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All spaces are assumed to be separated and uniformizable. A space is called pseudocompact if each continuous function is bounded, or equivalently, any locally finite family of non-void open sets is finite. By an n-cube of a space X, designated by X^n, we mean the product of n copies of X, more precisely the product of any family $\{X \mid a \in A\}$ where the cardinal of A is n, and also the n-fold product $X \times \cdots \times X$ of X by itself if n is finite. The purpose of this note is to exhibit the following two examples.

A. Given a positive integer n there exists a space X such that X^n is pseudocompact but X^{n+1} is not.

B. There exists a space Y such that each finite cube Y^k of Y is pseudocompact but $Y^{\aleph_0}$ is not.

To accomplish the picture and also to simplify the proof of Proposition E below we shall prove, see also [4,p.370].

C. The product of a family of spaces is pseudocompact provided that the product of each countable subfamily is so. (If the product is non-void, then evidently the converse holds.)

To prove C observe that if $\{U_n\}$ is a locally finite sequence of canonical open sets in a product space $P = X\{P_a \mid a \in A\}$ then there exists a countable $A_1 \subset A$ such that the projection of the sequence $\{U_n\}$ into the space $X\{P_a \mid a \in A_1\}$ is also locally finite. It should be remarked that this proves C in a more general setting, namely with pseudocompact replaced by countably H-closed, see [3].

Remark. According to the Glicksberg theorem, see [4] or [2], the properties of X in A can be formulated as follows: $(\beta X)^n$ is a Stone-Čech compactification of X^n, but $(\beta X)^{n+1}$ is not any Stone-Čech compactification of X^{n+1}; or equivalently (using the Stone-Weierstrass theorem), each continuous function of "n variables" admits arbitrarily close approximations by polynomials in bounded continuous functions of "one variable" but there exists a bounded continuous function of "n+1 variables", which does not. The same applies to B. In the case A there is also the following restatement: $C^*(X^n)$ is the n-fold tensor product of $C^*(X)$ by itself but $C^*(X^{n+1})$ is "larger" than the (n+1)-fold product of $C^*(X)$ by itself.

First we shall show that the exhibition of A and B reduces to the following examples A′ and B′. It should be noted that A′ for n = 2 and B′ answer the original problems of W.W. Comfort. Then we state proposition D, and prove A′ and B′ using D. Finally D will be proved; this is the main step in the proof.

A′. Given a positive integer n there exist spaces $X(k), k = 1, \dots, n+1$, such that any cube of any product $X(k_1) \times \dots \times X(k_n)$ is pseudocompact, but the product $X(1) \times \dots \times X(n+1)$ is not pseudocompact.

B′. There exists a sequence $\{Y(k)\}$ of spaces such that the product of any finite subfamily is pseudocompact but the product of every infinite subfamily is not pseudocompact.

Proof of A (using A′). For X take the sum of the family

$\{X(k)\}$, where X(k) are spaces with properties in A'.

Proof of B (using B'). Similarly let the sum Z of a family $\{Y(k)\}$ with properties in B' be an open subspace of a space Y such that Y - Z is a singleton, say (z), with neighborhoods of z defined to be all $U \ni z$ which contain all $(k) \times Y(k)$ except for a finite number of n.

D. Proposition. There exists an infinite disjoint collection $\mathcal{A}$ of subsets of βN (βN designates a Stone-Čech compactification of the discrete space N of natural numbers) such that every cube of $N \cup A$, $A \in \mathcal{A}$, is pseudocompact.

Exhibition of X(k) in A'. Choose a one-to-one family

$\{A(j) \mid j = 1, \ldots, n+1\}$ in A and put

$$B(k) = \bigcup\{A(j) \mid j \neq k\},\ X(k) = N \cup B(k)$$

for $k = 1, \ldots, n+1$. The product of $\{X(k)\}$ is not pseudocompact because $\bigcap\{B(k)\} = \emptyset$ and so the diagonal is closed, which implies that the family $\{(\{l \mid k = 1, \ldots, n+1\}) \mid l \in N\}$ of non-void open sets is locally finite. On the other hand if $k_i \neq k$ for $i = 1, \ldots, n$, then $\bigcap\{B(k_i)\} \supset A(k)$, and so any partial product is pseudocompact because it contains a cube of some $N \cup A(k)$ as a dense subspace, and every cube of $N \cup A(k)$ is pseudocompact.

