Josef Kolomý

Application of some existence theorems for the solutions of Hammerstein integral equations

Commentationes Mathematicae Universitatis Carolinae, Vol. 7 (1966), No. 4, 461--478

Persistent URL: `http://dml.cz/dmlcz/105079`

Commentationes Mathematicae Universitatis Carolinae
7,4 (1966)

APPLICATION OF SOME EXISTENCE THEOREMS FOR THE SOLUTIONS OF HAMMERSTEIN INTEGRAL EQUATIONS

Josef KOLOMÝ, Praha

The purpose of this note is to present some applications of the existence theorems [1],[2] concerning the solutions of Hammerstein equations. For recent investigations of these topics see for instance [3],[4],[5],[6],[7].

1. Throughout this paper I denotes the identity mapping of a real or complex separable and complete Hilbert space X, $X \neq (0)$. A linear continuous mapping $A: X \to X$ of a Hilbert space X into X is said to be normal if $AA^* = A^*A$, where A^* denotes the mapping adjoint to A. For the convenience of reader we recall the theorems which we shall use in the sequel.

<u>Theorem 1</u> [1]. Let $F: X \to X$ be a mapping of a Hilbert space X into X which has the Gâteaux derivative $F'(x)$ for every $x \in X$. Let $PF'(x)$ be a normal mapping for every $x \in X$ and such that $(PF'(x)h, h) \geq 0$ for every $x \in X$, $h \in X$, where P is a linear mapping of X into X having an inverse P^{-1}, $\|P\| \leq (\sup_{x \in X} \|F'(x)\|)^{-1}$. If there exist positive numbers α, γ, $\gamma < 1$ such that $\|x - PF(x)\| \leq \gamma \|x\|$ whenever $\|x\| \geq \alpha$, then the equation $F(x) = y$ has at least one solution for every $y \in X$.

Remark 1. Let $F: X \to X$ be a mapping of a Hilbert space X into X. If the number

$$|F| = \inf_{0<\rho<+\infty} \{ \sup_{\|x\| \geq \rho} \frac{\|F(x)\|}{\|x\|} \}$$

is finite, then F is linearly upper bounded (i.e. there exist $\alpha, \gamma > 0$ such that $\|F(x)\| \leq \gamma \|x\|$ whenever $\|x\| \geqq \alpha$); also conversely. It is easy to prove the

Theorem 2. Let $F: X \to X$, $P: X \to X$, $\varphi: X \to X$ be mappings of a Hilbert space X into X, φ, P be linear continuous having bounded inverses P^{-1}, φ^{-1}. Let the inequality

$$\|PF(u) - PF(v) - \varphi(u-v)\| \leq \alpha \|u - v\|$$

hold for every $u, v \in X$ with $\alpha \|\varphi^{-1}\| \leqq 1$. If there exist positive numbers α, γ, $\gamma < \|\varphi^{-1}\|$ such that $\|\varphi(u) - PF(u)\| \leq \frac{\gamma}{\|\varphi^{-1}\|} \|u\|$ whenever $\|u\| \geq \alpha$, then the equation $F(u) = y$ has at least one solution for every $y \in X$.

Theorem 3 [2]. Let $F: X \to X$ be a mapping of a real Hilbert space X into X such that for every $x \in E \subset X$ it has the Gâteaux derivative $F'(x)$ and that

(1) $\quad (P_1 F'(x)h, h) \geqq m \|h\|^2, \quad m > 0$

holds for every $x \in E, h \in X$, where E is a convex closed subset of X and P_1 is a linear continuous mapping of X into X having an inverse P_1^{-1}. Let the closed ball $D = \{x \in X: \|x - x_1\| \leq r_\vartheta\}$ be contained in E, where $x_{n+1} = x_n - PF(x_n) + Pf$, $(n = 0, 1, 2, \ldots$ $\ldots)$, $P = \vartheta P_1$, $0 < \vartheta < 2mk^{-1}$, $k = \sup_{x \in E} \|P_1 F'(x)\|^2 <$

$< +\infty$, $r_\vartheta \geq \alpha_\vartheta (1-\alpha_\vartheta)^{-1} \| x_1 - x_0 \|$, $\alpha_\vartheta = \sup_{x \in E} \| I - PF'(x) \|$, x_0 is an arbitrary element from E.

Then the equation $F(x) = f$ has a unique solution x^* in D. Furthermore, $\lim_{n \to \infty} \| x_n - x^* \| = 0$ and $\| x^* - x_n \| \leq \alpha_\vartheta^n (1 - \alpha_\vartheta)^{-1} \| x_1 - x_0 \|$. If $\vartheta = \vartheta_0 = mk^{-1}$, then $\alpha_{\vartheta_0} \leq (1 - m^2 k^{-1})^{1/2}$.

Remark 2. If (1) is fulfilled for every $x, h \in X$, then we obtain a global existence theorem. In this case the assumption $D \subset E$ is unnecessary.

According to [8], [9], [10] we can state the theorem 5 [2] in the following form.

Theorem 4. Let $F: X \to X$ be a mapping of a Hilbert space X into X such that in a convex closed bounded set $E \subset X$ it has the Gâteaux derivative $F'(x)$. Let $PF'(x)$ be a normal mapping for every $x \in E$ and such that $(PF'(x)h, h) \geq 0$ for every $x \in E$, $h \in X$, where P is a linear mapping of X into X having an inverse P^{-1} and $\| P \| \leq (\sup_{x \in E} \| F'(x) \|)^{-1}$. If $(I - PF) E \subset E$, then the equation $F(x) = 0$ has at least one solution x^* in E. Moreover, $x_n \to x$ weakly in X, where $x_{n+1} = x_n - \beta PF(x_n)$ $0 < \beta < 1$ $(n = 0, 1, 2, \dots)$ and x_0 is an arbitrary element from E.

