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ON CATEGORIAL EMBEDDINGS OF TOPOLOGICAL STRUCTURES INTO ALGEBRAIC

Zdeněk HEDRLÍN and Aleš PULTR, Praha

J.R. Isbell investigated in [3] the categories which can be fully embedded into a category of algebras and - later on - he has proposed to call such categories boundable.

The aim of the present paper is to prove that categories of a certain type are boundable. Among these categories are (under an assumption on non-existence of measurable cardinals): the category of topological spaces with continous mappings, category of uniform spaces with uniformly continuous mappings, the category of proximity spaces with proximity mappings, category of topological algebras of a given type with continuous homomorphisms, trivial category of ordinals etc.

To show the main idea of this paper we shall discuss as an example the category of topological spaces with continuous mappings. Denote by P^- the contravariant functor associating with every set X its power set $P(X)$ and with every mapping $f: X \to Y$ a mapping $\tilde{f}: P(Y) \to P(X)$ defined by $\tilde{f}(Y_1) = f^{-1}(Y_1)$ for every $Y_1 \subset Y$. A topology τ on X may be considered as a unary relation κ on $P(X)$, namely, $X_1 \in \kappa$ if and only if X_1 is

open; a mapping $f:(X,\tau)\rightarrow(Y,\sigma)$ is continuous if and only if $P^{-}(f)$ is compatible with relations just described. Similarly, some other categories studied in mathematics are given in the following way:
given a set functor F and a type Δ, the objects are couples (X,R), where R is a system of relations of the type Δ on $F(X)$ and $f:(X,R)\rightarrow(Y,S)$ is a morphism if and only if $F(f)$ is compatible. We shall show that if the functor F has a certain property - we call it selectivity - then the category described above is boundable. Roughly speaking, a set functor F is selective, if there is a canonical relational system on the sets $F(X)$ such that the compatibility with respect to this system selects exactly the mappings of the type $F(f)$ among all mappings $g: F(X)\rightarrow F(Y)$.

The categories defined by relational systems have been studied in [1]. There has been proved that any such a category can be fully embedded into the category of algebras with two unary operations - denoted by $\mathcal{U}(1,1)$ - or into the category $\mathcal{R}$ (the objects of $\mathcal{R}$ are sets each with a binary relation and morphisms are all compatible mappings). In fact, $\mathcal{R}$ is the category of directed graphs and their graph-homomorphisms. This result will be helpfull for the proof that categories we have mentioned are boundable. We remark that from [1] also follows that a category is boundable if and only if it is isomorphic with a full subcategory of $\mathcal{R}$ ($\mathcal{U}(1,1)$, resp.).

The paper is divided into four paragraphs. Paragraph 1 contains conventions concerning notation. In the paragraph 2 we define the notion of a selective functor and prove a few theorems about it. Further, in the paragraph 3 we show that certain functors are selective. It is also given an example of a functor which is not selective. Paragraph 4 contains some consequences of the previous ones with applications of the theory to some often discussed categories.

Among the results of this paper there are also new proofs of two theorems by J.R. Isbell - to whom we thank for a very stimulating correspondence - namely, that a dual of a boundable category is boundable and that the trivial category of ordinals is boundable. We are indebted to P. Vopěnka for valuable advice and to L. Bukovský, who called our attention to a paper [4], one part of which concerns the selectivity of the functor P^{-}.

§ 1. Conventions concerning notation. Throughout this paper we mean under a set functor any functor from the category of sets into the category of sets. The identical functor from the category of sets onto itself will be denoted by I.

If $\mathcal{A}$, $\mathcal{L}$ are categories, we write $\mathcal{A} \Rightarrow \mathcal{L}$, if there exists a full embedding of $\mathcal{A}$ into $\mathcal{L}$, i.e. $\mathcal{A} \Rightarrow \mathcal{L}$ means that there exists a one-to-one covariant functor which maps $\mathcal{A}$ onto a full subcategory of $\mathcal{L}$. If we want to express that a functor Φ has this property, we write $\Phi : \mathcal{A} \Rightarrow \mathcal{L}$. If there is any (covariant or contravariant) one-to-one functor which maps $\mathcal{A}$ onto a full subcategory of $\mathcal{L}$, we write $\mathcal{Q} \simeq\!\!\!\to \mathcal{L}$, The dual category to a category $\mathcal{Q}$ is denoted by $d\mathcal{Q}$. Evidently, it holds

$$\mathcal{Q} \Rightarrow \mathcal{L} \Longleftrightarrow d\mathcal{Q} \Rightarrow d\mathcal{L} ,$$

$$\mathcal{Q} \simeq\!\!\!\to \mathcal{L} \Longleftrightarrow \mathcal{Q} \Rightarrow \mathcal{L} \quad \text{or} \quad d\mathcal{Q} \Rightarrow \mathcal{L} .$$

A type Δ means a sequence $\Delta = \{\alpha_\beta \mid \beta < \gamma\}$, where α_β, β, γ are ordinals. The sum of the type Δ, $\Sigma\Delta$, means $\sum_{\beta<\gamma} \alpha_\beta$ in the usual sense of the sum of ordinals. If $\Delta_\iota = \{\alpha^\iota_\beta \mid \beta < \gamma^\iota\}$ are types indexed by ordinals $\iota < \sigma$, then the sum of these types $\dot{\sum_\iota} \Delta_\iota$ is a type $\Delta = \{\alpha_\beta \mid \beta < \Sigma\, \gamma^\iota\}$, where $\alpha_{\varkappa+\beta} = \alpha^\iota_\beta$ holds for every $\varkappa = \Sigma\{\gamma^\lambda \mid \lambda < \iota\}$.

Let r be an α-nary relation on a set X, s an α-nary relation on a set Y. A mapping $f: X \to Y$ is called rs-compatible, if the following implication holds:

$\{x_\iota \mid \iota < \alpha\} \in r \Longrightarrow \{f(x_\iota) \mid \iota < \alpha\} \in s$.

Under a relational system R of a type $\Delta = \{\alpha_\beta \mid \beta < \gamma\}$ on a set X we mean a system $R = \{r_\beta \mid \beta < \gamma\}$, where every r_β is a α_β-nary relation on X. If $R = \{r_\beta\}$ ($S = \{s_\beta\}$, resp.) is a relational system of the type Δ on a set X (Y, resp.), then $f: X \to Y$ is called RS-compatible, if it is $r_\beta s_\beta$-compatible for every $\beta < \gamma$.

The following category will play an important role in this paper:

<u>Category</u> $\mathcal{T}(\{F_\iota, \Delta_\iota \mid \iota \in J\})$: Let J be a set, F_ι ($\iota \in J$) set functors, Δ_ι types. The objects of $\mathcal{T}(\{F_\iota, \Delta_\iota \mid \iota \in J\})$ are systems $(X, \{R_\iota \mid \iota \in J\})$, where

X is a set and R_ι are relational systems of the type Δ_ι on $F_\iota(X)$. Morphisms from $(X, \{R_\iota\})$ into $(Y, \{S_\iota\})$ are all mappings $f: X \to Y$ such that $F_\iota(f)$ is $R_\iota S_\iota$-compatible if F_ι is covariant or $F_\iota(f)$ is $S_\iota R_\iota$-compatible, if F_ι is contravariant - for every $\iota \in J$. Exactly, we should say that morphisms are triples $\langle (X, \{R_\iota\}), f, (Y, \{S_\iota\}) \rangle$, but certainly there is no danger of misunderstanding.

