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ONE MORE REMARK ON REFLECTIONS

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First, I must apologize for an unpleasant blunder in [4]. It is incorrectly stated on page 249 that lemma 2.6 in [5] does not hold under the given conditions.

There is also a misprint in Theorem 3 in [4]. There must be $\mathcal{K}_X$ instead of $\mathcal{K}$ in the first sentence.

If we investigate the existence of a reflection in $\mathcal{K}'$ of an object X from $\mathcal{K}$, we look for the objects of $\mathcal{K}'$ such that each morphism f from X into $\mathcal{K}'$ (i.e. $f \in \mathfrak{M}_X$) can be decomposed through them. Among these objects we must find that one with the unique decompositions. If $\mathcal{K}'$ is product-admitting and the embedding $\mathcal{K}' \to \mathcal{K}$ preserves products, then in order to solve the first part it is necessary and sufficient to find a set $\mathfrak{N}_X \subset \mathfrak{M}_X$ such that each morphism from $\mathfrak{M}_X$ can be decomposed through a morphism from $\mathfrak{N}_X$. To find out the uniqueness we need further conditions. It is possible to require either $\mathcal{K}'$ to be left complete and the embedding $\mathcal{K}' \to \mathcal{K}$ to preserve inverse limits (see [2], p. 84) or $\mathfrak{N}_X$ to be a set of <u>epimorphisms with respect to</u> $\mathcal{K}'$ (i.e. $\mathcal{E}f \in \mathcal{K}'$ and $g_1 \circ f = g_2 \circ f$, $g_1 \in \mathcal{K}'$, $g_2 \in \mathcal{K}'$ implies $g_1 = g_2$) - see [4].

In this note we shall point out some cases in which the investigation of reflections is easier. Assume the following situation: $\mathcal{G}$ is a faithful covariant functor from a category $\mathcal{K}_1$ into a category $\mathcal{K}$, $\mathcal{K}_1'$ is a subcategory of $\mathcal{K}_1$, $\mathcal{K}'$ is a replete subcategory of $\mathcal{K}$ and $\mathcal{G}[\mathcal{K}_1'] \subset \mathcal{K}'$. There are many questions concerning relations between reflections in $\mathcal{K}_1'$ and in $\mathcal{K}'$. We restrict ourselves to the following three questions:

(1) Is $\langle \mathcal{G}Y, \mathcal{G}f \rangle$ a reflection of $\mathcal{G}X$ in $\mathcal{K}'$ provided that $\langle Y, f \rangle$ is a reflection of X in $\mathcal{K}_1'$?

(2) Is $\mathcal{K}'$ reflective in $\mathcal{K}$ provided that $\mathcal{K}_1'$ is reflective in $\mathcal{K}_1$?

(3) Is $\mathcal{K}_1'$ reflective in $\mathcal{K}_1$ provided that $\mathcal{K}'$ is reflective in $\mathcal{K}$?

The aim of this note is to find conditions under which the answers are in the affirmative.

Now, we recall the definition of S-functor from [3] and the main Theorem 3 from [4].

The functor $\mathcal{G}$ is called an S-<u>functor</u> (and then $\mathcal{K}_1$ is called an S-category with respect to $\mathcal{G}$) if the following conditions are fulfilled:

(a) $\mathcal{G}f = \varphi_1 \cdot \varphi_2$ implies $f = f_1 \circ f_2$ where $\mathcal{G}f_i = \varphi_i$;

(b) if $\varphi \in \mathcal{K}$, $\mathcal{G}X = \mathcal{E}\varphi$ (or $\mathcal{G}X = \mathcal{D}\varphi$) then there exists an $f \in \mathcal{K}_1$ such that $\mathcal{G}f = \varphi$ and $X = \mathcal{E}f$ (or $X = \mathcal{D}f$, respectively);

(c) for each object A in $\mathcal{K}$ the class $\mathcal{G}^{-1}[A] \cap \operatorname{obj} \mathcal{K}_1$ is the complete set in the quasi order $<_{\mathcal{G}} = \mathcal{E}\{\langle X, Y\rangle \mid 1_{\mathcal{G}X} \in \mathcal{G}[\operatorname{Hom}_{\mathcal{K}_1}\langle X, Y\rangle]\}$;

(d) if $\{f_i\}$ is a family in $\mathcal{K}_1$ such that $\mathcal{G}f_i = \varphi$ for each i, then there is an $f \in \operatorname{Hom}_{\mathcal{K}_1}\langle \sup\{\mathcal{D}f_i\}, \sup\{\mathcal{E}f_i\}\rangle$

with $\mathcal{G}f = \varphi$, and similarly for inf.

