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A NOTE ON DETERMINATION OF EIGENVALUES AND EIGENFUNCTIONS OF BOUNDED SELF-ADJOINT OPERATORS

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In [1] we gave some results concerning the determination of eigenvalues and eigenfunctions of bounded self-adjoint operators in a real Hilbert space X. In Section 1 we recall some assertions from [1]. The purpose of Section 2 of this note is to establish some estimates for the methods presented in [1].

1. Suppose that $A : X \to X$ is a linear self-adjoint positive ($(Ax, x) > 0$ for every $x \neq 0$, $x \in X$) mapping of a real Hilbert space X into X. Let $\tilde{\lambda}_1$ be the greatest element and m the smallest element of the spectrum $\sigma(A)$ of A. Denote by $\{E_\lambda\}$ the spectral family of A. If $E_\lambda x_0 \neq x_0$, $x_0 \in X$ for $\lambda < \tilde{\lambda}_1$, then $\lambda_n \to \tilde{\lambda}_1$, where

$$(1) \quad \lambda_{n+1} = (Ax_n, x_n)\|x_n\|^{-2}, \quad x_{n+1} = \lambda_{n+1}^{-1} Ax_n .$$

Suppose that $\tilde{\lambda}_1$ (not necessarily an isolated point of $\sigma(A)$) is an eigenvalue of A, $X_{\tilde{\lambda}_1}$ is the eigenspace corresponding to $\tilde{\lambda}_1$, and that the projection of $x_0 \in X$ on $X_{\tilde{\lambda}_1}$ is $\xi_1^{(0)} e_1$, where $e_1 \in X_{\tilde{\lambda}_1}$, $\|e_1\| = 1$, $\xi_1^{(0)} > 0$. Then $x_n \to N e_1$ in the norm topology of X, where $N = \sup_{n=1,2,\dots} \|x_n\| < +\infty$.

Now, if $\tilde{\lambda}_1$ is an isolated point of $\sigma(A)$ $(m \leq \lambda \leq M < \tilde{\lambda}_1)$, and $E_\lambda x_0 \neq x_0$ for $\lambda < \tilde{\lambda}_1$, then there exists a real q $(0 < q < 1)$ such that for n_0 sufficiently large, $(p = 1, 2, \dots)$

$$(2) \quad \tilde{\lambda}_1 - (Ax_{n_0+p}, x_{n_0+p})\|x_{n_0+p}\|^{-2} \leq q^{2p}(\tilde{\lambda}_1 - (Ax_{n_0}, x_{n_0})\|x_{n_0}\|^{-2}),$$

$$(3) \quad \begin{aligned} \|x_{n_0+p} - e_1 \|x_{n_0+p}\| \| &\leq \sqrt{2}\, q^p [\|x_{n_0+p}\| (\|x_{n_0}\| - \\ &- (x_{n_0}, e_1))]^{\frac{1}{2}} . \end{aligned}$$

Similar results hold for the sequence $\{y_n\}$, where

$$(4) \quad y_{n+1} = \mu_{n+1} A y_n, \quad \mu_{n+1} = (Ay_n, y_n)\|Ay_n\|^{-2} .$$

2. The inequalities (2),(3) state asymptotic estimates for (1). Using some facts from [2] we shall give estimates for finite number of steps of (1) or (4).

Suppose again that $A : X \to X$ is a linear self-adjoint positive mapping of a real Hilbert space X into X. Let $\tilde{\lambda}_1$ be the greatest and m the smallest element of the spectrum $\sigma(A)$. Suppose that $\tilde{\lambda}_1$ is an isolated point of $\sigma(A)$ $(m \leq \lambda \leq M < \tilde{\lambda}_1)$. Then $\tilde{\lambda}_1$ is an eigenvalue of A. Denote by $X_{\tilde{\lambda}_1}$ the eigenspace corresponding to $\tilde{\lambda}_1$ and e $(\|e\| = 1)$ the projection of $x_0 \bar{\in} X_2$, $x_0 \neq 0$, where X_2 is the orthogonal complement of $X_{\tilde{\lambda}_1}$. Then $X = X_{\tilde{\lambda}_1} \oplus X_2$, and for every x_n $(n = 0, 1, 2, \dots)$ defined by (1) we have a unique decomposition

$$(5) \quad x_n = \xi_n e + h_n , \quad \text{where } h_n \in X_2 \text{ and } (e, h_n) = 0.$$

Now set $\cos(x,y) = (x,y)\|x\|^{-1}\|y\|^{-1}$, $\sin(x,y) = (1 - \cos^2(x,y))^{\frac{1}{2}}$

for every $x, y \in X$. Then $\sin(x_n, e) = \|h_n\| \, \|x_n\|^{-1}$ for every n $(n = 0, 1, 2, \dots)$.