<u>Remarks</u>. 1) Sometimes we shall write $\mathcal{T}(\{F_\iota, \Delta_\iota\})$ instead of $\mathcal{T}(\{F_\iota, \Delta_\iota \mid \iota \in J\})$, if it is clear which set J is meant. If J is a one-point set, we write simply $\mathcal{T}(F, \Delta)$. If $J' = J \cup \{\iota_0\}$ we write often $\mathcal{T}(\{F_\iota, \Delta_\iota \mid \iota \in J\}, (F_{\iota_0}, \Delta_{\iota_0})\})$ instead of $\mathcal{T}(\{F_\iota, \Delta_\iota \mid \iota \in J'\})$ etc. A void type, i.e. $\{\alpha_\beta \mid \beta < 0\}$, is denoted by $\emptyset$. Evidently, $\mathcal{T}(I, \emptyset) = \mathcal{T}$, where $\mathcal{T}$ denotes the category of sets.

2) The category $\mathcal{T}(I, \Delta)$ is the same category as $\mathcal{R}(\Delta)$ in the notation of [1].

3) If $F_\iota = F$ for $\iota \in J' \subset J$, then $\mathcal{T}(\{F_\iota, \Delta_\iota \mid \iota \in J\})$ is isomorphic with the category $\mathcal{T}(\{F_\iota, \Delta_\iota \mid \iota \in J \setminus J'\}, (F, \dot{\Sigma}\{\Delta_\iota \mid \iota \in J'\}))$, where the last sum is taken by a well ordering of the set J'.

4) Evidently, the category of topological spaces with continuous mappings is isomorphic with a full subcategory of $\mathcal{T}(P^-, \{1\})$.

§ 2. <u>Selective functors</u>. The symbol $\square$ will denote the obvious forgetful functor from the category $\mathcal{T}(\{F_\iota, \Delta_\iota \mid \iota \in J\})$

into $\mathcal{T}$.

Definition 1. One-to-one set functor F into will be called Δ-selective, if there is Δ' and a functor $\Phi : \mathcal{T}(I, \Delta) \xrightarrow{\sim} \mathcal{T}(I, \Delta')$ such that

$$\Box \circ \Phi = F \circ \Box \quad .$$

The functor F will be called selective, if it is Δ-selective for every type Δ .

Theorem 1. Let functors F and G be naturally equivalent and the functor F be Δ-selective. Then G is Δ-selective.

Proof. Let $T : F \to G$ and $T' : G \to F$ be transformations such that $T \circ T'$ is the identity transformation of G and $T' \circ T$ is the identity transformation of F. Let $\Phi : \mathcal{T}(I, \Delta) \xrightarrow{\sim} \mathcal{T}(I, \Delta')$ and $\Box \circ \Phi = F \circ \Box$. If (X, R) is an object of $\mathcal{T}(I, \Delta)$, we have $\Phi(X, R) = (F(X), \bar{R})$. Let $\Delta' = \{\beta_\alpha \mid \alpha < \gamma\}$. If $\bar{R} = \{\bar{r}_\iota\}$, then $\bar{r}_\alpha \subset (F(X))^{\beta_\alpha}$. Put

$$\bar{\bar{r}}_\alpha = T_X^{\beta_\alpha}(\bar{r}_\alpha), \qquad \bar{\bar{R}} = \{\bar{\bar{r}}_\alpha\} \quad ,$$

and define $\Psi(X, R) = (G(X), \bar{\bar{R}})$, $\Psi(f) = G(f)$. Further, we shall give the proof for G contravariant (and, hence, F contravariant). For covariant G the proof would be similar.

Let $f : (X, R) \to (Y, S)$ be a morphism. Then $F(f) : (F(Y), \bar{S}) \to (F(X), \bar{R})$ is a morphism. Let $\{x_\iota\} \in \bar{\bar{s}}_\alpha$. Then $x_\iota = T_Y(y_\iota)$, where $\{y_\iota\} \in \bar{S}_\alpha$ and we get $\{G(f)(x_\iota)\} = \{G(f) \circ T_Y(y_\iota)\} = \{T_X \circ F(f)(y_\iota)\} \in \bar{\bar{r}}_\alpha$ as $\{F(f)(y_\iota)\} \in \bar{r}_\alpha$.

Thus, Ψ is a functor from $\mathcal{T}(I, \Delta)$ into $\mathcal{T}(I, \Delta')$. Ψ is evidently one-to-one and it remains only to prove that it maps $\mathcal{T}(I, \Delta)$ onto a full subcategory of $\mathcal{T}(I, \Delta')$. Let $g : (G(Y), \bar{\bar{S}}) \to (G(X), \bar{\bar{R}})$ be a morphism. Put $\bar{g} = T_X \circ g \circ T_Y$. If $\{x_\iota\} \in \bar{s}_\alpha$, then $\{g \circ T_Y(x_\iota)\} \in \bar{\bar{r}}_\alpha$. As T and T' are mutually inverse transformations, $\{T'_X \circ g \circ T_Y(x_\iota)\} \in \bar{r}_\alpha$. Hence, $\bar{g} : (F(Y), \bar{S}) \to (F(X), \bar{R})$ is a morphism and $\bar{g} = F(f)$, where $f : (X, R) \to (Y, S)$ is a morphism. We get $F(f) = T'_X \circ g \circ T_Y$ and

$$g = T_X \circ F(f) \circ T'_Y = G(f) \circ T_Y \circ T'_Y = G(f) .$$

The proof is finished.

<u>Theorem 2</u>. A composition of a finite number of selective functors is a selective functor.

<u>Proof</u>. It suffices to consider only two functors. Let F and G be selective functors, Δ a type. There exist

$\Phi : \mathcal{T}(I, \Delta) \xrightarrow{\simeq} \mathcal{T}(I, \Delta')$ and $\Psi : \mathcal{T}(I, \Delta') \xrightarrow{\simeq} \mathcal{T}(I, \Delta'')$

such that $\square \circ \Phi = F \circ \square$ and $\square \circ \Psi = G \circ \square$. We have

$$\Psi \circ \Phi : \mathcal{T}(I, \Delta) \xrightarrow{\simeq} \mathcal{T}(I, \Delta'')$$

and $\square \circ \Psi \circ \Phi = G \circ \square \circ \Phi = G \circ F \circ \square$.

