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INTERDEPENDENCE OF WEAKENED FORMS OF THE AXIOM OF CHOICE

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1. Introduction

The aim of the present paper [1] is to discuss the interdependence of weakened forms of the general axiom of choice in Gödel-Bernays axiomatic set theory Σ (cf.[2]):

(E) There is a choice-function on the universal class, i.e. there is a function F such that $F(x) \in x$ for every non-void set x.

It is well known that the following axiom of choice (in classical form) and the well ordering principle are equivalent (a number of set-theoretical statements equivalent to these is stated in [10]):

(AC) On every family of non-void sets there is a choice function.

(WE) Every set can be well ordered.

Let us consider their weakened forms (these are, if α is a special ordinal number [2], statements of the set theory):

($AC\aleph_\alpha$) On every family of cardinality $\aleph_\alpha$ of non-void sets there is a choice function.

($WE\aleph_\alpha$) Every cardinal number is comparable with $\aleph_\alpha$ (i.e. equal, less or greater than $\aleph_\alpha$)

1) read in Vopěnka's Seminar on set theory at the Carolina University in Prague in March 1966.

2) i.e. a special class (cf.[2]) which is an ordinal number.

Furthermore, let us consider the <u>ordering principle</u>, which is a consequence of the axiom of choice:

(OP) Every set can be ordered .

And finally, let us consider the <u>principle of dependent choices</u> (considered by A. Tarski in [12]) and its generalization (A. Lévy [6]):

(PDC) If R is a relation on the set a such that $(\forall x \in a)(\exists y \in a)[\langle x\, y\rangle \in R]$, then there is a sequence $x_1, x_2, \dots, x_n, \dots$ of elements of a such that $\langle x_n\, x_{n+1}\rangle \in R$ for $n = 1, 2, \dots$.

($PDC\,\aleph_\alpha$) Let a be a set and R a relation such that for every $\gamma \in \omega_\alpha$ and every $g \in a^\gamma$ (function of γ into a) there is a function $f \in a^{\omega_\alpha}$ with $\langle f \wedge \gamma,\ f(\gamma)\rangle \in R$ for every $\gamma \in \omega_\alpha$.

It is known that $(AC) \equiv (WE) \equiv (\forall\gamma)(WE\,\aleph_\gamma) \equiv (\forall\gamma)(PDC\,\aleph_\alpha)$. Moreover, it is apparent that, for $\gamma \in \delta$, $(AC\,\aleph_\alpha) \to (AC\,\aleph_\gamma), (WE\,\aleph_\delta) \to (WE\,\aleph_\gamma)$ and $(PDC\,\aleph_\delta) \to (PDC\,\aleph_\gamma)$.

All these weakened forms of the axiom of choice are independent on the axioms of the set theory Σ . The independence of $(WE\,\aleph_0)$ (and therefore also of the axiom of choice) was shown by Hájek and Vopěnka [3], the independence of the other forms by Jech and Sochor [4],[5]. The following form of the axiom of choice is weaker than all the statements statedaabove (e.g. $(OP) \to (e)$ is shown below):

(e) Every denumerable family of pairs contains a denumerable subfamily, on which there is a choice-function .

The statement (e) is also independent on axioms of the set theory Σ . This follows from mentioned papers of Jech and Sochor.

The interdependence of weakened forms of the axiom of choice has been thoroughly investigated in axiom systems where the axiom of regularity - Fundierungsaxiom - is not considered, viz. where the existence of individuals (or urelements or non-founded sets) is permitted. Fraenkel showed in [1] the independence of the axiom of choice on the existence of choice-function on every denumerable family of finite sets. Mostowski [7],[8] showed the independence of the axiom of choice on the ordering principle and on the principle of dependent choices, and the independence of $(\forall\gamma<\alpha)(AC\,\aleph_\gamma) \rightarrow (AC\,\aleph_\alpha)$ for regular special $\aleph_\alpha$ 3). The most thorough investigation was carried out by Lévy in [6].