2. Consider the equation

$$(2) \qquad x(s) - \lambda \int_G K(s,t) g(x(t), t) dt = f(s),$$

where the measurable function $K(s, t)$ is defined on

$G \times G$, G is a measurable subset of E_s (E_s denotes the euclidean s-space). Throughout this paper we assume that a linear operator A maps a real space $L_2(G)$ into itself and is defined on $L_2(G)$ by

$$Ax(t) = \int_G K(s,t)\,x(t)\,dt \,. \tag{3}$$

In the sequel, $\phi(x(t)) = g(x(t), t)$ denotes an operator of Nemytzkij (cf.[11],chapt.VI.), λ is a real parameter and $f(s)$ an arbitrary function from $L_2(G)$.

Theorem 5. Let the following conditions be fulfilled: 1° A function $g(x,t)$ measurable in $t \in G$ has a partial derivative $g'_x(x,t)$ which is continuous in $x \in (-\infty, +\infty)$ and for every $x \in (-\infty,+\infty)$ and almost every $t \in G$ there is $-N \leq g'_x(x,t) \leq M$ $(0 < N, M < +\infty,\ N, M = const.)$. 2° A linear mapping A is self-adjoint positive definite $(A \geq mI,\ m > 0)$ in $L_2(G)$ and such that $|\lambda| M \|A\| \leq 1$. 3° $|g(x,t)| \leq \leq \sum_{k=1}^{n} \varphi_k(t)|x|^{1-\alpha_k} + \psi(t)$ $(t \in G,\ x \in (-\infty,+\infty))$, where $\varphi_k(t) \in L_{\frac{2}{\alpha_k}}(G)$, $0 < \alpha_k < 1$ $(k = 1,2,\dots,n)$, $\psi(t) \in L_2(G)$.

Then the equation (2) has at least one solution in $L_2(G)$ for every $f \in L_2(G)$.

Proof. We rewrite the equation (2) in the form

$$x - \lambda A\phi(x) = f \,. \tag{4}$$

According to 2° there exist the mappings $A^{\frac{1}{2}}$, $A^{-\frac{1}{2}}$ and they are bounded self-adjoint positive definite in $L_2(G)$ and $R(A^{\frac{1}{2}}) = L_2(G)$, where $R(A^{\frac{1}{2}})$ denotes the range of $A^{\frac{1}{2}}$. Instead of (4), we shall solve the equation

$$x - \lambda A^{\frac{1}{2}} \phi(A^{\frac{1}{2}} x) = A^{-\frac{1}{2}} f \,. \tag{5}$$

Now, if x^* is a solution of (5), then $A^{\frac{1}{2}} x^*$ is a solution of (4). According to the assumption 1^o (cf.[11], § 20), the mapping $\phi : L_2(G) \to L_2(G)$ is continuous and it has a linear bounded Gâteaux differential $D\phi(x,h) = g'_x(x(t),t)h(t) = \phi'(x)h$ for every x, $h \in L_2(G)$. Therefore, if ε is an arbitrary positive number, then there exists a $\delta(\varepsilon) > 0$ such that, for every t with $|t| < \delta(\varepsilon)$ and $h \in L_2(G)$ we have $\| \frac{1}{t} \omega(A^{\frac{1}{2}}x, tA^{\frac{1}{2}}h)\| < \varepsilon \| A^{\frac{1}{2}} \|^{-1}$, where $\omega(A^{\frac{1}{2}}x, tA^{\frac{1}{2}} v) = \phi(A^{\frac{1}{2}}x + tA^{\frac{1}{2}}h) - \phi(A^{\frac{1}{2}}x) - tD\phi(A^{\frac{1}{2}}x, A^{\frac{1}{2}}h)$.
Set $Q(x) = A^{\frac{1}{2}} \phi(A^{\frac{1}{2}}x)$; then
$\frac{1}{t}[Q(x+th) - Q(x)] - A^{\frac{1}{2}} g'_x(A^{\frac{1}{2}}x, t)A^{\frac{1}{2}}h =$
$= \frac{1}{t} A^{\frac{1}{2}} \omega(A^{\frac{1}{2}}x, tA^{\frac{1}{2}}h)$.
If $|t| < \delta(\varepsilon)$, we obtain that $\| \frac{1}{t} A^{\frac{1}{2}} \omega(A^{\frac{1}{2}}x, tA^{\frac{1}{2}}h)\| < \varepsilon$.
Hence

$$\lim_{t \to 0} \frac{Q(x+th) - Q(x)}{t} = A^{\frac{1}{2}} g'_x(A^{\frac{1}{2}}x(t), t) A^{\frac{1}{2}} h(t).$$

Thus the mapping $Q : L_2(G) \to L_2(G)$ has a linear bounded Gâteaux differential