<u>Theorem 3</u>. If there exists a contravariant Δ-selective functor, then

$$d\,\mathcal{T}(I, \Delta) \Rightarrow \mathcal{R} \qquad (\Rightarrow \mathcal{C}\ell(1, 1) \text{ etc.}).$$

<u>Proof</u>. Let F be the contravariant Δ-selective functor, Φ the corresponding functor from $\mathcal{T}(I, \Delta)$ into $\mathcal{T}(I, \Delta')$. Φ must be also contravariant, and we get

$$d\,\mathcal{T}(I, \Delta) \Rightarrow \mathcal{T}(I, \Delta') \quad (= \mathcal{R}(\Delta') \text{ in [1]}).$$

It is proved in [1] that $\mathcal{R}(\Delta') \Rightarrow \mathcal{R}$. We get $d\,\mathcal{T}(I, \Delta) \Rightarrow R$.

Theorem 4. If F_ι are Δ_ι-selective functors, then

$$\mathcal{T}(\{G_\iota, \Delta_\iota\}) \longrightarrow \mathcal{T}(\{F_\iota \circ G_\iota, \Delta'_\iota\})$$

for some types Δ'_ι.

Proof. Let $\Phi_\iota : \mathcal{T}(I, \Delta_\iota) \xrightarrow{\sim} \mathcal{T}(I, \Delta'_\iota)$ be functors such that $\Box \circ \Phi_\iota = F_\iota \circ \Box$. Let $(X, \{R_\iota\})$ be an object in $\mathcal{T}(\{G_\iota, \Delta_\iota\})$. For any relational system R_ι on $G_\iota(X)$ we choose R'_ι such that

$$\Phi_\iota((G_\iota(X), R)) = ((F_\iota \circ G_\iota)(X), R'_\iota).$$

Put $\Phi((X, \{R_\iota\})) = (X, \{R'_\iota\})$, which is an object in $\mathcal{T}(\{F_\iota \circ G_\iota, \Delta'_\iota\})$, and $\Phi(f) = f$. If f is a morphism from $(X, \{R_\iota\})$ into $(Y, \{S_\iota\})$, then $G_\iota(f)$ is either $R_\iota S_\iota$-compatible (G_ι covariant) or $S_\iota R_\iota$-compatible (G_ι contravariant). Hence, $(F_\iota \circ G_\iota)(f) = \Phi_\iota(G_\iota(f))$ is either $R'_\iota S'_\iota$-compatible or $S'_\iota R'_\iota$-compatible. Hence, $\Phi(f) = f$ is a morphism from $(X, \{R'_\iota\})$ into $(Y, \{S'_\iota\})$ and Φ is a functor from $\mathcal{T}(\{G_\iota, \Delta_\iota\})$ into $\mathcal{T}(\{F_\iota \circ G_\iota, \Delta'_\iota\})$. Φ is evidently one-to-one. It remains to prove that its image is a full subcategory.

Let $f : (X, \{R'_\iota\}) \to (Y, \{S'_\iota\})$ be a morphism. Then $\Phi_\iota(G_\iota(f)) = (F_\iota \circ G_\iota)(f)$ is either $R'_\iota S'_\iota$-compatible or $S'_\iota R'_\iota$-compatible. There must be $g_\iota : G_\iota(X) \to G_\iota(Y)$ ($g_\iota : G_\iota(Y) \to G_\iota(X)$, if G_ι is contravariant, resp.) such that g_ι is either $R_\iota S_\iota$-compatible or $S_\iota R_\iota$-compatible and $\Phi_\iota(G_\iota(f)) = \Phi_\iota(g_\iota)$. Since Φ_ι is a one-to-one functor, we get $G_\iota(f) = g_\iota$ and $G_\iota(f)$ is either $R_\iota S_\iota$-compatible or $S_\iota R_\iota$-compatible. Thus, $S = \Phi(f)$, where $f : (X, \{R_\iota\} \to (Y, \{S_\iota\})$ is a morphism. The proof is finished.

__Theorem 5.__ Let F be a $\emptyset$-selective functor, F_ι be arbitrary set functors. Then

$$\mathcal{T}(\{F_\iota \circ F, \triangle_\iota\}) \overset{\sim}{\to} \mathcal{T}(\{F_\iota, \triangle_\iota\}, (I, \triangle)) \text{ for some } \triangle.$$

If F is covariant, we may write $\Rightarrow$ instead of $\overset{\sim}{\to}$.

__Proof.__ Let $\Phi : \mathcal{T} \Rightarrow \mathcal{T}(I, \triangle)$ be a functor such that $\square \circ \Phi = F$ (F is $\emptyset$-selective!). Denote by R_X a relational system such that $\Phi(X) = (F(X), R_X)$. Let $(X, \{R_\iota\})$ be an object in $\mathcal{T}(\{F_\iota \circ F, \triangle_\iota\})$. Put

$$G((X, \{R_\iota\})) = (F(X), \{R_\iota\}, R_X), \quad G(f) = F(f).$$

Evidently, $G((X, \{R_\iota\}))$ is always an object in $\mathcal{T}(\{F_\iota, \triangle_\iota\}, (I, \triangle))$. If $f : (X, \{R_\iota\}) \to (Y, \{S_\iota\})$ is a morphism, then $F(f)$ is $R_X R_Y$-compatible (or $R_Y R_X$-compatible) and $(F_\iota \circ F)(f) = F_\iota(F(f))$ is $R_\iota S_\iota$-compatible (or $S_\iota R_\iota$-compatible, if F is contravariant). Thus, G is evidently one-to-one functor into $\mathcal{T}(\{F_\iota, \triangle_\iota\}, (I, \triangle))$. Let $g : (F(X), \{R_\iota\}, R_X) \to (F(Y), \{S_\iota\}, R_Y)$ (if F is contravariant, then $g : (F(Y), \{S_\iota\}, R_Y) \to (F(X), \{R_\iota\}, R_X)$) be a morphism. As g is $R_X R_Y$-compatible ($R_Y R_X$-compatible, resp.), $g = F(f)$ for some $f : X \to Y$. Since $F_\iota(g) = F_\iota \circ F(f)$ is $R_\iota S_\iota$-compatible ($S_\iota R_\iota$-compatible, resp.), we get $f : (X, \{R_\iota\}) \to (Y, \{S_\iota\})$ and $g = G(f)$.

__Theorem 6.__ Let F_ι be selective functors. If there exist selective functors G_ι such that $G_\iota \circ F_\iota = F$, then $\mathcal{T}(\{F_\iota, \triangle_\iota\} \Rightarrow \mathcal{R}$ ($\Rightarrow \mathcal{U}(1, 1)$ etc.).

__Proof.__ By theorem 4 and remark 3 in the paragraph 1, $\mathcal{T}(\{F_\iota, \triangle_\iota\}) \Rightarrow \mathcal{T}(F, \triangle)$, where $\triangle = \dot{\sum} \triangle'_\iota$.
By theorem 5, $\mathcal{T}(F, \triangle) \overset{\sim}{\to} \mathcal{T}((I, \triangle), (I, \triangle''))$.

