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SUM OF CATEGORIES WITH AMALGAMATED SUBCATEGORY

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I. Preliminaries.

The following theorem is well known [4, p.41],[5],[2, p.224]: If $\{G_\alpha ; \alpha \in A\}$ is a set of groups and if H is a subgroup of all G_α with $G_\alpha \cap G_{\alpha'} = H$ for all $\alpha, \alpha' \in A$, $\alpha \neq \alpha'$, then there exists a group G such that

(1) all G_α are subgroups of G;

(2) if G' is a group and $\psi_\alpha : G_\alpha \to G'$ are homomorphisms such that $(\mu)\psi_\alpha = (\mu)\psi_{\alpha'}$ whenever $\alpha, \alpha' \in A$, $\mu \in H$, then there exists exactly one homomorphism $\psi : G \to G'$ such that $\iota_\alpha \cdot \psi = \psi_\alpha$ for every $\alpha \in A$ (where $\iota_\alpha : G_\alpha \to G$ denotes the inclusion-homomorphism).

The group G is usually called a free product of the groups G_α with the amalgamated subgroup H.

Analogous questions for semigroups are considered in [1]. In the present note there is solved an analoguous question concerning couples of small categories. However, if $\mathfrak{k}_1, \mathfrak{k}_2$ are small categories, $\mathfrak{l}$ is subcategory of both and $\mathfrak{k}_1 \cap \mathfrak{k}_2 = \mathfrak{l}$, then a category $\mathfrak{k}$ for which the statements (1),(2) hold (writing "category", "subcategory", "functor" instead of "group", "subgroup", "homomorphism", respectively) need not exist. But if the statement (1) is replaced by the weaker statement (1′) (cf. the definition below), then there always exists a category $\mathfrak{k}$ satisfying (1′),(2). We obtain the stronger result if we suppose that $\mathcal{l}$ is a full subcategory of $\mathcal{k}_1$ (cf. the Theorem), and if $\mathcal{l}$ is a full subcategory of both $\mathcal{k}_1, \mathcal{k}_2$, then $\mathcal{k}$ satisfies (1),(2) and $\mathcal{k}_1, \mathcal{k}_2$ are full subcategories of $\mathcal{k}$. The results may be easily extended to a set $\{\mathcal{k}_\alpha ; \alpha \in A\}$ of small categories or, in some theory of sets in which every class may be so well-ordered that all initial segments are sets, to arbitrary two categories $\mathcal{k}_1, \mathcal{k}_2$ ($\mathcal{l}$, however, must necessarily be small).

<u>Conventions</u>: a) If $\langle a_1, \dots, a_n \rangle$ and $\langle b_1, \dots, b_m \rangle$ are, respectively, n-tuples or m-tuples of elements, and if $\langle a_1, \dots, a_n \rangle = \langle b_1, \dots, b_m \rangle$, then $m = n$ and $a_i = b_i$ for $i = 1, \dots, n$.

b) If $\mathcal{k}$ is a category, denote by $\mathcal{k}^\sigma$ the class of its objects, $\mathcal{k}^m$ the class of its morphisms. If $a, b \in \mathcal{k}^\sigma$, denote by $H_{\mathcal{k}}(a, b)$ the set of all morphisms of $\mathcal{k}$ from a to b. A subcategory h of $\mathcal{k}$ is called full if $H_h(a, b) = H_{\mathcal{k}}(a, b)$ whenever $a, b \in h^\sigma$.

c) If $\alpha \in H_{\mathcal{k}}(a, b)$, $\beta \in H_{\mathcal{k}}(b, c)$, then their composition is denoted by $\alpha \cdot \beta$. The composition of mappings or functors is also written from right to left. The image of an element μ under a mapping φ is denoted by $(\mu)\varphi$.