(6) $DQ(x,h) = Q'(x)h = A^{\frac{1}{2}} g'_x(A^{\frac{1}{2}}x(t), t) A^{\frac{1}{2}} h(t)$,

on the space $L_2(G)$. Moreover, assuming 2^o,
$\lambda(Q'(x)h, h) = \lambda \int_G g'_x(A^{\frac{1}{2}}x(t), t)(A^{\frac{1}{2}}h(t))^2 dt \leq$
$\leq |\lambda| M(A^{\frac{1}{2}}h, A^{\frac{1}{2}}h) = |\lambda| M(Ah, h) \leq |\lambda| M \|A\| \|h\|^2 \leq \|h\|^2$

for every x, $h \in L_2(G)$. Thus $(F'(x)h, h) \geq 0$ for every x, $h \in L_2(G)$. According to 1^o and [11, § 20.2]], cf. also [12], § 5, we see that $D\phi(x,h)$ is continuous in $x \in L_2(G)$. But $\phi(x)$ is a potential operator in $L_2(G)$, $\phi(x) = \operatorname{grad} \varphi(x)$, where

$$\varphi(x) = \varphi_0 + \int_G dt \int_0^{x(t)} g(u,t)\,du .$$

Using Theorem 5.1 [11, § 5] we get that

$$(D\phi(x,h_1),h_2) = (D\phi(x,h_2),h_1).$$

Hence

$$(A^{\frac12} D\phi(A^{\frac12}x, A^{\frac12}h_1),h_2) = (D\phi(A^{\frac12}x, A^{\frac12}h_1), A^{\frac12}h_2) =$$
$$= (D\phi(A^{\frac12}x, A^{\frac12}h_2), A^{\frac12}h_1) = (A^{\frac12} D\phi(A^{\frac12}x, A^{\frac12}h_2), h_1) =$$
$$= (h_1, A^{\frac12} D\phi(A^{\frac12}x, A^{\frac12}h_2)).$$

Thus $DQ(x,h) = Q'(x)h$ is a self-adjoint operator in $L_2(G)$ for every $x \in L_2(G)$. Hence $F'(x)$ is a self-adjoint mapping for every $x \in L_2(G)$.

By (6) and according to [13,pp.250],

$$\|Q'(x)h\|^2 \leq \|A\| \, \|g'_x(A^{\frac12}x(t),t)A^{\frac12}h\|^2 \leq$$
$$\leq N_1^2 \, \|A\|^2 \, \|h\|^2 ,$$

where $N_1 = \max(M,N)$. Hence $\sup_{x \in L_2} \|Q'(x)\| \leq N_1 \|A\|$.

Using 3^0 we have

$$\frac{\|\phi(x)\|}{\|x\|} \leq \|x\|^{-1}\Big\{\sum_{k=1}^{n} \Big(\int_G \varphi_k^{\frac{2}{\alpha_k}}(t)\,dt\Big)^{\frac{\alpha_k}{2}} \|x\|^{1-\alpha_k} + \|\psi\|\Big\}.$$

Thus ϕ is asymptotically close to zero and is bounded on $L_2(G)$ (cf.[11], chapt.VI). According to lemma 1 [14], $\phi(A^{\frac12})$ and obviously also $A^{\frac12}\phi(A^{\frac12})$ are asymptotically close to zero. Set $P = \vartheta I$, where ϑ is a fixed number satisfying the inequality $0 < \vartheta < (1+|\lambda|\,N_1\,\|A\|)^{-1}$. Then $\|x - \vartheta F(x)\| \leq (1-\vartheta)\|x\| + \vartheta|\lambda|\,\|A^{\frac12}\phi(A^{\frac12}x)\|$. Taking $0 < \varepsilon < \vartheta$, there exists a positive number N_2 such that, for every $x \in L_2$ with $\|x\| \geq N_2$, we have

$\vartheta|\lambda| \, \| A^{\frac{1}{2}} \phi(A^{\frac{1}{2}} x) \| < \varepsilon \, \| x \|$. Clearly, for every $x \in L_2$ with $\| x \| \geq N_2$ there is $\| x - \vartheta F(x) \| \leq \gamma \, \| x \|$, where $\gamma = 1 - \sigma + \varepsilon < 1$. According to Theorem 1, the equation (5) has at least one solution x^* in $L_2(G)$. Therefore $A^{\frac{1}{2}} x^*$ is a solution of (4). This proves the existence and, thus, concludes the proof of the theorem.

Corollary 1. Under the assumptions 1°, 3° of Theorem 1, let A be a self-adjoint positive ($(Ax, x) \geq 0$ for every $x \in L_2$) mapping in $L_2(G)$ such that $M \| A \| \leq 1$. Then the equation

$$x(s) - \int_G K(s,t) \, g(x(t),t) \, dt = 0$$

has at least one solution in $L_2(G)$.

Theorem 6. Let the following conditions be fulfilled: 1° A is a positive ($(Ax, x) \geq 0$ for every $x \in L_2(G)$ self-adjoint mapping from $L_2(G)$ into $L_2(G)$ such that $\operatorname*{vrai\,sup}_{s,t \in G} |K(s,t)| = d^2 < +\infty$, where G is a subset of E_n with $mes(G) < +\infty$. 2° The function $g(x,t)$ measurable in $t \in G$ has a continuous partial derivative $g'_x(x,t)$ in $x \in \langle -c, c \rangle$, $(c > 0)$ and for every $x \in \langle -c, c \rangle$ and almost every $t \in G$ there is $0 \leq g'_x(x,t) \leq M < +\infty$ and $g(0,t) \in L_2(G)$.

