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ONE GENERALIZATION OF PLANAR PARTITIONS OF ABELIAN GROUPS

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In the present paper, partitions of Abelian groups are investigated which correspond naturally to certain André parallel systems closely related to translation planes. We shall endow these partitions with coordinatizing systems having two compositions which are shown to be quasifields (the case of translation planes is characterized by the well-known "planarity" condition).

Definition 1. A partition $\mathbb{P}$ of a nonzero [1] Abelian group $(S,+)$ is a set of nonzero subgroups $(S_\iota, +)$, $\iota \in \mathcal{J}$ in $(S,+)$ (called components) where

i) $card\ \mathcal{J} > 1$,

ii) $S_{\iota_1} \cap S_{\iota_2} = \{0\}$ for $\iota_1 \neq \iota_2$,

iii) $\bigcup_{\iota \in \mathcal{J}} S_\iota = S$.

The pair $\{S_\alpha, S_\beta\}$ is said to be generating if $S_\alpha + S_\beta = S$. If there exist pairwise distinct indices $\alpha, \beta, \gamma \in \mathcal{J}$ such that all $\{S_\alpha, S_\lambda\}$, $\lambda \in \mathcal{J}\setminus\{\alpha\}$ and also $\{S_\beta, S_\gamma\}$ are generating pairs, then $\mathbb{P}$ will be called a τ-partition. If each pair $\{S_{\iota_1}, S_{\iota_2}\}$ with $\iota_1 \neq \iota_2$ is generating,

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(1) A group $(S,+)$ is said to be nonzero if $card\ S > 1$. The neutral element of an additive group will be denoted by 0.

then $\mathbb{P}$ is called a planar partition [2].

Proposition 1 (André). Let $\mathbb{P}=\{(S_\iota,+)\mid \iota\in\mathcal{J}\}$ be a partition of a nonzero Abelian group $(S,+)$. Then

i) $\{\{(S_\iota+x \mid x\in S\}\mid \iota\in\mathcal{J}\}$ is a decomposition [3] of the set $\{S_\iota+x \mid \iota\in\mathcal{J}, x\in S\}$ [4],

ii) $card\,(S_\iota+x) > 1$ for all $\iota\in\mathcal{J}$, $x\in S$,

iii) there are three elements of S which do not belong to the same $S_\iota+x$,

iv) if $\tau_a : x\to x+a$ is a (left) translation of $(S,+)$, then $\tau_a(S_\iota+x)=S_\iota+(x+a)$ holds for all $\iota\in\mathcal{J}$, $a\in S$, $x\in S$ and $\{\tau_a \mid a\in S\}$ acts simply transitively on S,

v) if $\{S_\alpha, S_\beta\}$ is a generating pair, then $card\,((S_\alpha+x)\cap(S_\beta+y))=1$ for all $x, y\in S$.

Proof: [1], pp.163-165 and [2], pp.156-158.

Definition 2. A quasifield is a triplet $(F,+,\cdot)$, where

i) $(F,+)$ is a nonzero Abelian group [5],

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(2) Cf.[3],p.75. In [1],p.163, a planar partition is called a congruence.

(3) A decomposition of a set $A\neq\emptyset$ is a set of pairwise disjoint nonempty subsets of A which cover A.

(4) The sets $S_\iota+x$ $(\iota\in\mathcal{J}, x\in S)$ are called lines. Two lines of the same $\{S_\iota+x\mid x\in S\}$, $\iota\in\mathcal{J}$ are termed mutually parallel.

(5) The neutral element of $(F,+)$ is denoted by 0, whereas the neutral element of $(F\setminus\{0\},\cdot)$ is denoted by 1.

ii) $(F\setminus\{0\},\cdot)$ is a loop [5],

iii) $0\cdot x = x\cdot 0 = 0$ for all $x\in F$,

iv) $x\cdot(y+z)=x\cdot y+x\cdot z$ for all $x,y,z\in F$ (right distributivity).

We say that $(F,+,\cdot)$ is planar if

v) the map $x\to -(a\cdot x)+b\cdot x$ is a bijection $S\to S$ for each pair of distinct $a,b\in S$.

Definition 3. Let $Q=(F,+,\cdot)$ be a quasifield. Then we construct $(F\times F,+)=(F,+)\oplus(F,+)$ and $\mathcal{P}(Q)=\{(\{(x,y)\mid y=ax\},+)\mid a\in F\}\cup\{(\{x,y\}\mid x=0\},+)\}$.

Proposition 2. Let $Q=(F,+,\cdot)$ be a quasifield. Then $\mathcal{P}(Q)$ is a τ-partition of $\mathcal{F}=(F\times F,+)$. Q is planar iff each pair $\{\{(x,y)\mid y=\alpha x\},\{(x,y)\mid y=\beta x\}\}$ with $\alpha\neq\beta$ is generating.