d) If $\alpha \in H_{\mathcal{k}}(a, b)$, then α is always a triple, with first member a and third member b. Thus if for example $\alpha_1 \in H_{\mathcal{k}_1}(a_1, b_1)$, $\alpha_2 \in H_{\mathcal{k}_2}(a_2, b_2)$ and either $a_1 \neq a_2$ or $b_1 \neq b_2$, then $\alpha_1 \neq \alpha_2$; if $a \in \mathcal{k}_1^\sigma - \mathcal{k}_2^\sigma$, then $\mathcal{k}_2^m \cap (\bigcup_{b \in \mathcal{k}_1^\sigma} H_{\mathcal{k}_1}(a, b)) = \emptyset$; if $\mathcal{k}_1^\sigma \cap \mathcal{k}_2^\sigma = \emptyset$, then $\mathcal{k}_1^m \cap \mathcal{k}_2^m = \emptyset$.

The relation $\alpha \in \mathcal{k}^\sigma \cup \mathcal{k}^m$, will be equivalently expressed by $\alpha \in \mathfrak{k}$.

e) If $A \subset B$, $\varphi : B \to M$, $\psi : A \to N$ are mappings, then $\varphi/_A = \psi$ is to mean that $(x)\psi = (x)\varphi$ for all $x \in A$. If $\mathbb{A}$ is a monotone system of sets, $\varphi_A : A \to B_A$ is a mapping such that $\varphi_A/_{A'} = \varphi/_A$ whenever $A \in \mathbb{A}$, $A \in \mathbb{A}$, $A' \subset A$, denote by $\varphi = \bigcup_{A \in \mathbb{A}} \varphi_A$ the mapping of $\bigcup \mathbb{A}$ into $\bigcup_{A \in \mathbb{A}} B_A$ such that $\varphi/_A = \varphi_A$.

f) If R is an equivalence on a set A, $B \subset A$ is such that $x \in B$, $x R y$ imply $y \in B$, then the meaning of $B/_R$ is evident.

II. Definition and Theorem.

Definition. Let $\mathfrak{k}_1$, $\mathfrak{k}_2$ be small categories, ℓ a subcategory of both. Let $\langle \mathfrak{k}, \varphi_1, \varphi_2 \rangle$ be a triple such that

(1′) $\mathfrak{k}$ is a small category, $\varphi_1 : \mathfrak{k}_1 \to \mathfrak{k}$, $\varphi_2 : \mathfrak{k}_2 \to \mathfrak{k}$ are functors such that $(\mu)\varphi_1 = (\mu)\varphi_2$ whenever $\mu \in \ell$; $\varphi_1/_{\mathfrak{k}_1^\sigma}$, $\varphi_2/_{\mathfrak{k}_2^\sigma}$ are one-to-one mappings;

(2) if H is a category, $\psi_1 : \mathfrak{k}_1 \to H$, $\psi_2 : \mathfrak{k}_2 \to H$ are functors such that $(\mu)\psi_1 = (\mu)\psi_2$ whenever $\mu \in \ell$, then there exists exactly one functor $\psi : \mathfrak{k} \to H$ with $\varphi_1 \cdot \psi = \psi_1$, $\varphi_2 \psi = \psi_2$.

Then $\langle \mathfrak{k}, \varphi_1, \varphi_2 \rangle$ is called the sum of the categories $\mathfrak{k}_1$ and $\mathfrak{k}_2$ with the amalgamated subcategory ℓ.

Remark. It is easy to see that, for $\ell^\sigma = \emptyset$, there always exists such a $\langle \mathfrak{k}, \varphi_1, \varphi_2 \rangle$. In this case write

$k = k_1 \cup k_2$.

Theorem. Let k_1 , k_2 be small categories, ℓ a subcategory of both. Then the sum $\langle k , \varphi_1 , \varphi_2 \rangle$ of the categories k_1 and k_2 with the amalgamated subcategory ℓ exists. Moreover, if ℓ is a full subcategory of k_1 , then φ_2 is a full iso-functor and $(k_1^m)\varphi_1 \cap (k_2^m)\varphi_2 = (\ell^m)\varphi_1$, $(k_1^\sigma)\varphi_1 \cap (k_2^\sigma)\varphi_2 = (\ell^\sigma)\varphi_1$.