Then the equation

$$x(s) + \mu \int_G K(s,t) \, g(x(t),t) \, dt = 0 \tag{7}$$

with $0 < \mu < R \| A^{\frac{1}{2}} \phi(0) \|^{-1}$, $\phi(0) = g(0,t)$, has at least one solution x^* in $A^{\frac{1}{2}}(D_R) \subset L_2(G)$, where $D_R = \{ x \in L_2 : \| x \| \leq R, \, R = c d^{-1} \}$. Moreover, if

$$0 < \vartheta < Min(k^{-1}, 2Ra\,b^{-1}), \tag{8}$$

where $k = (1+\mu M\|A\|)^2$, $a = R-\mu\|A^{\frac{1}{2}}\phi(0)\|$, $b = R^2k-$ $-\mu^2\|A^{\frac{1}{2}}\phi(0)\|^2$, then $\lim_{n\to\infty}\|x_n - x^*\| = 0$,
where

(9) $x_n = A^{\frac{1}{2}}\tilde{x}_n$, $\tilde{x}_{n+1} = (1-\vartheta)\tilde{x}_n - \mu\vartheta A^{\frac{1}{2}}\phi(A^{\frac{1}{2}}\tilde{x}_n)$, $\tilde{x}_0 = 0$
and $\|x_n - x^*\| \leq \mu\vartheta\tilde{\alpha}_\vartheta^n(1-\tilde{\alpha}_\vartheta)^{-1}\|A\|\|\phi(0)\|$ with $0<\tilde{\alpha}_\vartheta<1$.

Proof. By the Golomb-Vajnberg theorem [11, lemma 24.1] we have that $\operatorname{vrai}\sup_{t\in G}|A^{\frac{1}{2}}x| \leq d\|x\|$ for every $x\in L_2(G)$, where $A^{\frac{1}{2}}$ denotes a positive square root of A. Thus for every $x\in D_R$, $R = cd^{-1}$ there is $\operatorname{vrai}\sup|A^{\frac{1}{2}}x| \leq c$ and

(10) $$0 \leq g'_x(A^{\frac{1}{2}}x(t),t) \leq M < +\infty .$$

Using 2° we have
$|g(x,t)| \leq M|x| + |g(0,t)| \leq 2cM + |g(0,t)| = \varphi(t)\in L_2(G)$,
for every $x\in\langle -c, c\rangle$ and almost every $t\in G$. Thus we can extend the function $g(x,t)$ outside the line-segment $\langle -c, c\rangle$ in such a manner that, according to [11, theorem 19.2], $\phi(x) = g(x(t),t)$ is a continuous mapping from $L_2(G)$ into itself. Set $Q(x) = A^{\frac{1}{2}}\phi(A^{\frac{1}{2}}x)$. By (10) and using the results of [11, § 20] we see that the mapping $Q: L_2\to L_2$ has, for every $x\in D_R$, a linear bounded Gâteaux differential
$DQ(x,h) = Q'(x)h = A^{\frac{1}{2}}g'_x(A^{\frac{1}{2}}x,t)A^{\frac{1}{2}}h$, $h\in L_2(G)$.
Put $F(x) = x + \mu Q(x)$. Then for every $x\in D_R$ and $h\in L_2(G)$
$(F'(x)h,h) = \|h\|^2 + \mu\int_G g'_x(A^{\frac{1}{2}}x,t)(A^{\frac{1}{2}}h)^2dt \geq \|h\|^2$,
and $\sup_{x\in D_R}\|F'(x)\|^2 \leq (1+\mu M\|A\|)^2$. We shall use theorem 3 with $E = D_R$, $\tilde{x}_0 = 0$, $m = 1$, $P_1 = I$ and

$k = (1+\mu M \| A \|)^2$. It remains to prove that $D_{r_\vartheta} = \{x \in L_2 : \| x - \tilde{x}_1 \| \leqq r_\vartheta\} \subset D_R$, where $\tilde{x}_1 = -\mu \vartheta A^{\frac{1}{2}} \phi(0)$, $r_\vartheta = \alpha_\vartheta (1-\alpha_\vartheta)^{-1} \mu \vartheta \| A^{\frac{1}{2}} \phi(0) \|$, $\alpha_\vartheta = \sup_{x \in D_R} \| I - \vartheta F'(x) \| \leqq (1-2\vartheta+\vartheta^2 k)^{1/2} = \tilde{\alpha}_\vartheta < 1$ (cf. [2] . Since $\| \tilde{x}_1 \| = \vartheta \mu \| A^{\frac{1}{2}} \phi(0) \| < \vartheta R < R$, then $\tilde{x}_1 \in D_R$. According to (8) we have that

$$\vartheta R^2 k - \mu^2 \vartheta \| A^{\frac{1}{2}} \phi(0) \|^2 \leqq 2R^2 - 2R \mu \| A^{\frac{1}{2}} \phi(0) \| .$$

Add R^2 and multiply by ϑ . Then

$R^2 \tilde{\alpha}_\vartheta^2 \leqq (R - \mu \vartheta \| A^{1/2} \phi(0) \|)^2$. Hence $\mu \vartheta \| A^{\frac{1}{2}} \phi(0) \| \leqq R(1-\tilde{\alpha}_\vartheta)$, and therefore

$$\mu \vartheta \| A^{\frac{1}{2}} \phi(0) \| (\tilde{\alpha}_\vartheta (1-\tilde{\alpha}_\vartheta)^{-1} + 1) \leqq R .$$