Hence, by remark 3 in § 1, $\mathcal{T}(F,\Delta) \rightrightarrows \mathcal{T}(I,\Delta')$, where it suffices to put $\Delta' = \Delta \dot{+} \Delta''$. If F is covariant, then $\mathcal{T}(F,\Delta) \Rightarrow \mathcal{T}(I,\Delta')$, and by [1], $\mathcal{T}(F,\Delta) \Rightarrow \mathcal{R}$. If F is contravariant, then $d\mathcal{T}(F,\Delta) \Rightarrow \mathcal{T}(I,\Delta')$ and $\mathcal{T}(F,\Delta) = d^2\mathcal{T}(F,\Delta) \Rightarrow d\mathcal{T}(I,\Delta')$. As F is selective and contravariant, we obtain, by theorem 3, $d\mathcal{T}(I,\Delta') \Rightarrow \mathcal{R}$. The proof is finished.

__Corollary 1.__ Let $F_1, F_2, \dots, F_n$ be selective functors. Then

$$\mathcal{T}((F_1,\Delta_1),(F_2 \circ F_1,\Delta_2),\dots,(F_n \circ F_{n-1} \circ \dots \circ F_1,\Delta_n)) \Rightarrow \mathcal{R}.$$

§ 3. __Some special functors.__ __Functor__ Q_A : Let A be a non-void set. If X is a set, we put $Q_A(X) = X^A$ (i.e. the set of all mappings from A into X); if f is a mapping from X into Y we define $Q_A(f)$ by

$$Q_A(f)(\varphi) = f \circ \varphi .$$

Q_A is evidently one-to-one functor into.

__Remarks.__ 1) If A is a one-point set, then Q_A is naturally equivalent with the identical functor, if A is a two-point set, Q_A is naturally equivalent with the functor Q, which is defined by $Q(X) = X \times X$, $Q(f) = f \times f$.

2) Evidently, if $\mathrm{card}\, A = \mathrm{card}\, B$, then Q_A is naturally equivalent with Q_B.

__Theorem 7.__ Q_A is a selective functor.

__Proof.__ By previous remark and by theorem 1, we may assume that A is an ordinal number δ (i.e. the set of all ordinals less than δ). Let $\Delta = \{\beta_\alpha \mid \alpha < \gamma\}$, and let

Δ'' be sequence of the length σ, every element of which is the number 2. Let (X, R) be an object in $S(I, \Delta)$, $R = \{ r_\alpha \}$. For $\alpha < \gamma$, define $\bar{r}_\alpha$ by

$$\{ \varphi_\iota \} \in \bar{r}_\alpha \Longleftrightarrow \{ \varphi_\iota (0) \} \in r_\alpha .$$

(We remark that 0 is an ordinal, and therefore it is an element of A.) Put $\Delta' = \Delta \dotplus \Delta''$. Further, define $\bar{r}_{\gamma+a}$, for $a \in A$, by

$$(\varphi, \psi) \in \bar{r}_{\gamma+a} \Longleftrightarrow \varphi(a) = \psi(0) .$$

Thus, $\bar{R} = \{ \bar{r}_\alpha, \alpha < \gamma + \sigma \}$ is a relational system of the type Δ' on $Q_A(X)$.

Let $f : (X, R) \to (Y, S)$ be a morphism, and $\{\varphi_\iota\} \in \bar{r}_\alpha$, $\alpha < \gamma$. It means $\{ \varphi_\iota (0)\} \in r_\alpha$. Thus, $\{(Q_A(f)(\varphi_\iota))(0)\} = \{f \circ \varphi_\iota(0)\} = \{f(\varphi_\iota(0))\} \in s_\alpha$ and $\{Q_A(f)(\varphi_\iota)\} \in \bar{s}_\alpha$. Let $(\varphi, \psi) \in \bar{r}_{\gamma+a}$; hence, $\varphi(a) = \psi(0)$ and $f \circ \varphi(a) = f \circ \psi(0)$, i.e. $(Q_A(f)(\varphi), Q_A(f)(\psi)) \in s_{\gamma+a}$. If $f : (X, R) \to (Y, S)$ is a morphism, we put $\Phi(X, R) = (Q_A(X), \bar{R})$ and $\Phi(f) = Q_A(f)$. Evidently, Φ is a one-to-one functor from $\mathcal{T}(I, \Delta)$ into $\mathcal{T}(I, \Delta')$. As $\square \circ \Phi = Q_A \circ \square$, it remains to prove that Φ maps $\mathcal{T}(I, \Delta)$ onto a full subcategory of $\mathcal{T}(I, \Delta')$.

Let $g : (Q_A(X), \bar{R}) \to (Q_A(Y), \bar{S})$, $\varphi(a) = \psi(b)$.
Take $\chi \in Q_A(X)$ such that $\chi(0) = \varphi(a)$. Hence, $(\varphi, \chi) \in \bar{r}_{\gamma+a}$, $(\psi, \chi) \in \bar{r}_{\gamma+b}$. Since g is $\bar{R}\bar{S}$-compatible, $(g(\varphi), g(\chi)) \in \bar{s}_{\gamma+a}$, $(g(\psi), g(\chi)) \in \bar{r}_{\gamma+b}$.
We have $g(\varphi)(a) = g(\chi)(0) = g(\psi)(b)$. If $x = \varphi(a)$, we define a mapping $f : X \to Y$ by $f(x) = g(\varphi)(a)$. Evidently, for every x there exist φ and a such that $x = \varphi(a)$. By previous considerations, $g(\varphi)(a)$ is defined uniquely. It holds: $f \circ \varphi(a) = g(\varphi)(a)$ for any φ and a.

Hence, $g = Q_A(f)$. It remains to prove that the mapping f we have constructed is RS-compatible. Let $\{x_\iota\} \in r_\alpha$. Define a mapping $\varphi_\iota : A \to X$ by $\varphi_\iota(a) = x_\iota$ for all $a \in A$. Hence, $\{\varphi_\iota\} \in \bar{r}_\alpha$ and $\{f \circ \varphi_\iota\} = \{g(\varphi_\iota)\} \in \bar{s}_\alpha$. Finally, $\{f(x_\iota)\} = \{f \circ \varphi_\iota(0)\} \in s_\alpha$. The proof is finished.

Definition. If X is a set, we put $P^-(X) = \{X_1 \mid X_1 \subset X\}$. If $f: X \to Y$ is a mapping, we define $P^-(f): P^-(Y) \to P^-(X)$ by $P^-(f)(A) = f^{-1}(A)$, where $f^{-1}(A)$ denotes, as usual, the preimage of the set A by the mapping f.

Remark. The functor P^- is naturally equivalent with functor F, which associates with every set X the set 2^X of all mappings from X into $2 = \{0, 1\}$, and with every $f: X \to Y$ a mapping from 2^Y into 2^X defined by $F(f)(\varphi) = \varphi \circ f$ for all $\varphi \in 2^Y$.

Theorem 8. Designate by (M) the following assertion:

(M) There exists a cardinal σ such that every σ-additive two-valued measure is γ-additive, γ any cardinal.

If (M) holds, then P^- is a selective functor.

Proof. First, we remark that the condition (M) may be formulated also in the following way: There exists a cardinal σ such that any ultrafilter, which is closed under the intersections of σ sets, is trivial (in another terminology fixed).