Proof. $\mathcal{Y}=(Y,+)=(\{(x,y)\mid x=0\},+)$ is a group isomorphic to $\mathcal{F}=(F,+)$. Next, each $\mathcal{F}_a=(F_a,+)=(\{(x,y)\mid y=ax\},+)$, $a\in F$, is an Abelian semigroup with neutral element $(0,0)$. By right distributivity, it follows that $(a,au)+(-a,-(au))=(0,0)$, so that $\mathcal{F}_a$ must be a subgroup of $\mathcal{F}$. Each element $(u,v)\in(F\setminus\{0\})\times F$ belongs to exactly one F_a, $a\in F$: indeed, $(u,v)=(u,au)$ holds for a uniquely determined $a\in F$ such that $a\cdot u=v$. Thus $\mathcal{P}(Q)$ is a partition of $\mathcal{F}$. Now, we shall show that each pair $\{Y,F_a\}$, $a\in F$ is generating:

The equation $Y+F_a=F$ is valid iff each pair $(u,v)\in F\times F$ can be expressed in the form $(0,v_1)+(u_1,au_1)$ for some $u_1,v_1\in F$. But $u=0+u_1$, $v=v_1+au_1$ hold for $u_1=u$, $v_1=v-au$.

Furthermore, $\{F_0, F_1\}$ is generating: $F_0 + F_1 = F$ holds iff each $(u, v) \in F \times F$ can be expressed as $(u_1, 0) + (u_2, u_2)$ for certain $u_1, u_2 \in F$. But $u = u_1 + u_2$, $v = 0 + u_2$ is obviously satisfied for $u_1 = u - v$ and $u_2 = v$. Finally, $\{F_a, F_b\}$, $a \neq b$, is generating iff each $(u, v) \in F \times F$ can be expressed as $(u_1, au_1) + (u_2, bu_2)$ for certain $u_1, u_2 \in F$. But the equations $u = u_1 + u_2$, $v = au_1 + bu_2$ have a (unique) solution $(u_1, u_2) \in F \times F$ iff there exists an $u_1 \in F$ such that $-au_1 + bu_1 = bu - v$ (or, in other words, iff $bu - v$ has a (unique) inverse image relative to the map $x \to -a \cdot x + b \cdot x$, $a \neq b$.) This concludes the proof.

__Definition 4__. Let $\mathbb{P} = \{(S_\iota, +) \mid \iota \in \mathcal{J}\}$ be a τ-partition of a nonzero Abelian group $(S, +)$. Let α, β, γ be indices with the same meaning as in Definition 1. Then every element $a \in S$ admits a unique expression in the form $\xi a + \eta b$, where $\xi a \in S_\beta$, $\eta b \in S_\alpha$. Let π be a bijection $S_\alpha \to S_\beta$ such that $x + \pi x \in S_\gamma$ for all $x \in S_\alpha$. Let $\varkappa$ be a bijection $S_\alpha \to \mathcal{J} \setminus \{\beta\}$ such that $(\varkappa \circ \eta)c = \theta$ where $\{c\} = (Y + \pi e) \cap S_\theta$ for a fixed element e in $S_\alpha \setminus \{0\}$. Let there be given a multiplication $\cdot$ on S_α such that $a \cdot b = \eta e$, where $\{c\} = S_{\varkappa a} \cap (S_\alpha + \pi b)$. Finally, let $\mathcal{Q}(P)$ denote the triplet $(S_\alpha, +, \cdot)$.

__Proposition 3__. If $\mathbb{P} = \{(S_\iota, +) \mid \iota \in \mathcal{J}\}$ is a τ-partition of a nonzero Abelian group $(S, +)$, then $\mathcal{Q}(\mathbb{P})$ is a quasifield. $\mathbb{P}$ is planar iff $\mathcal{Q}(P)$ is planar.

__Proof__. The definition of the multiplication $\cdot$ implies $0 \cdot a = a \cdot 0 = 0$, $e \cdot a = a \cdot e = a$ for all $a \in S_\alpha$. The quasigroup properties of $(S_\alpha \setminus \{0\}, \cdot)$ follow from the following facts: 1) each element of $S_\alpha \setminus \{0\}$ lies in exactly one component of $\mathbb{P}$ and 2) $card((S_\alpha + x) \cap S_{\alpha a}) = 1$ for all $x \in S$ and $a \in S_\alpha$. Part 2) follows from $S_\alpha + S_{\varkappa a} = S$ by Proposition 1V. Now verify right distributivity. First, $\pi a + a$ and $\pi b + b$ belong to S_γ, so that $(\pi a + a) + (\pi b + b) = (\pi a + \pi b) + (a + b) \in S_\gamma$ with $\pi a + \pi b \in S_\beta$ and $a + b \in S_\alpha$. Thus $\pi a + \pi b = \pi(a + b)$. Therefore $\pi a + ca, \pi b + cb \in S_{\varkappa c}$, where $\pi(a+b) \in S_\beta$ and $ca + cb \in S_\alpha$.