Thus we have $\| \tilde{x}_1 \| + \tilde{r}_\vartheta \leqq R$ with $r_\vartheta \leqq \tilde{r}_\vartheta$, where $\tilde{r}_\vartheta = \mu \vartheta \tilde{\alpha}_\vartheta (1-\tilde{\alpha}_\vartheta)^{-1} \| A^{\frac{1}{2}} \phi(0) \|$. Hence $D_{r_\vartheta} \subset D_R$. According to theorem 3, the equation

$$x + \mu A^{\frac{1}{2}} \phi(A^{\frac{1}{2}} x) = 0$$

has a unique solution x_1^* in D_{r_ϑ} and also in $D_{\tilde{r}_\vartheta}$. Moreover, $\lim_{n \to \infty} \| \tilde{x}_n - x_1^* \| = 0$, where $\{\tilde{x}_n\}$ is defined by (9) and $\| \tilde{x}_n - x_1^* \| \leqq \mu \vartheta \alpha_\vartheta^n (1-\alpha_\vartheta)^{-1} \| A^{\frac{1}{2}} g(0,t) \|$. Set $x^* = A^{\frac{1}{2}} x_1^*$, so that x^* is a solution of (7) and $x^* \in A^{\frac{1}{2}}(D_R) \subset L_2(G)$. Now

$$\| x^* - x_n \| = \| A^{\frac{1}{2}} x_1^* - A^{\frac{1}{2}} \tilde{x}_n \| \leqq \| A^{\frac{1}{2}} \| \, \| \tilde{x}_n - x_1^* \| \leqq$$
$$\leqq \mu \vartheta \tilde{\alpha}_\vartheta^n (1-\tilde{\alpha}_\vartheta)^{-1} \| A \| \, \| g(0,t) \| .$$

The proof is complete.

<u>Corollary 2</u>. Let the following conditions be fulfilled:

1° A is a positive self-adjoint continuous mapping from $L_2(G)$ into itself, where G is a measurable subset of E_k.

2° A function $g(x,t)$ measurable in $t \in G$ has a continuous partial derivative $g'_x(x, t)$ in $x \in$ $\in(-\infty,+\infty)$ and for every $x \in (-\infty, +\infty)$ and almost every $t \in G$ there is $0 \leqq g'_x(x, t) \leqq M < +\infty$, $(M = const.)$.

Then the equation (7) has at least one solution x^* in $L_2(G)$ for every μ $(0 < \mu < \infty$. Moreover, if $0 < \vartheta < 2(1+\mu M \| A \|)^2$, then $x_n \to x^*$ in the norm topology of $L_2(G)$ with the rate of a geometric sequence, where $x_n = A^{\frac{1}{2}}\tilde{x}_n$, $\tilde{x}_{n+1} = (1-\vartheta)\tilde{x}_n - \mu\vartheta A^{\frac{1}{2}}\phi(A^{\frac{1}{2}} x_n)$ and x_0 is an arbitrary element of $L_2(G)$.

Theorem 7. Let the following conditions be fulfilled: 1° A function $g(x,t)$ measurable in $t \in G$ has a continuous partial derivative $g'_x(x, t)$ in $x \in$ $\in \langle -c, c \rangle, (c > 0)$, and for every $x \in \langle -c, c \rangle$ and almost every $t \in G$ with $mes(G) < +\infty$ there is $-N \leqq g'_x(x,t) \leqq M$, $(0 < N, M < +\infty$, M, N are constants) and $g(0, t) \in L_2(G)$. 2° A is a positive definite $(A \geqq m I, m > 0)$ self-adjoint mapping from $L_2(G)$ into itself such that $\operatorname{vrai\,sup}_{s,t \in G} |K(s,t)| = d^2 < +\infty$ and $|\lambda| \, \| A \| M < 1$. If

$$(11) \qquad 0 < \vartheta < Min(m_1 k^{-1}, 2Ra b^{-1}),$$

where $m_1 = 1 - |\lambda| M \| A \|$, $k^{\frac{1}{2}} = 1 + N_1 \| A \|$, $N_1 = max(N, M)$, $R = cd^{-1}\sqrt{m}$, $a = Rm_1 - \| \lambda A \phi(0) \|$,

$b = kR^2 - \| \lambda A \phi(0) \|^2$, $\phi(0) = g(0, t)$, then the equation

$$x(s) - \lambda \int_G K(s,t) g(x(t), t) dt = 0$$

with $|\lambda| < \dfrac{Rm_1}{\| A\phi(0) \|}$ has a unique solution x^* in

the ball D_{R_ϑ}, where $D_{R_\vartheta} = \{x : \|x - x_1\| \leq R_\vartheta\}$
$x_1 = \vartheta\lambda A\phi(0)$, $R_\vartheta = \tilde{\alpha}_\vartheta(1-\tilde{\alpha}_\vartheta)^{-1}\|x_1\|$, $\tilde{\alpha}_\vartheta = 1 - 2m_1\vartheta + \vartheta^2 k$.
Moreover, $\lim_{n\to\infty}\|x_n - x^*\| = 0$, $\|x_n - x^*\| \leq$
$\leq \vartheta\tilde{\alpha}_\vartheta^m(1-\tilde{\alpha}_\vartheta)^{-1}|\lambda|\|A\phi(0)\|$, where $x_{n+1} = x_n -$
$-\vartheta(x_n - \lambda A\phi(x_n))$, $(n = 0,1,2,\dots)$ and $x_0 = 0$.