Theorem 3 from [4] (not in the full generality): Assume that $\mathcal{K}'$ is product-admitting. Then each $X \in obj\ \mathcal{K}$ has a reflection in $\mathcal{K}'$ if and only if:

(a) the embedding $\mathcal{K}' \to \mathcal{K}$ preserves products;

(b) $Hom_{\mathcal{K}} \langle X, Y \rangle \neq \emptyset$ for some $Y \in obj\ \mathcal{K}'$;

(c) there is a cofinal set in the class $\mathcal{H}_X$ of all epimorphisms with respect to $\mathcal{K}'$ with domains X (in the order: $f < g$ if $g = h \circ f$ for some $h \in \mathcal{K}'$);

(d) each $f \in Hom_{\mathcal{K}} \langle X, Y \rangle$, $Y \in obj\ \mathcal{K}'$, can be factorized through a morphism from $\mathcal{H}_X$.

(1) Let $\langle Y, f \rangle$ be a reflection of X in $\mathcal{K}_1'$. We want to know if $\langle \mathcal{G}Y, \mathcal{G}f \rangle$ is a reflection of $\mathcal{G}X$ in $\mathcal{K}'$. The answer will be in the affirmative if $\mathcal{G}$ fulfils some conditions similar to the condition (b) in the definition of the S-functor. $\mathcal{G}f$ has the factorization property if the following condition holds:

(α) if $\varphi : \mathcal{G}X \to A$, $A \in obj\ \mathcal{K}'$ then there is $g : X \to Z$, $Z \in obj\ \mathcal{K}_1'$, $\mathcal{G}g = \varphi$, $g \in \mathcal{K}_1'$. In order to get the uniqueness of this factorization one must require fulfilling of some conditions of the following type:

(β) if $\varphi_1, \varphi_2 \in Hom_{\mathcal{K}'} \langle A, B \rangle$, $\mathcal{G}Z = A$, $Z \in obj\ \mathcal{K}_1'$,then there is a $Z' \in obj\ \mathcal{K}_1'$ and $g_1, g_2 \in Hom_{\mathcal{K}_1'} \langle Z, Z' \rangle$ such that $\mathcal{G}g_i = \varphi_i$.

(β') if g is an epimorphism with respect to $\mathcal{K}_1'$, then $\mathcal{G}g$ is an epimorphism with respect to $\mathcal{K}'$.

(β'') $\mathcal{G}f$ is an epimorphism with respect to $\mathcal{K}'$.

In the case that $X <_{g} Y'$ for some $Y' \in obj\ \mathcal{K}_1'$ the condition (α) is a consequence of (β).

(2) Let each object from $\mathcal{G}^{-1}[A]$ (which is supposed to be a non-void set) have a reflection in $\mathcal{K}_1'$. We want to know whether A has a reflection in $\mathcal{K}'$. Let, for each $\varphi : A \to B$, $B \in \mathcal{K}'$, there exist f in $\mathcal{K}_1$ such that $\mathcal{E} f \in \mathcal{K}_1'$, $\mathcal{G} f = \varphi$. Then the images under $\mathcal{G}$ of the reflections of objects from $\mathcal{G}^{-1}[A]$ form a set $\mathcal{N}_A$ with the factorization property (see the introduction). But it is possible to use also the result of (1). If $\mathcal{G}^{-1}[A]$ has the least object X_A in $<_{\mathcal{G}}$, then the condition (α) is fulfilled for $X = X_A$. Thus, if $\mathcal{G}$ satisfies a condition of the type (β), A has a reflection in $\mathcal{K}'$. In a special case we shall get the following statement:

Theorem 1. Let the class $\mathcal{G}^{-1}[A]$ be a non-void set with a least object X_A for each $A \in \operatorname{obj} \mathcal{K}$ and $X_A \in \operatorname{obj} \mathcal{K}_1'$ for $A \in \operatorname{obj} \mathcal{K}'$.

If $\mathcal{G}$ satisfies the condition (b) for the ranges (i.e. the case $\mathcal{G} X = \mathcal{E} \varphi$ only) in the definition of S-functors, then the reflectivity of $\mathcal{K}_1'$ in $\mathcal{K}_1$ implies the reflectivity of $\mathcal{K}'$ in $\mathcal{K}$.

Remark. The existence of X_A and the fulfilling of the condition (b) for the ranges is equivalent to the existence of a full embedding of $\mathcal{K}$ onto a coreflective subcategory of $\mathcal{K}_1$.

(3) In what cases the reflectivity of $\mathcal{K}'$ in $\mathcal{K}$ does imply the reflectivity of $\mathcal{K}_1'$ in $\mathcal{K}_1$? For an answer to this question it is possible to use the inductive generation in $\mathcal{K}_1'$ by means of a morphism with domain in $\mathcal{K}_1$. The exis-

tence of this generation implies the reflectivity of $\mathcal{K}_1'$ in $\mathcal{K}_1$ in our case. But we shall investigate another way using Theorem 3 from [4].