By the definition of $\pi, \varkappa, \cdot$, we conclude that $ca + cb = c(a+b)$, as required. Note that the map $x \to -ax + bx$ $(a \neq b)$ is a bijection iff $card(S_{\varkappa a} \cap (S_{\varkappa b} + y)) = 1$ for all $y \in S$. Q.E.D.

<u>Definition 5</u>. Let $\mathbb{P} = \{(S_\iota, +) \mid \iota \in \mathcal{J}\}$ be a partition of a nonzero Abelian group $(S, +)$. Let E be the set of all $\mathbb{P}$ -endomorphisms of $(S,+)$ (that is, of all endomorphisms $\varkappa$ of $(S,+)$ for which $\varkappa S_\iota \subseteq S_\iota, \iota \in \mathcal{J}$). Suppose that two compositions $+$, $\circ$ are defined on E by $(\varkappa_1 + \varkappa_2)a = \varkappa_1 a + \varkappa_2 a$, $(\varkappa_1 \circ \varkappa_2)a = \varkappa_1(\varkappa_2 a)$, where $\varkappa_1, \varkappa_2 \in E$ and $a \in S$.

Then we shall call the triplet $\mathcal{K} = (E, +, \circ)$ the <u>kernel</u> of $\mathbb{P}$.(6)

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(6) It is well-known that $\mathcal{K}$ is a unitary ring without zero divisers. The neutral element of $(E, +)$ is the zero endomorphism $\underline{0}$ ($\underline{0}a = 0$ for all $a \in S$), and the neutral element of $(E \setminus \{0\}, \circ)$ is the identity automorphism $\underline{1}$ ($\underline{1}a = a$ for all $a \in S$). Cf.[2],p.162.If $\mathbb{P}$ is planar, then $\mathcal{K}$ is a skew-field ([1],p.167).

Definition 6. The kernel of a quasifield $Q=(F,+,\cdot)$ is the subsystem $(K,+,\cdot)$ with $K=\{s \mid (x+y)\cdot s = x\cdot s + y\cdot s,\ x\cdot(y\cdot s)=(x\cdot y)\cdot s\}$.

Proposition 4. Let there be given a quasifield $Q=(F,+,\cdot)$. Then the $\mathcal{P}(Q)$-endomorphisms of $(F\times F,+)$ are precisely the maps $(x,y)\to(x\cdot a, y\cdot a)$ with $a\in K$, and the kernel of Q is isomorphic to the kernel of $\mathcal{P}(Q)$ which is a skew-field.

Proof (coincides with [1],pp.173-175): a) Every map $\mu_k:(x,y)\to(x\cdot k, y\cdot k)$ with $k\in K\setminus\{0\}$ is a $\mathcal{P}(Q)$-automorphism of $(F\times F,+)$. Indeed, by a simple calculation one obtains $\mu_k((x,y)+(x',y'))=\mu_k(x,y)+\mu_k(x',y')$. Since $\mu_k(x,ax)=(xk,(ax)k)=(xk,a(xk))$ and $\mu_k(0,y)=(0,yk)$, it follows that each $\{(x,y)\mid y=ax\}$ or $\{(x,y)\mid x=0\}$, respectively, is mapped onto itself.

b) Each $\mathcal{P}(Q)$-endomorphism of $(F\times F,+)$ has the form $(x,y)\to(x\cdot k, y\cdot k)$ for a suitable $k\in K$. Indeed, let σ be a $\mathcal{P}(Q)$-endomorphism distinct from 0. Then $\sigma(x,y)=\sigma(x,0)+\sigma(0,y)=(\bar\sigma x, \bar{\bar\sigma} y)$, where $\bar\sigma$ and $\bar{\bar\sigma}$ are well-determined maps of F into F. As $\{(x,y)\mid y=x\}$ is sent into itself, it follows that $\bar\sigma=\bar{\bar\sigma}$, so that $\bar\sigma(ax)=a\cdot\bar\sigma x$ because $\{(x,y)\mid y=ax\}$ must be mapped into itself. Since σ is an endomorphism of $(F\times F,+)$, the additivity property $\bar\sigma(x+y)=\bar\sigma x+\bar\sigma y$ must be fulfilled. For $h=\bar\sigma(1)$ we obtain $\bar\sigma(x\cdot 1)=x\cdot\bar\sigma(1)=x\cdot h$, so that $(ax)h=\bar\sigma(ax)=a\bar\sigma x=a(xh)$ (first kernel property) and $(x+y)h=xh+yh$ (second kernel property).