Proof. Since $A \geq mI$, $(m > 0)$, then $A^{\frac{1}{2}} \geq m^{\frac{1}{2}} I$ and $\sqrt{m} \leq \|A^{\frac{1}{2}}x\|\,\|x\|^{-1}$ for every $x \in L_2(G)$. Hence $\sqrt{m} \leq \inf_{x\neq 0}\|A^{\frac{1}{2}}x\|\,\|x\|^{-1} = m(A^{\frac{1}{2}})$.
But

$$\|A^{-\frac{1}{2}}\| = \frac{1}{m(A^{\frac{1}{2}})} \leq \frac{1}{\sqrt{m}} .$$

Since $R(A^{\frac{1}{2}}) = L_2(G)$, then according to the Vajnberg-Golomb theorem for every $x \in L_2(G)$ there is

$$\operatorname*{vrai\,sup}_{t\in G}|x(t)| = \operatorname*{vrai\,sup}_{t\in G}|A^{\frac{1}{2}}(A^{\frac{1}{2}}x)| \leq$$
$$\leq d\,\|A^{-\frac{1}{2}}x\| \leq \frac{d}{\sqrt{m}}\|x\| .$$

Denote $D_R = \{x : \|x\| \leq R\}$, where $R = cd^{-1}\sqrt{m}$. Then for every $x \in D_R \subset L_2(G)$ we have that $\operatorname*{vrai\,sup}_{t\in G}|x(t)| \leq c$. Thus for every $x \in D_R$ we get

(12) $$-N \leq g'_x(x(t), t) \leq M .$$

Since
$|g(x,t)| \leq N_1|x| + |g(0,t)| \leq 2cN_1 + |g(0,t)| = \varphi(t) \in L_2(G)$
for every $x \in \langle -c, c\rangle$ and almost every $t \in G$, we extend the function $g(x,t)$ outside the line-segment $\langle -c, c\rangle$ in such a manner that, according to [11, theorem 19.2], $\phi(x) = g(x(t),t)$ is a continuous mapping from $L_2(G)$ into $L_2(G)$. In view of 1^0, 2^0 and (12) we have that $(F'(x)h, h) \geq m_1\|h\|^2$,
$\|F'(x)\| \leq 1 + N_1\|A\|$ for every $x \in D_R$ and

$h \in L_2(G)$. Since $\vartheta < 1, m_1 < 1$, then $\|x_1\| = \vartheta \|\lambda A \phi(0)\| < R m_1 \vartheta < R$. Therefore $x_1 \in D_R$. Using (11) it is easy to show that $D_{R_\vartheta} \subset D_R$. The assertions of our Theorem follow from Theorem 3. This concludes the proof.

Remark 3. Under the assumptions of Theorem 6 let be a positive definite $(A \geq mI, m > 0)$ operator in $L_2(G)$. Then the equation (7) has a unique solution x^* in the ball $D_{\tilde{\kappa}_\vartheta} = \{x : \|x - x_1\| \leq \tilde{\kappa}_\vartheta\}$, where $x_1 = \mu\vartheta A\phi(0)$, $\tilde{\kappa}_\vartheta = \mu\vartheta\tilde{\alpha}_\vartheta(1-\tilde{\alpha}_\vartheta)^{-1}\|A\phi(0)\|$, $\tilde{\alpha}_\vartheta = 1-2\vartheta+ +\vartheta^2 k$, $k = (1+\mu M\|A\|)^2$ and ϑ satisfies $0 < \vartheta < Min(k^{-1}, 2Rab^{-1})$, $a = R-\mu\|A\phi(0)\|$, $b = kR^2-\mu^2\|A\phi(0)\|^2$, $R = cd^{-1}\sqrt{m}$.
Moreover,
$\lim_{n\to\infty}\|x_n - x^*\| = 0$, $\|x_n - x^*\| \leq \mu\vartheta\tilde{\alpha}_\vartheta^n(1-\tilde{\alpha}_\vartheta)^{-1}\|A\phi(0)\|$,
where $x_{n+1} = x_n - \vartheta(x_n + \mu\int_G K(s,t)g(x_n(t),t)dt)$, $x_0 = 0, (n = 0,1,2,\ldots)$.

Corollary 3. Let the following conditions be fulfilled: 1° A function $g(x,t)$ measurable in $t \in G$ has a continuous partial derivative $g'_x(x,t)$ in $x \in (-\infty,+\infty)$ and for every $x \in (-\infty,+\infty)$ and almost every $t \in G$ there is $-N \leq g'_x(x,t) \leq M$, where M, N are constants, $0 < M, N < +\infty$ and G is a measurable subset of E_p. 2° A is a positive self-adjoint mapping from $L_2(G)$ into itself and such that $|\lambda| M\|A\| < 1$.

Then the equation (2) with $f \in L_2(G)$ has a unique solution x^* in $L_2(G)$. Moreover, if $0 < \vartheta < 2mk^{-1}$, where $m = 1-|\lambda|M\|A\|$, $k = (1+\|A\|N_1)^2$, $N_1 = Max(M,N)$, then $\lim_{n\to\infty}\|x_n - x^*\| = 0$, where
$x_{n+1}(s) = x_n(s) - \vartheta(x_n(s) - \lambda\int_G K(s,t)g(x_n(t),t)dt) + \vartheta f(s)$

and x_0 is an arbitrary element from $L_2(G)$. The error $\| x^* - x_n \|$ satisfies $\| x_n - x^* \| \leq \alpha_\vartheta^n (1-\alpha_\vartheta)^{-1} \| x_1 - x_0 \|$ with $0 < \alpha_\vartheta < 1$. If $\vartheta = \vartheta_0 = m k^{-1}$, then $\alpha_{\vartheta_0} \leq$ $\leq (1 - m^2 k^{-1})^{\frac{1}{2}}$.

For the result of Corollary 3 compare [11,th.10.2] and [15].

In the sequel we use the following

Lemma 1 [11,§ 19]. If $\phi(x) = g(x(t), t)$ is an operator of Nemytzkij and $\phi : L_2(G) \to L_2(G)$, then

$$(13) \qquad \int_G |g(x(t), t)|^2 \leq c_1 + b_1 \int_G |x(t)|^2 dt,$$

where c_1, b_1 are constants, $c_1, b_1 \geq 0$.