Set $x = x_n \|x_n\|^{-1}$, $h = h_n \|x_n\|^{-1}$, $x^{(1)} = x_{n+1} \|x_n\|^{-1}$. Then $x^{(1)} = (Ax, x)^{-1}(\tilde{\lambda}_1 \xi e + Ah)$, where $\xi = \xi_n \|x_n\|^{-1}$. Therefore $x^{(1)} = \alpha \xi e + g$, where $\alpha = \xi_{n+1} \|x_n\|^{-1} = \tilde{\lambda}_1 \cdot (Ax, x)^{-1}$, $g = h_{n+1} \|x_n\|^{-1} = (Ax, x)^{-1} Ah$. Since $x = \xi e + h$ and $\xi^2 = 1 - \|h\|^2$, it follows that $(Ax, x)^{-1} = (\tilde{\lambda}_1 - a)^{-1}$, where $a = ((\tilde{\lambda}_1 E - A)h, h)$, E denotes the identity mapping of X. Thus $\alpha = \tilde{\lambda}_1 (\tilde{\lambda}_1 - a)^{-1}$, $g = (\tilde{\lambda}_1 - a)^{-1} Ah$.

Now we have $D = \|g\|^2 \|x^{(1)}\|^{-2} \|h\|^{-2} =$

$$= \|g\|^2 b^{-1} \|h\|^{2} = 1 - \frac{b\|h\|^2 - \|g\|^2}{b\|h\|^2} \, ,$$

where $b = \alpha^2 \xi^2 + \|g\|^2$. Since $\xi^2 = 1 - \|h\|^2$, one has that

$$(6) \quad D = 1 - \beta(\alpha^2 - \beta)^{-1} \xi^2 \|h\|^{-2}, \text{ where } \beta = \alpha^2 \|h\|^2 - \|g\|^2 = (\tilde{\lambda}_1 - a)^{-2} (\tilde{\lambda}_1^2 \|h\|^2 - \|Ah\|^2) \geq \tilde{\lambda}_1 (\tilde{\lambda}_1 - a)^{-2} \cdot (\tilde{\lambda}_1 \|h\|^2 - (Ah, h)) = \tilde{\lambda}_1 (\tilde{\lambda}_1 - a)^{-2} ((\tilde{\lambda}_1 E - A)h, h) = \tilde{\lambda}_1 a (\tilde{\lambda}_1 - a)^{-2} \, .$$

Therefore

$$(7) \quad \beta(\alpha^2 - \beta)^{-1} \geq \tilde{\lambda}_1 a (\tilde{\lambda}_1^2 - \tilde{\lambda}_1 a)^{-1} > a \tilde{\lambda}_1^{-1} \, .$$

Since $h \in X_2$ and the spectrum $\sigma(A)$ of A in X_2 lies on the line-segment $\langle m, M \rangle$,

$$(8) \quad a = ((\tilde{\lambda}_1 E - A)h, h) \geq (\tilde{\lambda}_1 - M) \|h\|^2 \, .$$

According to (6), (7) and (8),
$D < 1 - (\xi_n \|x_n\|^{-1})^2 \, (1 - M \tilde{\lambda}_1^{-1})$. Thus we obtain the following

Theorem 1. Let $A: X \to X$ be a positive self-adjoint mapping in X. Suppose that $\tilde{\lambda}_1$ is an isolated point of $\sigma(A)$ and $x_0 \,\overline{\in}\, X_2$, $x_0 \neq 0$.
Then

(9) $\sin(x_{n+1}, e) < q_n \sin(x_n, e)$, where

(10) $q_n = [1-(\xi_n \| x_n \|^{-1})^2 (1-M\tilde{\lambda}_1^{-1})]^{\frac{1}{2}}$, $(n = 0, 1, 2, \dots)$.

Remark 1. The inequality (9) can be written in the form

(11) $\| h_{n+1} \| \| x_{n+1} \|^{-1} < q_n \| h_n \| \| x_n \|^{-1}$, $(n = 0, 1, 2, \dots)$.