We restrict ourselves to the case $obj\ \mathcal{K} = \mathcal{G}\,[obj\ \mathcal{K}_1]$, $\mathcal{K}_1' = \mathcal{G}^{-1}[\mathcal{K}']$. If $\mathcal{K}_1$ is product-admitting and $\mathcal{G}$ preserves products, then the embedding $\mathcal{K}_1' \rightarrow \mathcal{K}_1$ preserves products too and, hence, the condition (a) of Theorem 3 from [4] is fulfilled. The verification of (b) is often very easy and we shall not deal with it. The condition (d) will be satisfied if $\mathcal{G}$ fulfils the condition (a) from the definition of the S-functors. If $\mathcal{G}^{-1}[A]$ is a set for each $A \in obj\ \mathcal{K}$, then (c) is valid and, hence, $\mathcal{K}_1'$ is reflective in $\mathcal{K}_1$. In this case $\mathcal{G}$ preserves reflections (see question (1)).

The condition (a) from the definition of the S-functors is too strong (among consequences of this condition one obtains a preservation of subobjects and of quotients). But it is possible to avoid this condition as it is shown in the following statement:

__Theorem 2__. Let $obj\ \mathcal{K} = \mathcal{G}\,[obj\ \mathcal{K}_1]$, $\mathcal{K}_1' = \mathcal{G}^{-1}[\mathcal{K}']$, let $\mathcal{K}_1$ be product-admitting and $\mathcal{K}'$ reflective in $\mathcal{K}$. Then $\mathcal{K}_1'$ is reflective in $\mathcal{K}_1$ if the following conditions are satisfied:

(a′) $\mathcal{G}$ preserves products and fulfils the condition (b) for the ranges in the definition of the S-functors;

(b′) $\bigcup\{Hom_{\mathcal{K}_1}\langle X, Y\rangle \mid Y \in obj\ \mathcal{K}_1'\} \neq \emptyset$ for each $X \in obj\ \mathcal{K}_1$;

(c′) there exists a faithful functor $\mathcal{F}$ from $\mathcal{K}$ such that $\mathcal{F} \circ \mathcal{G}$ is an S-functor;

(d′) each $f \in \mathcal{K}$ with $\mathcal{E}f \in \mathcal{K}'$ can be factorized as $f_1 \circ f_2$, where $f_1 \in \mathcal{K}'$ is projectively $\mathcal{F}$-generating (see [3]), f_2 belongs to a set $\mathcal{M}_{\mathcal{D}f}$ of epimorphisms with respect to $\mathcal{K}'$.

Proof. The conditions (a′),(b′) imply the conditions (a), (b) of Theorem 3 from [4]. We shall prove that the remaining conditions (c),(d) are also valid. Let $X \in obj\ \mathcal{K}_1$, $f: X \to Y$, $Y \in obj\ \mathcal{K}_1'$. It follows from (d′) that $\mathcal{G}f$ can be factorized as $f_1 \circ f_2$, where f_1 is projectively $\mathcal{F}$-generating and $f_2 \in \mathcal{M}_{\mathcal{G}X}$. It is possible, by (c′), to factorize f as $g_1 \circ g_2$, where g_1 is projectively $\mathcal{F} \circ \mathcal{G}$-generating and $(\mathcal{F} \circ \mathcal{G})g_i = \mathcal{F}f_i$. Since $\mathcal{G}$ fulfils the condition (b) for the ranges, it preserves projectively generating mappings and, hence, $\mathcal{G}g_1 = f_1$. Consequently, $\mathcal{G}g_2 = f_2$ and, hence, also the condition (c) is fulfilled (it follows from (c′) that $\mathcal{G}^{-1}[\mathcal{M}_{\mathcal{G}X}]$ is a set).

Remark 1) It is not necessary for $\mathcal{F} \circ \mathcal{G}$ to satisfy all the conditions of the S-functors. 2) The condition (a′) is fulfilled in the case that $\mathcal{K}$ is a coreflective subcategory of $\mathcal{K}_1$ and $\mathcal{G}$ is the right adjoint of the embedding $\mathcal{K} \to \mathcal{K}_1$. The subcategory $\mathcal{K}_1'$ of $\mathcal{K}_1$ is then composed of those objects and morphisms the coreflection in $\mathcal{K}$ of which belongs to $\mathcal{K}'$.

Examples: Theorem 2 can be used e.g. in the case that $\mathcal{K}_1$ is the category of quasi-uniform spaces (see e.g.[1] for definition), $\mathcal{K} = \mathrm{Top}$ and $\mathcal{K}'$ is the category of $\mathcal{T}_i$-spaces $(i = 0, 1, 2, 3)$.