Let $\Delta = \{\beta_\alpha \mid \alpha < \gamma\}$, $\Delta'' = \{1, 2, \sigma + 1\}$; put $\Delta' = \Delta \dotplus \Delta''$. If (X, R) is an object in $S(I, \Delta)$, we define relations on $P^-(X)$ by: for $\alpha < \gamma$,

$\{X_\iota\} \in \bar{r}_\alpha \iff ((x_\iota \in X_\iota$ for every $\iota) \Rightarrow \{x_\iota\} \notin r_\alpha)$,

$$A \in \bar{r}_{\gamma} \Longleftrightarrow A = \emptyset ,$$

$$(A, B) \in \bar{r}_{\gamma+1} \Longleftrightarrow A = C(B) \quad (\text{i.e. } A = X \setminus B \;),$$

$$\{A_\iota\} \in \bar{r}_{\gamma+2} \Longleftrightarrow A_{\sigma} = \bigcap \{A_\iota \mid \iota < \sigma\} .$$

Put $\bar{R} = \{\bar{r}_\alpha \mid \alpha < \gamma + 3\}$, $\overline{P^-}(X,R) = (P^-(X), \bar{R})$. Let $f : (X, R) \to (Y, S)$ be a morphism, $\alpha < \gamma$, $\{Y_\iota\} \in \bar{s}_\alpha$. Let $\{f^{-1}(Y_\iota)\} \notin \bar{r}_\alpha$. Hence, there are $x_\iota \in f^{-1}(Y)$ such that $\{x_\iota\} \in r_\alpha$. Then we have $\{f(x_\iota)\} \in s_\alpha$ and $f(x_\iota) \in Y_\iota$ – a contradiction. The cases $\alpha = \gamma, \gamma + 1, \gamma + 2$ are obvious.

We get: $\overline{P^-}$ is a (evidently, one-to-one) functor from $\mathcal{T}(I, \triangle)$ into $\mathcal{T}(I, \triangle')$. It remains to prove that it maps $\mathcal{T}(I, \triangle)$ onto a full subcategory of $\mathcal{T}(I, \triangle')$. Let $g : (P^-(Y), \bar{S}) \to (P^-(X), \bar{R})$ be a morphism. Since g is $\bar{s}_\alpha \bar{r}_\alpha$-compatible, for $\alpha = \gamma, \gamma + 1, \gamma + 2$, we derive easily:

$$g(C(Y_1)) = C(g(Y_1)) ,$$

$$g(Y) = X,\; g(\bigcap\{Y_a \mid a \in A\}) = \bigcap (g(Y_a) \mid a \in A) \text{ for } \operatorname{card} A \leq \sigma,$$

$$g(Y_1 \cup Y_2) = g(Y_1) \cup g(Y_2),\; Y_1 \subset Y_2 \Rightarrow g(Y_1) \subset g(Y_2) ,$$

Y_1, Y_2 disjoint $\Rightarrow g(Y_1), g(Y_2)$ disjoint,

Thus, the family $\{g(\{y\}) \mid y \in Y\}$ contains only mutually disjoint sets. We shall prove that this family is a cover of X. Let $x \in X$. Put $J = \{Z \mid Z \subset Y,\; x \in g(Z)\}$. J is an ultrafilter on Y, closed under the intersection of σ sets. By the assumption, J is trivial and contains a one-point set $\{y\}$. We get $x \in g(\{y\})$. Now, define $f : X \to Y$ by $f(x) = y \Leftrightarrow x \in g(\{y\})$. By the previous considerations, f is well defined.

If $x \in f^{-1}(Y_1)$, then $f(x) = y \in Y_1$ and $x \in g(\{y\}) \subset g(Y_1)$. We get $f^{-1}(Y_1) \subset g(Y_1)$ for every $Y_1 \subset Y$. Especially, $C(f^{-1}(Y_1)) = f^{-1}(C(Y_1)) \subset g(C(Y_1)) = C(g(Y_1))$, and, finally, $g = P^-(f)$.

It remains to prove that f is $\kappa_\alpha \lambda_\alpha$-compatible for every $\alpha < \gamma$. Let $\{x_\iota\} \in \kappa_\alpha$, $\{f(x_\iota)\} \in \lambda_\alpha$. Then $\{(f(x_\iota))\} \in \bar{\lambda}_\alpha$, and, consequently, $\{f^{-1}(f(x_\iota))\} \in \bar{\kappa}_\alpha$. But $x_\iota \in f^{-1}(f(x_\iota))$, and we have got a contradiction. We remark that the last proof uses the same idea as the proofs in [5] and the proof of 2.5 in [2].

__Definition__. If X is a set, we put $P^+(X) = \{X_1 \mid X_1 \subset X\}$. If $f: X \to Y$, we define $P^+(f): P^+(X) \to P^+(Y)$ by $P^+(f)(X_1) = f(X_1) = \bigcup_{x \in X_1} \{f(x)\}$.

__Theorem 9__. The functor P^+ is not selective (even not $\emptyset$-selective).

__Remark__. At this point we must emphasize that we work in the Gödel-Bernays set theory with the axiom of infinity. Of course, if we assume the negation of the axiom of infinity, the functor P^+ would be selective.

__Proof__. Assume P^+ is selective. Then there exists a type $\Delta = \{\beta_\alpha \mid \alpha < \gamma\}$ and a functor $F: \mathcal{T} \Rightarrow \mathcal{T}(1, \Delta)$, such that $\square \circ F = P^+ \circ \square$. We denote $\delta = \operatorname{card} \sup \{\beta_\alpha \mid \alpha < \gamma\}$. Let X be an infinite set such that $\operatorname{card} X > 2^\delta$. Choose an arbitrary $x_0 \in X$ and a mapping $f: X \to X \setminus \{x_0\}$, which is one-to-one onto. Define $g: P^+(X) \to P^+(X)$ by:

$$g(X_1) = f(X_1) \quad \text{for } \operatorname{card} X_1 \leq 2^\delta,$$
$$g(X_1) = \delta(X_1) \cup \{x_0\} \quad \text{for } \operatorname{card} X_1 > 2^\delta.$$

Evidently, $g \neq P^+(h)$ for any $h: X \to X$.

Let $F(X)=(P^+X,\{r_\alpha\})$. There exists $\alpha<\gamma$ and $\{X_\iota\}\in r_\alpha$ such that $\{g(X_\iota)\}\notin r_\alpha$. If $card\, X_\iota \leq 2^\sigma$ for every ι, $P^+(f)$ would not be $r_\alpha r_\alpha$-compatible. Hence, $card\, X_\varkappa > 2^\sigma$ for some $\varkappa$.