The estimate (9) is not exact. A better estimate is given in the following

Theorem 2. Let the conditions of Theorem 1 be satisfied; then

(12) $\| h_n \| \| x_n \|^{-1} < q_{n-1}\, q_{n-2}\, q_{n-3} \cdots\cdots q_0 \| h_0 \| x_0 \|^{-1} (n = 1, 2, \dots)$,

where $q_{n-1} < q_{n-2} < \dots < q_0 < 1$, and q_k $(k = 0, 1, 2, \dots)$ is defined by (10).

Proof. Since $x_0 \,\overline{\in}\, X_2$, $x_0 \neq 0$, one has that $| \xi_0 | \| x_0 \|^{-1} > 0$; hence $q_0 < 1$. Because $\| h_1 \| \| x_1 \|^{-1} < \| h_0 \| \| x_0 \|^{-1}$ and $\xi_0^2 \| x_0 \|^{-2} + \| h_0 \|^2 \| x_0 \|^{-2} = \xi_1^2 \| x_1 \|^{-2} + \| h_1 \|^2 \| x_1 \|^{-2}$, we conclude that $\xi_1^2 \| x_1 \|^{-2} > \xi_0^2 \| x_0 \|^{-2}$; hence $q_1 < q_0 < 1$. Similarly $q_{n-1} < q_{n-2} < \dots < q_0 < 1$. This concludes the proof.

Remark 2. Denote by $y_n = \gamma_n e + g_n$, $(n = 0, 1, 2, \dots)$ the unique decomposition of y_n (defined by (4)), where $g_n \in X_2$. Under the assumptions of Theorem 1 we have that

(13) $\| g_n \| \| y_n \|^{-1} < r_{n-1}\, r_{n-2} \cdots\cdots r_0 \| g_0 \| \| y_0 \|^{-1}$,

where $r_{n-1} < r_{n-2} < \dots < r_0 < 1$, and r_k $(k = 0, 1, 2, \dots)$ is defined by

(14) $$r_k = [1 - (\gamma_k \| y_k \|^{-1})^2 (1 - M \tilde{\lambda}_1^{-1})]^{\frac{1}{2}} \quad .$$

A similar result also holds for Kellogg's method.

Theorem 3. Let the conditions of Theorem 1 be satisfied; then

(15) $$\tilde{\lambda}_1 - \lambda_n < q_{n-1}^2 \, q_{n-2}^2 \cdot \dots \cdot q_0^2 \, (\tilde{\lambda}_1 - m) \| h_0 \|^2 \| x_0 \|^{-2} \ ,$$

where $q_{n-1} < q_{n-2} < \dots < q_0 < 1$, and q_k $(k = 0, 1, 2, \dots)$ is defined by (10). Moreover, if $m = \inf_{\|x\| = 1} (Ax, x) > 0$, then

(16) $$\mu_n - \tilde{\lambda}_1^{-1} < r_{n-1}^2 \, r_{n-2}^2 \dots r_0^2 (M m^{-2} - \tilde{\lambda}_1^{-1}) \| \varphi_0 \|^2 \| y_0 \|^{-2} \ .$$

Proof. According to (1) and (5),

$$\tilde{\lambda}_1 - \lambda_n = [\tilde{\lambda}_1 \| x_n \|^2 - (A x_n, x_n)] \| x_n \|^{-2} =$$
$$= [\tilde{\lambda}_1 \| h_n \|^2 - (A h_n, h_n)] \| x_n \|^{-2} = ((\tilde{\lambda}_1 E - A) h_n, h_n) \| x_n \|^{-2} .$$

Since $h_n \in X_2$ and the spectrum $\sigma(A)$ of A in X_2 lies on the line-segment $\langle \tilde{\lambda}_1 - M, \tilde{\lambda}_1 - m \rangle$, one has that $\tilde{\lambda}_1 - \lambda_n < (\tilde{\lambda}_1 - m) \| h_n \|^2 \| x_n \|^{-2}$. Using Theorem 2 we obtain (15). Furthermore,

$$\mu_n - \tilde{\lambda}_1^{-1} = [(A \varphi_n, \varphi_n) - \tilde{\lambda}_1^{-1} (A^2 \varphi_n, \varphi_n)] \| A y_n \|^{-2} \le$$
$$\le ((E - \tilde{\lambda}_1^{-1} A) A \varphi_n, \varphi_n) m^{-2} \| y_n \|^{-2} \le$$
$$\le (M m^{-2} - \tilde{\lambda}_1^{-1}) \| \varphi_n \|^2 \| y_n \|^{-2} .$$

Using (13) we get (16). This concludes the proof.

R e f e r e n c e s

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