In present paper, similar results are obtained for the set theory Σ . The following assertion is proved (in section 4), if $\aleph_\alpha$ is any regular special cardinal number:

None of following statements: ordering principle(OP), restricted well-ordering principle (WE $\aleph_\alpha$), restricted axiom of choice (AC $\aleph_\alpha$) and generalized principle of dependent choices (PDC $\aleph_\alpha$) can be proved from the axioms of the

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3) A special aleph $\aleph_\alpha$ is called regular if it is regular under validity of the axiom of choice. E.g. $\aleph_1$ is regular, although it can be a union of denumerable collection of denumerable sets if the axiom of choice does not hold.

set theory Σ and the assumption that $(AC\ \aleph_\gamma)$, $(WE\ \aleph_\gamma)$ and $(PDC\ \aleph_\gamma)$ hold for every $\gamma \in \alpha$.

In [6] it is proved that $(PDC\ \aleph_\alpha)$ implies both $(AC\ \aleph_\alpha)$ and $(WE\ \aleph_\alpha)$, and that, for singular $\aleph_\alpha$, $(\forall\gamma\in\alpha)(AC\ \aleph_\gamma)$ implies $(AC\ \aleph_\alpha)$, and $(\forall\gamma\in\alpha)\ (PDC\ \aleph_\gamma)$ implies $(PDC\ \aleph_\alpha)$. The ordering principle implies that on every family of finite sets there is a choice function (indeed, if a is a family of finite sets and $\bigcup a$ is ordered, then every $x \in a$ has the least element which can be chosen).

The following questions remain open:

1. Does $(\forall\gamma\in\alpha)(WE\ \aleph_\gamma)$ imply $(WE\ \aleph_\alpha)$ for singular $\aleph_\alpha$?
2. What relation is there between $(AC\ \aleph_\alpha)$ and $(WE\ \aleph_\alpha)$?
3. Is the axiom of choice independent of the ordering principle?
4. Is the axiom of choice independent of $(\forall\gamma)(AC\ \aleph_\gamma)$?
5. Is the general axiom of choice independent of the "weak" axiom of choice (AC)?

If the validity of the axiom of regularity is not required, the answer to questions 3,4 and 5 is affirmative. The problem is whether the same holds for theory Σ .

The results of present paper are obtained by construction of a θ-model of set theory. The reader is assumed to be familiar with the papers [2],[14],[16] and [4]; the notation used in these papers is preserved here.

2. The model ∇ and the characteristic $\sigma(ct)$ of the topological space

The model ∇ (with parameters $ind, \langle c, t \rangle, G, \varkappa, j$) introduced by Vopěnka in [13] and [14] is the syntactic model of the theory Σ^* (Gödel's axioms A,B,C,D,E) in the theory Σ_{ind} (A,B,C,E with individuals).[4] In [15], the dependence of properties of the ∇-model upon the characteristics $\mu(ct)$ and $\nu(ct)$ of the topological space $\langle ct \rangle$ is investigated. For the purpose of present paper it is useful to consider a further characteristic of the space $\langle ct \rangle$:

Definition. $\sigma(ct)$ is the least cardinal number $\aleph_\sigma$ such that there is no basis t_0 of the topology t with the following property: The intersection $\bigcap_{\gamma \in \omega_\sigma} v_\gamma$ of any monotone (i.e. $v_\xi \geq v_\eta$ for $\xi \in \eta$) collection of elements of t_0 contains an open non-void subset. [5]

Lemma 1. Let $x \in Pol$, $b \subseteq Pol$, $\aleph_\eta < \sigma(ct)$, let $u \neq 0$ be an open set and let $u \subseteq F^r x \subseteq b \;\&\; card\, x = \aleph_\eta^{\urcorner}$. Then there exist $z \subseteq b$ and open $v \neq 0$ such that $card\, z = \aleph_\eta$, $v \subseteq u \cap F^r x = z^{\urcorner}$.

Proof. Let t_0 be a basis of the topology t such that $\bigcap_{\gamma \in \omega_\eta} v_\gamma$ contains an open non-void subset for every monotone

4) In the present paper, the operations, notions etc. in the ∇-model are provided with an asterisk.