Exhibition of Y(k) in B'. Let $\{A(k)\}$ be a disjoint sequence in A and let

$$B(k) = \bigcup\{A(j) \mid j \leq k\},\ Y(k) = N \cup B(k).$$

Clearly the intersection of any infinite subfamily of $\{B(k)\}$ is empty, and the intersection of every finite subfamily contains some A(k). Thus every cube of the product of any finite subfamily is pseudocompact because it contains a cube of some $N \cup A(k)$ as a dense subspace. To prove that the product Z of an infinite subfamily $\{ Y(k) \mid k \in K \}$ is not pseudocompact we shall show that the family of the canonical open sets

$U_k = E\{x = \{x(j)\} \mid x \in Z, x(j) = k$ for $j \leq k\}$, $k \in K$,

is locally finite. Pick a $y = \{y(k) \mid k \in K\}$ in Z. If some y(k) belongs to N then the set $E\{x \mid x \in Z$, $x(k) = y(k)\}$ is a neighborhood of y which intersects no U_m with $m > y(k)$. If no y(k) belongs to N, then $y(i) \neq y(j)$ for some $i \neq j$ in K because the intersection of $\{ B(k) \mid k \in K \}$ is empty. Choose disjoint neighborhoods U of y(i) and V of y(j) in βN. Clearly the neighborhood

$$E\{x \mid x \in Z, x(i) \in U, x(j) \in V\}$$

of y intersect no U_k with $k > i, j$. This concludes the proof. It should be remarked that one could show that each cluster point of $\{ U(k) \}$ is a cluster point of the diagonal of Z, and use the fact that the diagonal is closed.

<u>Proof of</u> D. Call a mapping $f : N \longrightarrow X$ eventually one-to-one (eventually constant) if $f : (N - M) \longrightarrow X$ is one-to-one (constant) for some finite set M. Consider the set P of all eventually one-to-one mappings of N into itself. For f in P let f^* denote the unique continuous extension of f to a mapping of β N into itself. Write $x \rho y$ iff $x, y \in \beta N - N$ and $f^* x = y$ for some f in P. It is easy to verify that ρ is an equivalence relation on $\beta N - N$. It should be remarked that the equivalence classes are the smallest P^*-invariant non-void subsets of $\beta N - N$. We shall prove that the collection Ⓐ of all equi-

valence classes has the properties stated in D .

E. <u>Proposition</u>. The collection $\mathfrak{A}$ has the properties stated in D , $\operatorname{card} \mathfrak{A} = \exp \exp \aleph_0$, and $\operatorname{card} A = \exp \aleph_0$ for any A in $\mathfrak{A}$.

<u>Proof</u>. The cardinal of any A in $\mathfrak{A}$ is at most $\exp \aleph_0$ because the cardinal of P is $\exp \aleph_0$ and all the points of A are images under mappings from P of any fixed point of A . On the other hand, A is dense in $\beta N - N$ and so the cardinal of A is at least $\exp \aleph_0$. The cardinal of βN is $\exp \exp \aleph_0$, and so the cardinal of $\mathfrak{A}$ is $\exp \exp \aleph_0$.

According to C to prove that any cube of $(N \cup A)$ is pseudocompact it will suffice to prove that $Z = (N \cup A)^N$ is pseudocompact. We shall prove that every sequence $\{x(k)\}$ in N^N has a cluster point in Z ; it will follow that Z is pseudocompact since N^N is dense. Given $\{x(k)\}$ choos a subsequence $\{y(k)\}$ such that each coordinate sequence y(k) is either eventually one-to-one or eventually constant. Pick any a in A and consider the point $z = \{z(k)\}$ in Z such that z(k) is the value of $(y(k))^*$ at a if y(k) is eventually one-to-one, and the eventual constant value of y(k) otherwise. We shall prove that z is a cluster point of $\{y(k)\}$, and so certainly of $\{x(k)\}$. Let U be a canonical neighborhood of z determined by neighborhoods $U(0), U(1), \dots, U(n)$ of $z(0), z(1), \dots, z(n)$, respectively. Since any f^* , with f in P , defines a homomorphism on $\beta N - N$, there exists a neighborhood V of a in βN such that

$$(y(k))^*[V] \cap (N \cup A) \subset U(n)$$

if $k \leq n$ and $y(k) \in P$. If $k \leq n$ and y(k) is eventually constant then we choose a residual set N_k in N such that y(k) is constant on N_k. The intersection N' of $V \cap N$ and all the N(k) is a non-void (infinite) subset of N and clearly $y(i) \in U$ if $i \in N'$.
The proof is complete.

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