We shall show a pattern of a construction of reflections in the case that $\mathcal{K}_1$ is the category of syntopogene spaces (see [1]), $\mathcal{K} = \mathit{Unif}$ and $\mathcal{K}'$ is the category of complete Hausdorff uniform spaces. This case is solved in [1] but that construction is rather complicated. It is very easy to use Theorem 2 and to apply it also to other definitions of completeness (not only to that used in [1]).

All the conditions of Theorem 2 are fulfilled (it is better to use Remark 2). Hence, every syntopogene space $\mathcal{P}$ has a complete separated reflection $\langle \nu \mathcal{P}, f_{\mathcal{P}} \rangle$. We want to know in what cases the mapping $f_{\mathcal{P}}$ is an embedding. Since $\nu \mathcal{P}$ is separated (in the sense of [1]), $\mathcal{P}$ must be separated too. We shall show that this is just the case. If $\mathcal{P}$ is separated, then $f_{\mathcal{P}}$ must be one-to-one (because for each $x, y \in \mathcal{P}$, $x \neq y$ there exists a continuous mapping f from $\mathcal{P}$ into a complete separated syntopogene space such that $fx \neq fy$). It follows easily from Theorem 12.41 in [1] that $f_{\mathcal{P}}$ is projectively generating (it is possible to find a continuous mapping $f_{<_0}$ from $\mathcal{P}$ into a complete separated space $\mathcal{Q}_{<_0}$ for each $<_0$ from the structure of $\mathcal{P}$ such that $<_0 \subset f_{<_0}^{-1}(<)$ for some $<$ in the structure of $\mathcal{Q}_{<_0}$). Hence $f_{\mathcal{P}}$ is an embedding.

If $\mathcal{P}$ is not separated then $f_{\mathcal{P}}$ is not one-to-one. In this case too, it is possible to find a completion with the extension property (of course, these extensions are not unique). Let us denote by $\langle \tau \mathcal{P}, \varphi_{\mathcal{P}} \rangle$ the separated reflection of $\mathcal{P}$ and define $\tilde{\nu} \mathcal{P} = \mathcal{P} \cup (\nu \tau \mathcal{P} - \tau \mathcal{P})$, $f : \mathcal{P} \rightarrow \tilde{\nu} \mathcal{P}$ to be the identity mapping on $\mathcal{P}$, $\tilde{f} : \tilde{\nu} \mathcal{P} \rightarrow \nu \tau \mathcal{P}$ to be equal to $\varphi_{\mathcal{P}}$ on $\mathcal{P}$ and being the identity mapping on

$(\nu\tau\mathcal{P}-\tau\mathcal{P})$. The structure of $\tilde{\nu}\mathcal{P}$ is that one projectively generated by $\tilde{f}$. Because the mapping $\tilde{f}\circ f = f_{\tau\mathcal{P}}\circ\varphi_{\mathcal{P}}$ is projectively generating, it follows that f is also projectively generating and, hence, an embedding. It is clear that the completion $\langle\tilde{\nu}\mathcal{P}, f\rangle$ is augmentation-separated (separated with respect to $\mathcal{P}$ in the terminology of [1]).
If $\mathcal{P}$ is separated, then $\tilde{\nu}\mathcal{P} = \nu\mathcal{P}$.

Now, we shall prove that this completion has the extension property. Let a continuous mapping g from $\mathcal{P}$ into a complete syntopogene space $\mathcal{Q}$ be given (i.e. the uniform coreflection of $\mathcal{Q}$ is complete which is equivalent to the double-completeness in the sense of [1]). We have the following commutative diagram.

$$\begin{tikzcd}
\nu\tau\mathcal{P} & \tau\mathcal{P} \arrow[l, "f_{\tau\mathcal{P}}"'] \arrow[r, "\tau g"] & \tau\mathcal{Q} \\
\tilde{\nu}\mathcal{P} \arrow[u, "\tilde{f}"] & \mathcal{P} \arrow[l, "f"] \arrow[r, "g"'] \arrow[u, "\varphi_{\mathcal{P}}"'] & \mathcal{Q} \arrow[u, "\varphi_{\mathcal{Q}}"']
\end{tikzcd}$$

There exists exactly one continuous mapping $h : \nu\tau\mathcal{P}\to\tau\mathcal{Q}$ such that $\tau g = h\circ f_{\tau\mathcal{P}}$. Now, it is easy to define a mapping $\tilde{h} : \tilde{\nu}\mathcal{P}\to\mathcal{Q}$ such that $g = \tilde{h}\circ f$, $\varphi_{\mathcal{Q}}\circ\tilde{h} = h\circ\tilde{f}$. Because $h\circ\tilde{f}$ is continuous and $\varphi_{\mathcal{Q}}$ is projectively generating, the mapping $\tilde{h}$ is continuous. This extension $\tilde{h}$ of g is unique if $\varphi_{\mathcal{Q}} = 1_{\mathcal{Q}}$ (i.e. if $\mathcal{Q}$ is separated).

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