5) For every space, $\sigma(ct) \leq \nu(ct)$ ($\nu(ct)$ is the least cardinal number $\aleph_\nu$ such that there is no open non-void set which can be covered by $\aleph_\nu$ closed nowhere dense sets). The present $\sigma(ct)$ is a minor modification of the characteristics considered in [13] (unpublished) and in [9].

collection $\{v_\gamma\}_{\gamma\in\omega_\eta}$ of elements of t_o. There is a polynomial g and $\bar{u}\in t_o$ such that $\bar{u}\subseteq u\cap F^\ulcorner g\ \mathit{Fn}\ \omega_\eta \,\&\, \mathcal{W}(g)=x^\urcorner$.

There exists a monotone collection $\{v_\gamma\}_{\gamma\in\omega_\eta}$, $v_\gamma\in t_o$ and a 1-1 sequence $\{y_\gamma\}_{\gamma\in\omega_\eta}$ of elements of b such that $v_\gamma\subseteq\bar{u}\cap F^\ulcorner\langle y_\gamma\gamma\rangle\in g^\urcorner$. Let $z=\{y_\gamma;\ \gamma\in\omega_\eta\}$, let $0\neq v\subseteq\bigcap_{\gamma\in\omega_\eta}v_\gamma$. Let us prove $v\subseteq F^\ulcorner x=z^\urcorner$. If there were $w\in t_o$ and $y\in Pol$ such that $w\subseteq v\cap F^\ulcorner y\in x \,\&\, y\notin z^\urcorner$, then there would exist $\gamma\in\omega_\eta$ and $\bar{w}\in t_o$ with $\bar{w}\subseteq w\cap F^\ulcorner\langle y\,\gamma\rangle\in g \,\&\, y\neq y_\gamma^\urcorner$, contradicting $\bar{w}\subseteq F^\ulcorner\langle y_\gamma\gamma\rangle\in g \,\&\, \mathit{Fnc}\,(g)^\urcorner$.

<u>Lemma 2</u>. Let $b\subseteq Pol$. Let $f\subseteq^* k_b$, $card^* f<^* k_{\aleph_\eta}$, $\aleph_\eta\leq\sigma(ct)$. Then there exists $g\in\widetilde{Pol}$ such that $g=^* f$ and $(\forall x\in\mathcal{D}(g))[g(x)\subseteq b \,\&\, card\ g(x)<\aleph_\eta]$.

<u>Proof</u>. We can assume that $\mathcal{D}(f)=\{x;\ x\in F^\ulcorner f(x)\subseteq b \,\&\, card\,f(x)<\aleph_\eta^\urcorner\}$. Evidently, $\mathcal{D}(f)$ is the union of pairwise disjoint open sets $f^{-1}(y)$ (for $y\in\mathcal{W}(f)$). Let $u=f^{-1}(x)$ be one of these, i.e. $u\subseteq F^\ulcorner x\subseteq b \,\&\, card\,x<\aleph_\eta^\urcorner$. According to the preceding lemma there exist $v(u)$ and $z(u)$ such that $card\ z(u)<\aleph_\eta$, $z(u)\subseteq b$ and $v(u)\subseteq F^\ulcorner x=z(u)^\urcorner$. Let us denote u by u_o. Let $u_\gamma=Int(u-\bigcup_{\xi\in\gamma}v(u_\xi))$. Let γ_o be the first ordinal such that $u_{\gamma_o}=0$. Let $u'=\bigcup_{\gamma\in\gamma_o}v(u_\gamma)$. Obviously, u' is dense in u. Let us define the function g on $u'\subseteq u$ as follows: $g(y)=z(u_\gamma)$ for $y\in v(u_\gamma)$. Similarly on other $u=f^{-1}(x)$, $x\in\mathcal{W}(f)$. Evidently $\mathcal{D}(g)$ is dense in $\mathcal{D}(f)$ and thus $\mathcal{D}(g)\in j$.
Then obviously $f=^* g$.

<u>Theorem 1</u>. Let $X\subseteq Pol$. Let $f\subseteq^*\tilde{X}$, $card^* f<^* k_{\aleph_\eta}$, $\aleph_\eta\leq\sigma(ct)$. Then there exists a g such that $g=^* f$ and

$(\forall x \in \mathfrak{D}(g))[g(x) \subseteq X \;\&\; \mathrm{card}\, g(x) < \aleph_\eta]$.