Thus $h \in K$ and $\sigma(x,y)=(xh, yh)$ for our $h \in K$.

c) The rest of Proposition 4 is a corollary of parts a)-b) and of the footnote [(6)].

Remark. The study of all partitions of Abelian groups the kernel of which is a skew-field may lead to useful results.

Proposition 5. There exists a non-planar quasifield. Thus, by Proposition 2, there is τ-partition which is not planar.

Proof. Let $(F, +, \bullet)$ be a transcendental Dickson nearfield constructed in [4], § 4 in such a manner that $(F, +, \cdot)$ is a field of rational expressions of one indeterminate t over the field of complex numbers and the new multiplication $\bullet$ is given by

$$\frac{a_1(t)}{a_2(t)} \bullet \frac{b_1(t)}{b_2(t)} = \frac{a_1(t)}{a_2(t)} \cdot \frac{b_1(t+\deg a_1 - \deg a_2)}{b_2(t+\deg a_1 - \deg a_2)}$$

where $\frac{a_1(t)}{a_2(t)}, \frac{b_1(t)}{b_2(t)} \in F$. The map $\frac{x_1(t)}{x_2(t)} \to -\left(1 \bullet \frac{x_1(t)}{x_2(t)}\right) + t \bullet \frac{x_1(t)}{x_2(t)}$ of F into F is not surjective because $-\frac{x_1(t)}{x_2(t)} + t \cdot \frac{x_1(t+1)}{x_2(t+1)} \neq 1$ for all $\frac{x_1(t)}{x_2(t)} \in F$.

It may be noted that Proposition 5 solves the question posed in M. Hall, Projective planes and related topics, California Institute of Technology 1954, p.48: Indeed, the condition (γ) "there is at most one permutation which displaces all elements and sends an element a into an element $b \neq a$ " is not a consequence of (α) " G is doubly transitive" and (β) "only the identity permutation fixes two distinct elements" in a suitable group G of permutations on a set S.

<u>Proposition 6</u>. Let $\mathbb{P}=\{(S_\iota,+) \mid \iota \in \mathcal{J}\}$ be a τ-partition of a nonzero Abelian group $(S,+)$ and $\mathcal{Q}(\mathbb{P}) = (S_\alpha,+,\cdot)$ the quasifield determined as in Proposition 3. Let $\theta_{(a,b)}: x \to a\cdot x + b$ be a map of S onto S where $a \in S\setminus\{0\}$, $b \in S$. Then in $\mathcal{Q}(\mathbb{P})$ the multiplication is associative iff the following condition holds:

(R) For any two $(a_1,b_1),(a_2,b_2) \in (S\setminus\{0\})\times S$, the composition $\varphi_{(a_2,b_2)}\circ\varphi_{(a_1,b_1)}$ coincides with some $\varphi_{(a_3,b_3)}$, $(a_3,b_3)\in(S\setminus\{0\})\times S$.

<u>Proof</u>. Let the multiplication of $\mathcal{Q}(\mathbb{P})$ be associative. Then $\varphi_{(a_2,b_2)}\circ\varphi_{(a_1,b_1)} = \varphi_{(a_2a_1,\,a_2b_1+b_2)}$. If (R) is fulfilled then $a_2\cdot(a_1\cdot x)+a_2\cdot b_1 + b_2 = a_3\cdot x + b_3$ holds for all $x \in S$, where a_i, b_i $(i=1,2,3)$ have the meaning described in (R). For $x=0$ one obtains $a_2b_1 + b_2 = b_3$ and consequently $a_2\cdot(a_1\cdot x) = a_3\cdot x$. For $x=1$, one obtains $a_2\cdot a_1 = a_3$ and consequently $a_2\cdot(a_1\cdot x) = (a_2\cdot a_1)\cdot x$. Thus $\mathcal{Q}(\mathbb{P})$ must have an associative multiplication. This completes the proof.

It is well-known (M. Hall, The Theory of Groups, New York 1959, p.382) that a) in any associative quasifield $(S,+,\cdot)$ the mentioned maps $\theta_{(a,b)}$, $(a,b)\in(S\setminus\{0\})\times S$, form a group which acts doubly transitively on S and which has the property that only the identity fixes two distinct elements of S, b) if $(G,\circ)$ is a group of permutations on a set S, $\operatorname{card} S > 1$, satisfying the two preceding conditions, then there is an associative quasifield $\mathcal{Q} = (S, +, \cdot)$ such that the cosets $\mathfrak{G}_a + \alpha$, $a \in S \setminus \{0\}$, $\alpha \in S \times S$ in $(S \times S, +)$ where $\mathfrak{G}_a = \{(x, y) \mid y = a \cdot x\}$, $a \in S \setminus \{0\}$, are precisely the sets $\{(s, \sigma s) \mid s \in S\}$, $\sigma \in \mathfrak{G}$.

R e f e r e n c e s

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