Put $X^*=\bigcup\{X_\iota \mid \iota<\beta\}$. We define a system $\mathcal{U}$ of subsets of X^* by:

$$A\in\mathcal{U} \iff A=\bigcap\{A_\iota \mid \iota<\beta_\alpha\},$$

where every A_ι is either X_ι or $X^*\setminus X_\iota$. It is easy to see that $\mathcal{U}$ forms a disjoint cover of X^*. Define $\mathcal{U}_\iota=\{A \mid A\in\mathcal{U}, A\subset X_\iota\}$. Evidently, $X_\iota = \bigcup\mathcal{U}_\iota$. Further, define

$\mathcal{U}_\iota^1=\{A \mid A\in\mathcal{U}_\iota,\ card\, A\leq 2^\sigma\}$, $\mathcal{U}_\iota^2=\mathcal{U}_\iota\setminus\mathcal{U}_\iota^1$, $\mathcal{U}^i=\bigcup\{\mathcal{U}_\iota^i \mid \iota<\beta_\alpha\}, i=1,2$. Since $card\,\mathcal{U}\leq 2^\sigma$, we have $card\, X_\iota>2^\sigma \Rightarrow \mathcal{U}_\iota^2\neq\emptyset$.

If $A\in\mathcal{U}$, $card\, A>2^\sigma$, choose any fixed $a\in A$. Since $card\, f(A)=card(A-\{a\})$, there exists $h_A: A\to f(A)\cup\{x_0\}$, which is onto and $h_A(a)=x_0$. Denote $Y=(X\setminus X^*)\cup\bigcup\mathcal{U}^1$ and define $h: X\to Y$ by:

$$h(x)=f(x) \text{ for } x\in Y,$$
$$h(x)=h_A(x) \text{ for } x\in A\in\mathcal{U}^2 .$$

If $card\, X_\iota\leq 2^\sigma$, we have $X_\iota=\bigcup\mathcal{U}_\iota^1$ and

$$P^+h(X_\iota)=h(\bigcup\{A \mid A\in\mathcal{U}_\iota^1\}=\bigcup\{h(A) \mid A\in\mathcal{U}_\iota^1\}=$$
$$=\bigcup\{f(A \mid A\in\mathcal{U}_\iota^1\}=f(X_\iota)=g(X_\iota).$$

If $card\, X_\iota>2^\sigma$, then $\mathcal{U}_\iota^2$ is non-void and we have

$$P^+h(X_\iota)=\bigcup\{h(A) \mid A\in\mathcal{U}_\iota^1\}\cup\bigcup\{h(A) \mid A\in\mathcal{U}_\iota^2\}=$$
$$=\bigcup\{f(A) \mid A\in\mathcal{U}_\iota^1\}\cup\bigcup\{f(A)\cup\{x_0\} \mid A\in\mathcal{U}_\iota^2\}=f(X_\iota)\cup\{x_0\}=g(X_\iota).$$

Thus, $P^+h(X_\iota)=g(X_\iota)$ for every ι and, hence, P^+h is not compatible. We have got a contradiction.

<u>§ 4. Applications.</u> As the assumption (M) will be frequently used in this paragraph, we remark that (M) is consistent with the Gödel-Bernays set theory and is not in contradiction with the existence of inaccessible cardinals.

<u>Theorem 10</u>. Assuming (M), the dual to a boundable category is boundable.

<u>Proof</u>. The proof follows immediately from theorems 3 and 8 .

<u>Theorem 11</u>. Assuming (M), the category of topological spaces and all their continuous mappings is boundable.

<u>Proof</u>. As we sketched in the introduction, this category may be considered as a full subcategory of $\mathcal{T}(P^-, \{1\})$.

<u>Theorem 12</u>. Assuming (M), the category $\mathcal{Y}(\Delta)$ of topological algebras of the type Δ and their continuous homomorphisms is boundable.

<u>Proof</u>. If $\Delta = \{\alpha_\beta \mid \beta < \gamma\}$, put $\Delta' = \{\alpha_\beta + 1 \mid \beta$ Evidently, an algebraical structure of the type Δ is a special case of a relational system of the type Δ', and the property "to be a homomorphism" is the same as the compatibility with respect to the corresponding relational system. Similarly as in the proof of the previous theorem, $\mathcal{Y}(\Delta)$ can be considered as a full subcategory of $\mathcal{T}((P^-, \{1\}, (1, \Delta'))$. Hence, by theorems 6 and 8, $\mathcal{Y}(\Delta) \Rightarrow \mathcal{R}$.

<u>Theorem 13</u>. The category $\mathcal{L}$ of closure spaces ([4]) and their continuous mappings is - under the assumption

(M) - boundable.

Proof. We shall show that $\mathcal{L} \to \mathcal{T}(P^-, \{2\})$. Then the theorem will follows from theorems 6 and 8 .

Let X be a set, u the closure function on X. We define a relation u' on $P^-(X)$ by:

$$(A, B) \in u' \Longleftrightarrow A \supset u(B).$$

By definition, a mapping $f:(X,u) \to (Y,v)$ is continuous if and only if $f^{-1}(v(A)) \supset u(f^{-1}(A))$ for every $A \subset Y$, i.e. $P^-f(v(A)) \supset u(P^-f(A))$. The proof will be finished, if we prove that f is continuous if and only if P^-f is $v'u'$-compatible.

Let f be continous, $(A, B) \in v'$. Hence $A \supset v(B)$ and $P^-f(A) \supset P^-f(v(B)) \supset u(P^-f(B))$. Thus, $(P^-f(A), P^-f(B)) \in u'$, and P^-f is $v'u'$-compatible. Now, let P^-f be $v'u'$-compatible, $A \subset Y$. We have $(v(A), A) \in v'$ and $(P^-f(v(A)), P^-f(A)) \in u'$, i.e. $f^{-1}(v(A)) \supset u(f^{-1}(A))$. The mapping f is continuous.

Theorem 14. Assuming (M), the category $\mathcal{P}$ of proximity spaces and all their proximal mappings is boundable.

Proof. If (X, δ_1), (Y, δ_2) are proximity spaces, (δ_i relations "to be proximal"), then, by definition, $f:(X,\delta_1) \to (Y, \delta_2)$ is a proximal mapping if and only if

(1) $(A, B) \in \delta_1 \Longrightarrow (f(A), f(B)) \in \delta_2$

for every $A, B \subset X$. Denote by $\overline{\delta_1}$, $\overline{\delta_2}$ the complementary relations to δ_1 and δ_2 . Evidently, $f:(X,\delta_1) \to (Y,\delta_2)$ is proximal if and only if

(2) $(A, B) \in \overline{\delta_2} \Rightarrow (f^{-1}(A), f^{-1}(B)) \in \overline{\delta_1}$ for all $(A, B) \in \overline{\delta_2}$.

Thus, if we describe the proximities by complementary relations "to be far", the category $\mathcal{P}$ can be considered as a full subcategory of $\mathcal{T}(P^-, \{2\})$. By theorems 6 and 8, $\mathcal{P} \longrightarrow \mathcal{R}$.

Theorem 15. Assuming (M), the category $\mathcal{U}$ of uniform spaces and their uniformly continuous mappings is boundable.