Proof. Let $f \subseteq^* X$. There is a subset z of the class $\tilde{X}$ such that $(\forall h \in^* f)(\exists h_1 \in z)[h =^* h_1 \;\&\; \mathcal{W}(h_1) \subseteq X]$. Let $k = \bigcup_{h \in z} \mathcal{W}(h)$. Then $f \subseteq^* k_k$ and the assertion follows from lemma 2.

3. Permutation submodels of the model ∇

The reader is assumed to be familiar with both permutation models and permutation submodels of the ∇-model, and with the notation used in [16] and [4]. G is a group of permutations of the set a, F a filter on G, $Q = Q(a, G, F)$ a subclass of $\Pi(a)$ determining a permutation model (model of the set theory without the axiom of regularity). [6] $\mathcal{G}$ is a group of permutations of the set ind, $\mathcal{F}$ a filter on $\mathcal{G}$, $P = P(\mathcal{G}, \mathcal{F})$ a subclass of Pol. The class $\tilde{P}$ determines an inner complete submodel (denoted as ∇_P) of the model ∇. This model is called a permutation submodel of the model ∇ and axioms of the theory Σ hold in it (Vopěnka and Hájek [16]).

For $x \in \Pi(a)$, $H(x)$ is the group of all $q \in G$ such that $qx = x$, and $K(x)$ is the group of all $q \in G$ such that q is identical on x. The subgroups $\mathcal{H}(x)$ and $\mathcal{K}(x)$ of $\mathcal{G}$ for $x \in Pol$ have a similar meaning.

Definition. Let G be a group of permutations of a, let F be a filter on G, γ an ordinal. F is called ω_γ-multiplicative if the intersection $\bigcap_{\xi \in \omega_\gamma} H_\xi$ of

6) This is a useful generalization (due to Specker, cf. [11]) of Fraenkel's and Mostowski's methods.

any collection $\{H_\xi\}_{\xi\in\omega_\eta}$ of elements of F belongs to F.

Lemma 3. Let $\aleph_\eta$ be a cardinal number. Let the filter $\mathcal{F}$ be ω_γ-multiplicative for all $\gamma\in\eta$. Let $P = P(\mathcal{G},\mathcal{F})$. Then, if $x \subseteq P$ and $card\, x < \aleph_\eta$, then $x \in P$.

Proof. Since $\mathcal{H}(x) \supseteq \mathcal{K}(x) = \bigcap_{y\in x} \mathcal{H}(y)$, the assertion is obvious.

Theorem 2. Let $\aleph_\eta$ be a cardinal number, let $\aleph_\eta \leq \sigma(c\,t)$. Let the filter $\mathcal{F}$ be ω_γ-multiplicative for all $\gamma \in \eta$. Let $P = P(\mathcal{G},\mathcal{F})$. Then, if $f \subseteq^* \tilde{P}$ and $card^* f <^* k_{\aleph_\eta}$, then $f \in \tilde{P}$.

Proof. According to theorem 1 there is $g =^* f$ such that $(\forall x \in \mathcal{D}(g))[g(x) \subseteq P \,\&\, card\, g(x) < \aleph_\eta]$. By the preceding lemma, $g(x)$ belongs to P and thus $f \in \tilde{P}$.

Theorem 3. Let the axiom of choice be true. Let M be a perfect class determining an inner complete model $\mathfrak{M}$. Let $\aleph_\alpha$ be a cardinal number. Let $(x)[x \subseteq M \,\&\, card\, x \leq \aleph_\alpha \to x \in M]$. Then (PDC $\aleph_\alpha$) holds in $\mathfrak{M}$.

Proof. Let R be a relation in the model $\mathfrak{M}$, $a \in M$, and for every $\gamma \in \omega_\alpha$ and $g \in (a^\gamma)_{\mathfrak{M}} = a^\gamma \cap M$ let there exist an $x \in a$ such that $\langle g\, x\rangle \in R$. It follows from the assumption that $a^\gamma \cap M = a^\gamma$ (because $g \subseteq M$ and $card\, g \leq \aleph_\alpha$ if $g \in a^\gamma$). Thus the assumptions of (PDC $\aleph_\alpha$) are satisfied by R, a and (since the axiom of choice holds) there is an $f \in a^{\omega_\alpha}$ such that $\langle f \upharpoonright \gamma,\, f(\gamma)\rangle \in R$ for every $\gamma \in \omega_\alpha$. Since $card\, f \leq \aleph_\alpha$, f belongs to M.