Let us solve the equation (2) with $\lambda = 1$, $f(s) = 0$. We establish the following

Theorem 8. Let the following conditions be fulfilled: 1^0 A function $g(x, t)$ measurable in $t \in G$ has a continuous partial derivative in $x \in (-\infty, +\infty)$ and for every $x \in (-\infty, +\infty)$ and almost every $t \in G$ (G is a measurable subset of E_s) there is $-N \leq$ $\leq g'_x(x, t) \leq M$ for constants M, N, $0 < M, N < +\infty$. 2^0 A is a positive self-adjoint mapping from $L_2(G)$ into itself and such that $M \| A \| \leq 1$. 3^0 $\frac{1}{2} g(x,t) x \leq$ $\leq a x^2 + b(t) |x|^{2-\gamma} + c(t)$, $(x \in (-\infty, +\infty)$, $t \in G)$, where $0 < \gamma < 2$, $b(t) \in L_{\frac{2}{\gamma}}(G)$, $c(t) \in L_1(G)$, $a < \frac{1}{2} \| A \|^{-1}$.

Then the equation

$$(14) \qquad x(s) - \int_G K(s,t) g(x(t), t) dt = 0$$

has at least one solution x^* in $L_2(G)$. Moreover, if

$$(15) \quad 0 < \vartheta < \mathrm{Min}\left((1 + N_1 \| A \|)^{-1}, \frac{2(1 - a \| A \|)}{1 + b_1 \| A \|^2 - 2a \| A \|} \right),$$

where $N_1 = Max(M, N)$, b_1 is the constant from lemma 1, then the sequence $\{x_n\}$, where $x_n = A^{\frac{1}{2}}\tilde{x}_n$, $\tilde{x}_{n+1} = \tilde{x}_n - \beta\vartheta(\tilde{x}_n - A^{\frac{1}{2}}\phi(A^{\frac{1}{2}}\tilde{x}_n)), (n = 0, 1, 2, \dots)$, $0 < \beta < 1$, $\tilde{x}_0 = 0$, converges weakly in $L_2(G)$ to any solution of (14).

<u>Proof</u>. We shall solve the equation

$$x - A^{\frac{1}{2}}\phi(A^{\frac{1}{2}}x) = 0, \tag{16}$$

where $A^{\frac{1}{2}}$ denotes a positive square root of A. Set $Q(x) = A^{\frac{1}{2}}\phi(A^{\frac{1}{2}}x)$, $F(x) = x - A^{\frac{1}{2}}\phi(A^{\frac{1}{2}}x)$, $P = \vartheta I$, where ϑ satisfies (15). The mapping $Q(x)$ is continuous from $L_2(G)$ into itself and for every $x \in L_2(G)$ has a linear bounded Gâteaux differential

$DQ(x,h) = Q'(x)h = A^{\frac{1}{2}}g'_x(A^{\frac{1}{2}}x,t)A^{\frac{1}{2}}h$, $h \in L_2(G)$.

Furthermore, $(Q'(x)h, h) = (g'_x(A^{\frac{1}{2}}x,t)A^{\frac{1}{2}}h, A^{\frac{1}{2}}h) \leqq$
$\leqq M\int_G (A^{\frac{1}{2}}h)^2 dt \leqq M\|A\|\,\|h\|^2 \leqq \|h\|^2$.

Hence $(PF'(x)h, h) \geqq 0$ for every $x \in L_2(G)$ and $h \in L_2(G)$. Clearly, $PF'(x)$ is a self-adjoint mapping in $L_2(G)$. We shall apply Theorem 4. It is sufficient to prove that the mapping $\psi = (1-\vartheta)I + \vartheta A^{\frac{1}{2}}\phi(A^{\frac{1}{2}})$ maps some closed ball $D = \{x \in L_2(G): \|x\| \leqq \rho\}$ into itself (ϑ is a number satisfying (15)). If $x \in D$, then

$$\|\psi(x)\| \leqq (1-\vartheta)^2\rho^2 + 2\vartheta(1-\vartheta)\int_G g(A^{\frac{1}{2}}x,t)A^{\frac{1}{2}}x(t)\,dt + \vartheta^2\|A^{\frac{1}{2}}\phi(A^{\frac{1}{2}}x)\|^2 .$$

Assuming 3°, we have that

$$\int_G g(A^{\frac{1}{2}}x,t)A^{\frac{1}{2}}x(t)\,dt \leqq a\|A^{\frac{1}{2}}x\|^2 + \int_G b(t)|A^{\frac{1}{2}}x(t)|^{2-\gamma}dt + \int_G c(t)\,dt \leqq$$

$$\leqq a \| A \| \varrho^2 + b \| A^{\frac{1}{2}} \|^{2-\gamma} \varrho^{2-\gamma} + c ,$$

where $b = (\int_G |b(t)|^{\frac{2}{\gamma}} dt)^{\frac{\gamma}{2}}$, $c = \int_G c_1(t)\, dt$.