Proof. If we describe the uniformities by means of systems of neighbourhoods of diagonals, then the uniformity on X is a unary relation on $P^- \circ Q\,(X)$. Since $f:(X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous if and only if $P^- \circ Q\,(f)$ is $\mathcal{V}\mathcal{U}$-compatible, $\mathcal{U}$ can be considered as a full subcategory of $\mathcal{T}(P^- \circ Q, \{1\})$. Hence, by theorems 2,6,7 and 8, $\mathcal{U} \longrightarrow \mathcal{R}$.

Remark. Theorems 13,14,15 could be strengthened in the same way as theorem 11 in theorem 12. Thus, e.g. the category of uniform algebras of the given type with all their uniformly continuous homomorphisms is boundable etc.

Theorem 16. Let $\mathcal{M}$ denote the trivial category of ordinals, i.e. the objects are all ordinals, morphisms all couples (α, β) where $\alpha \leq \beta$, $(\beta, \gamma) \circ (\alpha, \beta) = (\alpha, \gamma)$. Assuming (M), $\mathcal{M}$ is boundable.

Proof. Denote by $\mathcal{M}'$ the full subcategory of $\mathcal{T}(P^-, \{2\})$ generated by objects (α, κ_α), where α are non-zero ordinals, κ_α binary relation on $P^-(\alpha)$ defined by:

$$(m, n) \in \kappa_\alpha \iff \text{either } n = 0 \text{ and } m = \alpha \text{ or } m = \gamma + 1 \ (\gamma < \alpha) \text{ and } n \cap m = \{\gamma\}.$$

We are going to prove that $\mathcal{M}'$ is isomorphic with $\mathcal{M}$. It suffices to prove that, for $f : \alpha \to \beta$, $P^-(f)$ is $\kappa_\beta \kappa_\alpha$-compatible if and only if $\alpha \leq \beta$ and $f(\gamma) = \gamma$ for all $\gamma < \alpha$.

Let $\alpha \leq \beta$, $f: \alpha \to \beta$ defined by $f(\gamma) = \gamma$ for all $\gamma < \alpha$. Hence, $P^- f(a) = \alpha \cap a$ for every $a \subset \beta$. Let $(m, n) \in \varkappa_\beta$. If $n = 0$, then $m = \beta$ and further $(\alpha \cap m, \alpha \cap u) = (\alpha, 0) \in \varkappa_\alpha$. If $n \subset \beta - \alpha$, we have $m = \gamma + 1$, $\gamma \in n$. Thus, $\gamma \geq \alpha$ and $(\alpha \cap m, \alpha \cap n) = (\alpha, 0) \in \varkappa_\alpha$. Finally, let $m = \gamma + 1$, $\gamma < \alpha$, $m \cap n = \{\gamma\}$. Then $(\alpha \cap m, \alpha \cap n) = (m, \alpha \cap n) \in \varkappa_\alpha$, as $\{\gamma\} \supset m \cap \alpha\, n \in \gamma$ and we get $m \cap (\alpha \cap n) = \{\gamma\}$.

Let $f: \alpha \to \beta$ be a mapping such that $P^- f$ is $\varkappa_\beta \varkappa_\alpha$-compatible. First, we shall show that $\gamma < \delta < \alpha$ implies $f(\gamma) < f(\delta)$. We have $(f(\gamma) + 1, \{f(\gamma)\}) \in \varkappa_\beta$. Since $\gamma \in P^- f(\{f(\gamma)\})$, $\delta \notin P^- f(f(\gamma) + 1)$, i.e. $f(\delta) \notin f(\gamma) + 1$, and consequently, $f(\gamma) < f(\delta)$. Thus $\alpha \leq \beta$. Let γ be the least element in α such that $f(\gamma) \neq \gamma$. As f is increasing, we get $f(\gamma) > \gamma$, $P^- f(\{\gamma\}) = 0$ and $P^- f(\gamma + 1) = \gamma$. Since $(\gamma + 1, \{\gamma\}) \in \varkappa_\beta$, we get $(\gamma, 0) \in \varkappa_\alpha$ and $\gamma = \alpha$. This is a contradiction, as $\gamma \in \alpha$. Hence $f(\gamma) = \gamma$ for all $\gamma < \alpha$. The proof is finished.

<u>Definition</u>. Let A be a non-void set. We define a category $\mathcal{Y}(A)$ by: The objects are couples $(X, \varkappa)$, where X is a set and $\varkappa \subset A^X$ (i.e. $\varkappa$ is a set of mappings from X into A; the set $\varkappa$ will be called an inverse A-relation on X); a mapping $f: X \to Y$ will be a morphism from $(X, \varkappa)$ into (Y, ρ) if and only if $\varphi \circ f \in \varkappa$ for every $\varphi \in \rho$.

1) The notion of the inverse relation and the corresponding choice of mappings can be considered in some sense as a dual notion to the relations and compatible mappings. Actually, an α-nary relation is a subset of X^α (here

X^{α} is the set of all mappings from α into X), and the compatible mappings are then exactly the mappings $f : (X, \varkappa) \to (Y, \wp)$ which fulfill the condition $f \circ \varphi \in \in \wp$ for every $\varphi \in \varkappa$.

2) Topology can be considered as an inverse binary relation. Let $A = 2 = \{0, 1\}$. If τ is a topology on a set X , put $\bar{\tau}$ is the set of all characteristic functions of open sets. Evidently, $f : (X, \tau) \to (Y, \sigma)$ is continuous if and only if f is a morphism from $(X, \bar{\tau})$ into $(Y, \bar{\sigma})$ in $\mathcal{Y}(2)$.

3) Similarly, a differential structure on a manifold is essentially an inverse E_1 -relation. Let M and N be two differentiable manifolds, $\varkappa$ ($\wp$ resp.) the set of all differentiable mappings from M (N resp.) into the real line E_1 . $f : M \to N$ is differentiable if and only if $\varphi \circ f \in \varkappa$ for every $\varphi \in \wp$.

<u>Theorem 17</u>. Assuming (M), $\mathcal{Y}(A)$ is boundable for any set A .

<u>Proof</u>. Evidently, if $card\ A = card\ B$, then $\mathcal{Y}(A)$ and $\mathcal{Y}(B)$ are isomorphic. Thus it suffices to prove that $\mathcal{Y}(\alpha)$ is boundable, where α is any ordinal number.

