

Metatheorem 1. Let $\varphi(x \ldots y)$ be an elementary formula. Then the following is provable in the set theory: For every $f, \ldots, g \in C^P(B)$,

$$F\ulcorner\varphi(f \ldots g)\urcorner \in z \equiv \varphi^{W_z, E}(w(f, z), \ldots, w(g, z)).$$

Proof. If φ is atomic, i.e. φ is $f \in g$, then the theorem is proved inductively with respect to $max(\rho(f), \rho(g))$ (and using the fact that if the theorem holds for all f, $g \in C(B)$ such that $\rho(f) < \rho(g) = \alpha$ then

$F^{<}\ulcorner f = g \urcorner \in z \equiv w(f,z) = w(g,z)$ holds for all $f, g \in C(B)$ such that $max(\rho(f), \rho(g)) = \alpha$). Further, the Metatheorem is proved for every φ by the metamathematical induction with respect to the recursive construction of φ.

Theorem 1. Let z be a free ultrafilter on $B \in P$, where P is a model-class. Then $Mcl(W_z)$.

Denotation. Let $\hat{\Sigma}$ be an extension of the set theory such that the existence of a model-class P (such that the axiom of choice holds in sense of P), a Boolean algebra B complete in sense of P and an ultrafilter z on B free with respect to P is provable in $\hat{\Sigma}$.

$\Delta(P, B, z)$ denotes the specification of the complete parametric model of Σ in $\hat{\Sigma}$ by the condition $X = W_z^B$. (Let us stress the fact that $\Delta(P, B, z)$ is a standard model.)

Theorem 2. Under the assumptions of Theorem 1, let $x \in P$, $y \subseteq x$. Then $y \in W_z$ if and only if there is a (B, x)-function $f \in P$ such that $y = \{x_1, f(x_1) \in z\}$.

Proof. Define g on $\{k_x, x \in \mathcal{D}(f)\}$ by $g(k_x) = f(x)$. Evidently $g \in C^P(B)$, $w(g, z) = y$.

Remark. After completing this paper I was informed by Prof. A. Mostowski in September 1966 that Prof.D. Scott proved certain independence in Zermelo-Fraenkel set theory using also the notion of complete Boolean algebra. I don't know whether our approaches are similar.

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