Since ϕ is a continuous mapping and $\phi : L_2(G) \to L_2(G)$ [11, § 20], then according to lemma 1

$$\| A^{\frac{1}{2}} \phi(A^{\frac{1}{2}} x \|^2 \leqq \| A \| \, \| \phi(A^{\frac{1}{2}} x)\|^2 \leqq \| A \| \, (c_1 + b_1 \| A^{\frac{1}{2}} x \|^2)$$
$$\leqq \| A \| \, (c_1 + b_1 \| A \| \varrho^2)$$

for every $x \in D$. We can now infer that, for every $x \in D, \|\psi(x)\|^2 \leqq d\varrho^2 + 2\vartheta(1-\vartheta) b \| A^{\frac{1}{2}} \|^{2-\gamma} \cdot \varrho^{2-\gamma} + c_2$, where $d = (1-\vartheta)^2 + 2\vartheta(1-\vartheta) a \| A \| + \vartheta^2 b_1 \| A \|$, $c_2 = 2\vartheta(1-\vartheta)c + \vartheta^2 \| A \| c_1$. *Because* $a < \frac{1}{2} \| A \|^{-1}$, there is $1 - a \| A \| > 0$, $1 + b_1 \| A \|^2 - 2a \| A \| > 0$. According to (15) we obtain that $d < 1$. Now, let ϱ be a positive number such that $d\varrho^2 + 2\vartheta(1-\vartheta) b \| A^{\frac{1}{2}} \|^{2-\gamma} \varrho^{2-\gamma} + c_2 < \varrho^2$. Then $\psi(D) \subset D$. Hence the equation (16) has at least one solution x_1^* in $L_2(G)$, and thus $x^* = A^{\frac{1}{2}} x_1^*$ is a solution of (14) in $L_2(G)$. The second part of our theorem is evident. This completes the proof.

Now we shall state a theorem concerning the solution of Hammerstein equations of the first kind. Consider the equation

$$A\phi(x) = f , \tag{17}$$

where f is an arbitrary element from $L_2(G)$ (G is a measurable subset of E_s).

<u>Theorem 9</u>. Let the following conditions be fulfilled: 1° A function $g(x, t)$ measurable in $t \in G$ has a

continuous partial derivative $g'_x(x, t)$ in $x \in$ $\in(-\infty,+\infty)$ and for every $x \in (-\infty, +\infty)$ and almost every $t \in G$ there is $0 \leqq g'_x(x, t) \leqq M < +\infty$ (M is a constant). 2° A linear continuous self-adjoint mapping A maps $L_2(G)$ into itself and $(Ax, x) \geqq$ $\geqq m \|x\|^2$, $(m > 0)$ holds for every $x \in L_2(G)$. 3° $\frac{1}{2} g(x,t)x \geqq a x^2 - b(t)|x|^{2-\gamma} - c(t), (x \in (-\infty,+\infty), t \in G)$, where $0 < \gamma < 2, b(t) \in L_{\frac{2}{\gamma}}(G), c(t) \in L_1(G), a > 0$.

Then the equation

(18) $$\int_G K(s,t)\, g(x(t),t)\,dt = f(s)$$

has at least one solution x^* in $L_2(G)$.

Remark. For some result concerning the solution of the equations of the first kind see [11, § 11]. For A one may set in theorems of § 2 an integral operator with the Carlemann kernel.

References

[1] J. KOLOMÝ: Some existence theorems for nonlinear problems. Comment.Math.Univ.Carolinae 7(1966), 207-217.

[2] J. KOLOMÝ: Remark to the solution of nonlinear functional equations in Banach spaces. Comment. Math.Univ.Carolinae 5(1964),97-116.

[3] М.М. ВАЙНБЕРГ, Р.И. КАЧУРОВСКИЙ: К вариационной теории нелинейных операторов и уравнений. Докл.Ак.н.СССР,т.29(1959),№ 6,1199-1202.

[4] WANG-SHENG-WANG, LOU Yu-TONG, FENG BAO-QI: On the solutions of the Hammerstein integral equations.III.Chinese Math.Vol.6(1965),No 3, 363-374.

[5] C.L. DOLPH, G.J. MINTY: On nonlinear integral equations of the Hammerstein type. Nonlinear integral equations. Proceedings of an Advanced Seminar, 1963. The University of Wisconsin Press, 1964, 99-154.

[6] I.I. KOLODNER: Equations of Hammerstein type in Hilbert space. Journ. of Math. and Mech. Vol. 13(1964), 701-750.

[7] М.М. ВАЙНБЕРГ, И.В. ШРАГИН: Оператор Гаммерштейна в пространствах Орлича I, II. Изв. в. уч. зав., Математика 1965, № 1(44), 17-27, № 3 (46), 32-37.

[8] М.С. БРОДСКИЙ, Д.И. МИЛЬМАН: О центре выпуклого множества. Докл. ак. н. СССР т.59(1948), № 5, 837-840.

[9] F.E. BROWDER: Fixed-point theorems for noncompact mappings in Hilbert space. Proc. Nat. Acad. Sci. U.S.A. 53(1965), 1272-1276.

[10] D. GÖHDE: Zum Prinzip der kontraktiven Abbildung. Math. Nachr. 30(1965), 251-258.

[11] М.М. ВАЙНБЕРГ: Вариационные методы исследования нелинейных операторов. Москва 1956.

[12] М.М. ВАЙНБЕРГ: Некоторые вопросы дифференциального исчисления в линейных пространствах. Усп. мат. наук Т. VII, (1952), в.4, 55-102.

[13] A.E. TAYLOR: Introduction to Functional Analysis. Wiley, New York, 1958.

[14] M.D. GEORGE: Completely well posed problems for non-

linear differential equations.Proc.Am.Math. Soc.15(1964),No 1,96-100.

[15] М.М. ВАЙНБЕРГ: О минимуме выпуклых функционалов. Усп.мат.наук,20(121),№ 1, 239-240,(1965).

[16] М.М. ВАЙНБЕРГ: О сходимости процесса наискорейшего спуска для нелинейных уравнений. Сиб.мат.ж. т. II (1961),№ 2, 201-220.

[17] М.А. КРАСНОСЕЛЬСКИЙ: Топологические методы в теории нелинейных интегральных уравнений. Москва 1956.

(Received May 2, 1966)