Corollary. If $\aleph_\eta \leq \sigma(c\,t)$ is a cardinal number and

$\mathcal{F}$ is ω_γ-multiplicative for all $\gamma \in \eta$, then ($PDC\ \aleph_\gamma$), ($AC\ \aleph_\gamma$) and ($WE\ \aleph_\gamma$) hold in ∇_p for all cardinals (of the model ∇_p) less than $\aleph_{\aleph_\eta}$.

4. The model θ.

The model θ (with parameters $\beta, \sigma, a, \mathcal{G}, \mathcal{F}$) is a permutation submodel of the model ∇ (cf. [4], [5]).

Lemma 4. If the model θ has parameters $\beta, \sigma, a, \mathcal{G}, \mathcal{F}$ then $\bar{\sigma}(ct) \geq \aleph_\beta$. ($\langle ct \rangle$ is the space from the definition of the model θ.)

In this section the following theorem is proved for any regular special cardinal number $\aleph_\alpha$:

Theorem 4. The parameters $\beta, \sigma, a, \mathcal{G}, \mathcal{F}$ can be chosen such that the statement $(\forall \gamma \in \alpha)[(AC\,\aleph_\gamma) \,\&\, (WE\,\aleph_\gamma) \,\&\, \&\, (PDC\,\aleph_\gamma)] \,\&\, \neg(AC\,\aleph_\alpha) \,\&\, \neg(WE\,\aleph_\alpha) \,\&\, \neg(PDC\,\aleph_\alpha) \,\&\, \neg(OP)$ holds in the model θ.

In the proof the method described in [4] and [5] will be used. There the following assertion was proved:

Let η be a special ordinal, let φ be a η-boundable[7] formula. If there exist a, G, F such that φ holds in the permutation model determined by $Q(a, G, F)$, then φ holds in a θ-model with suitably chosen parameters, i.e. with β sufficiently large, $\sigma \geq \beta$ and $\langle \mathcal{G}, \mathcal{F} \rangle$ feasible in reference to $\langle G, F \rangle$[7].

Since the formula $\varphi = \neg(AC\,\aleph_\alpha) \,\&\, \neg(WE\,\aleph_\alpha) \,\&\, \neg(PDC\,\aleph_\alpha) \,\&\, \neg(OP)$ is η-boundable (η is at most $\omega_\alpha + 3$), it is suffi-

7) The meaning of the expressions " η -boundable formula" and " $\langle \mathcal{G}, \mathcal{F} \rangle$ is feasible in reference to $\langle G, F \rangle$ " is explained in [5].

cient (according to preceding section, lemma 4 and the fact that $k_{\aleph_\alpha}$ is $\aleph_\alpha^*$ in ∇-model, if $\aleph_\alpha \leq \sigma(ct)$,

(i) to find a permutation model (i.e. the parameters a, G, F) in which φ holds, and

(ii) to choose sufficiently large β and find $\langle \mathcal{G}, \mathcal{F} \rangle$ feasible in reference to $\langle G, F \rangle$ such that $\mathcal{F}$ is ω_γ-multiplicative for all $\gamma \in \alpha$.

Remark. Let G be a group of permutations of the set a. Let γ be an ordinal. All subgroups $K(e)$ of G with $e \subseteq a$ and $card\ e < \aleph_\gamma$ generate a filter on G which is denoted by $F(\omega_\gamma)$. The filter $\mathcal{F}(\omega_\gamma)$ on $\mathcal{G}$ has a similar meaning.

The parameters a, G, F are chosen as follows (cf. Mostowski [8]): a is the union of ω_α pairs $\{x_\gamma, y_\gamma\}$, $\gamma \in \omega_\alpha$, G is the group of all permutations of a preserving every pair $\{x_\gamma, y_\gamma\}$, $F = F(\omega_\alpha)$, $Q = Q(a, G, F)$.

That φ holds in the permutation model determined by Q follows (as shown in section 1) from the following theorem.