We shall prove that $\mathcal{Y}(\alpha) \Rightarrow \mathcal{T}(P^-, \{\alpha\})$. Let $(X, \varkappa)$ be an object in $\mathcal{Y}(\alpha)$. We define on $P^-(X)$ an α -nary relation $\bar{\varkappa}$ by:

$$\{m_\beta \mid \beta < \alpha\} \in \bar{\varkappa} \iff \text{there exists } \varphi \in \varkappa \text{ such that } m_\beta = \varphi^{-1}(\beta) \text{ for every } \beta < \alpha .$$

Now, let $f : (X, \varkappa) \to (Y, \wp)$, $\{m_\beta\} \in \bar{\wp}$. Then there

exists $\varphi \in \mathfrak{s}$ such that $m_\beta = \varphi^{-1}(\beta)$ for every $\beta < \alpha$. We have $P^-(f)(m_\beta) = P^-(f)(\varphi^{-1}(\beta)) = P^-(f)P^-(\varphi)((\beta)) = P^-(\varphi \circ f)((\beta)) = (\varphi \circ f)^{-1}(\beta)$. As $\varphi \circ f \in \mathfrak{r}$, we get $\{P^-(f)(m_\beta)\} \in \bar{\mathfrak{r}}$.
Let $f : (X, \bar{\mathfrak{r}}) \to (Y, \bar{\mathfrak{s}})$, $\varphi \in \mathfrak{s}$. Then $\{\varphi^{-1}(\beta) \mid \beta < \alpha\} \in \bar{\mathfrak{s}}$, and $\{P^-(f)(\varphi^{-1}(\beta))\} = \{(\varphi \circ f)^{-1}(\beta)\} \in \bar{\mathfrak{r}}$. Thus there exists $\psi \in \mathfrak{r}$ such that $(\varphi \circ f)^{-1}(\beta) = \psi^{-1}(\beta)$ for every $\beta < \alpha$. Now, we easily derive $\varphi \circ f = \psi$.
The proof is finished.

<u>Corollary</u>. Assuming (M), the category of differentiable manifolds and all their differentiable mappings is boundable.

<u>Amendment</u>. It follows from § 2 that the selective functors play an important role in full embeddings into categories of algebras. However, the fact that P^+ is not selective does not mean e.g. that $\mathcal{T}(P^+, \Delta)$ cannot be fully embedded into $\mathcal{R}$. Now, we are going to show that the concept of selectivity can be generalized in a natural way. By this generalization we shall show as an example that $\mathcal{T}(P^+, \Delta)$ is boundable.

Return for a moment to the definition of a selective functor. The reason why we have used the categories $\mathcal{T}(I, \Delta)$ in the definition is the fact that we knew beforehead that $\mathcal{T}(I, \Delta)$ is boundable. Now we have a wider supply of boundable categories. It turns out to be worthwhile to define a more general notion.

Let $\mathcal{Q} = \mathcal{T}(\{F_\iota, \Delta_\iota \mid \iota \in J\})$, $\mathcal{L} = \mathcal{T}(\{G_\varkappa, \Delta_\varkappa \mid \varkappa \in \mathcal{Q}\})$ be categories. A set functor F is called selective from

$\mathcal{A}$ by means of $\mathcal{L}$, if there is $\Phi : \mathcal{A} \xrightarrow{\sim} \mathcal{L}$ such that $\Box \circ \Phi = F \circ \Box$ and F is one-to-one.

An evident analogon of theorems 5 and 6 is:

Theorem 18. Let F be a selective functor from $\mathcal{T}\{F_\iota, \Delta_\iota \mid \iota \in J\})$ by means of $\mathcal{T}(\{G_\varkappa, \Delta_\varkappa \mid \varkappa \in K\})$. Then $\mathcal{T}(\{F_\iota, \Delta_\iota \mid \iota \in J\}, (F, \Delta)) \xrightarrow{\sim} \mathcal{T}(\{G_\varkappa, \Delta_\varkappa \mid \varkappa \in K\}, (I, \Delta))$.

Theorem 19. P^+ is a selective functor from $\mathcal{T}$ by means of $\mathcal{T}((P^-, \{1\}), (I, \{1, 3\}))$.

Proof. If X is a set, $f : X \to Y$ a morphism in $\mathcal{T}$, we put $\Phi(X) = (P^+(X), \kappa_1, \kappa_2, \kappa^*)$, $\Phi(f) = P^+(f)$, where: κ_1 is a unary relation on $P^+(X)$ defined by $X_1 \in \kappa_1 \Longleftrightarrow X_1 = \{x\}$ (i.e. X_1 is a one point set; κ_2 is a ternary relation on $P^+(X)$ defined by:

$$\{X_1, X_2, X_3\} \in \kappa_2 \Longleftrightarrow X_1 \cup X_2 = X_3 ;$$

κ^* is a unary relation on $P^-(P^+(X))$ defined by: if $\mathcal{U} \in P^-(P^+(X))$ (i.e. $\mathcal{U}$ is a system of subsets of X), then $\{\mathcal{U}\} \in \kappa^*$ if and only if $\{x_i\} \in \mathcal{U}$, $i \in J$, implies $\bigcup_{i \in J} \{x_i\} \in \mathcal{U}$ (i.e. if $\mathcal{U}$ contains a family of one point sets, then it contains also its union).

Evidently, if $f : X \to Y$, then $P^+(f)$ is compatible with κ_1 and κ_2, $P^-(P^+(f))$ is compatible with κ^*.

Let $\tilde{f} : P^+(X) \to P^+(Y)$ be compatible with κ_1 and κ_2, $P^-(f)$ be compatible with κ^*. Considering compatibility with κ_1, we get that the image of every one point set under $\tilde{f}$ is a one point set. If $a \in A$, then $A \cup \{a\} = A$, and - using compatibility with κ_2 - $\tilde{f}(A) \cup \tilde{f}(\{a\}) = \tilde{f}(A)$, i.e. $\tilde{f}(A) \supset \bigcup_{a \in A} \tilde{f}(\{a\})$ for every $A \subset X$. The proof will be finished if we show that

$\tilde{f}(A) = \bigcup_{a \in A} \tilde{f}(\{a\})$ for every $A \subset X$. Assume $\tilde{f}(A) \setminus$

$\setminus \bigcup_{a \in A} \tilde{f}(\{a\}) \neq \emptyset$. Let $\mathcal{L}$ be a system of all non-void subsets of $\bigcup_{a \in A} \tilde{f}(\{a\})$. Evidently, $\{\mathcal{L}\} \in \kappa^*$. Hence the image of $\mathcal{L}$ under $P^-(\tilde{f})$ must be also in the unary relation κ^*. But $\{a\} \in P^-(\tilde{f})(\mathcal{L})$ for every $a \in A$ and $A \notin P^-(\tilde{f})(\mathcal{L})$. We have got a contradiction. The theorem is proved.

It follows from boundability of P^-, theorems 18,19 and corollary 1:

<u>Theorem 20</u>. Assuming (M), $\mathcal{T}(P^+, \triangle)$ is boundable for any type

Observe, that this way we have obtained a new proof of Theorem 14, as the category of proximity spaces is a full subcategory of $\mathcal{T}(P^+, \{2\})$.

A very general notion is defined in [4], namely a merotopic space. It is easy to see that the category of merotopic spaces and all their merotopically continuous mappings is a full subcategory of $\mathcal{T}(P^+ \circ P^+, \{1\})$. We are going to sketch a proof that also this category is (assuming (M)) boundable. It is possible to show - applying two times a slightly modified proof of Theorem 19 - that $P^+ \circ P^+$ is selective from $\mathcal{T}$ by means of $\mathcal{T}((P^-, \{1,1\}), (I, \{1,1,1,3,3\}))$. First, we consider only systems containing one set (we distinguish them by means of a unary relation on I), then we distinguish systems containing only one one-point set and repeat the proof of Theorem 19. Then we proceed to systems containing more sets and apply once more the proof of theorem 19.

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