Theorem 5. a) There is no function $f \in Q$ choosing one element from every pair $\{x_\gamma, y_\gamma\}$, $\gamma \in \omega_\alpha$.

b) If $x \subseteq a$ and $card\ x = \aleph_\alpha$, then there is no $f \in Q$ mapping ω_α onto x.

Proof. Let us prove b) (a) is analogous). Let $x \subseteq a$, $card\ x = \aleph_\alpha$, let $f \in Q$ map ω_α onto x. There exist $e \subseteq a$, $card\ e < \aleph_\alpha$ such that $H(f) \supseteq K(e)$. There is a $\gamma \in \omega_\alpha$ such that $x_\gamma \in x$ and neither x_γ nor y_γ belongs to e. The permutation q which exchanges x_γ and y_γ and is identical otherwise preserves e but not f, because if $\xi = f^{-1}(x_\gamma)$, then $q' \langle x_\gamma\ \xi \rangle = \langle y_\gamma\ \xi \rangle$

which cannot belong to f.

Remark. The set of all pairs $\{x_\gamma, y_\gamma\}, \gamma \in \omega_\alpha$ is well orderable in this permutation model and has cardinality $\aleph_\alpha$.

Now, choose sufficiently large β, $\delta \geq \beta$ and consider x_γ, y_γ as pairwise disjoint sets of individuals, $card\ x_\gamma = card\ y_\gamma = \aleph_\beta$.

It remains to find $\mathcal{G}$ and $\mathcal{F}$.

Lemma 5. A filter F is ω_γ-multiplicative iff there is a basis B of the filter F such that the intersection of any collection $\{H_\xi\}_{\xi \in \omega_\gamma}$ of elements of B belongs to F.

Lemma 6. Let G be a group of permutations of the set a, let $\aleph_\eta$ be a regular cardinal number. Then $F(\omega_\eta)$ is ω_γ-multiplicative for all $\gamma \in \eta$.

Proof. It suffices to prove that $\bigcap_{\xi \in \omega_\gamma} K(e_\xi) \in F(\omega_\eta)$ if $e_\xi \subseteq a$ and $card\ e_\xi < \aleph_\eta$ for all $\xi \in \omega_\gamma$. But $\bigcap_{\xi \in \omega_\gamma} K(e_\xi) = K(\bigcup_{\xi \in \omega_\gamma} e_\xi)$ and $card \bigcup_{\xi \in \omega_\gamma} e_\xi < \aleph_\eta$ ($\aleph_\eta$ is regular).

The parameters $\mathcal{G}$, $\mathcal{F}$ are chosen as follows: $\mathcal{G}$ is the group of all permutations p of ind which, extended to a (let us denote this extension by $ext(p)$), are permutations of a and belong to G. Let $H^{\mathcal{H}} = \mathcal{H} \cap \{p;\ ext(p) \in H\}$ for $H \in F(\omega_\alpha)$ and $\mathcal{H} \in \mathcal{F}(\omega_\alpha)$. $\mathcal{F}$ is the filter generated by all subgroups $H^{\mathcal{H}}$ of $\mathcal{G}$, where $H \in F(\omega_\alpha)$ and $\mathcal{H} \in \mathcal{F}(\omega_\alpha)$. According to [4], $\langle \mathcal{G}, \mathcal{F} \rangle$ is feasible in reference to $\langle G, F \rangle$.

Lemma 7. $\mathcal{F}$ is ω_γ-multiplicative for all $\gamma \in \alpha$.

Proof. Let $\gamma \in \alpha$, let $H_\xi \in F(\omega_\alpha)$, $\mathcal{H}_\xi \in \mathcal{F}(\omega_\alpha)$ for $\xi \in \omega_\gamma$ and let $H = \bigcap_{\xi \in \omega_\gamma} H_\xi$ and $\mathcal{H} = \bigcap_{\xi \in \omega_\gamma} \mathcal{H}_\xi$. It is obvious that $\bigcap_{\xi \in \omega_\gamma} H_\xi^{\mathcal{H}_\xi} = H^{\mathcal{H}}$ and, since, by lemma 6, $H \in F(\omega_\alpha)$ and $\mathcal{H} \in \mathcal{F}(\omega_\alpha)$, $\mathcal{F}$ is ω_γ-multiplicative by lemma 5.

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