III. Proof of the Theorem and Examples.

Lemma 1. Let k_1 , k_2 be small categories, ℓ a full subcategory of both, $k_1^\sigma \cap k_2^\sigma = \ell^\sigma$, and let the sets $k_1^\sigma - \ell^\sigma$, $k_2^\sigma - \ell^\sigma$ contain exactly one element. Then the sum $\langle k, \iota_1, \iota_2 \rangle$ of k_1 and k_2 with the amalgamated subcategory ℓ exists. $_1$ and $_2$ are identical and k_1 and k_2 are full subcategories of k .

Proof. Set $k_i^\sigma - \ell^\sigma = \{a_i\}$ for $i = 1, 2$, $a_1 \neq a_2$. We shall construct the category k . Put $k^\sigma = k_1^\sigma \cup k_2^\sigma$, k_1 and k_2 are to be full subcategories of k . Now we must define the sets $H_k(a_1, a_2)$, $H_k(a_2, a_1)$ and the composition in k . Denote by $\overset{n}{\cdot}$ the composition in k_n $(n = 1, 2)$. Let $\{i, j\} = \{1, 2\}$. Put $S_i = \bigcup_{b \in \ell^\sigma} \{H_{k_i}(a_i, b) \times H_{k_j}(b, a_j)\}$. Denote by R_i the following relation on S_i : $\langle \alpha , \beta \rangle R_i \langle \gamma , \delta \rangle$ if and only if there exists a $\rho \in \ell^m$ such that $\alpha \overset{i}{\cdot} \rho = \gamma$, $\beta = \rho \overset{j}{\cdot} \delta$. Let R_i^* be the smallest equivalence on S_i which contains R_i ; put $S_i^* = S_i / R_i^*$. If $\langle \alpha , \beta \rangle \in S_i$, denote by $\langle \alpha , \beta \rangle^*$ the element of S_i^* which contains $\langle \alpha , \beta \rangle$. Put $H_k(a_i , a_j) = \{a_i\} \times S_i^* \times \{a_j\}$.

For $\langle \alpha, \beta \rangle \in S_i$ set $\langle \overline{\alpha, \beta} \rangle = \langle a_i, \langle \alpha, \beta \rangle^*, a_j \rangle$. Now we define the composition $\cdot$ in k : Let $\{i, j\} = \{1, 2\}$.

1) If $\mu, \nu \in k_n^m$ $(n = 1, 2)$, then $\mu \cdot \nu$ is defined if and only if $\mu \overset{n}{\cdot} \nu$ is defined; obvioual y $\mu \cdot \nu = \mu \overset{n}{\cdot} \nu$.

2) For $\mu \in H_{k_i}(b, a_i)$, $\nu \in H_{k_j}(a_j, c)$, $\rho \in H_k(a_i, a_j)$, $\rho = \langle \overline{\alpha, \beta} \rangle$, $b, c \in \ell^\sigma$ set $\mu \cdot \rho = (\mu \overset{i}{\cdot} \alpha) \overset{j}{\cdot} \beta$, $\rho \cdot \nu = \alpha \overset{i}{\cdot} (\beta \overset{j}{\cdot} \nu)$.

3) For $\tau_n \in H_k(a_n, a_n)$, $n = 1, 2$, $\sigma \in H_k(a_i, a_j)$, $\sigma = \langle \overline{\alpha, \beta} \rangle$, set $\tau_i \cdot \sigma = \langle \overline{\tau_i \overset{i}{\cdot} \alpha, \beta} \rangle$, $\sigma \cdot \tau_j = \langle \overline{\alpha, \beta \overset{j}{\cdot} \tau_j} \rangle$.

4) For $\sigma_1 \in H_k(a_1, a_2)$, $\sigma_2 \in H_k(a_2, a_1)$, $\sigma_n = \langle \overline{\alpha_n, \beta_n} \rangle$, $n = 1, 2$, set $\sigma_i \cdot \sigma_j = \alpha_i \overset{i}{\cdot} (\beta_i \overset{j}{\cdot} \alpha_j) \overset{i}{\cdot} \beta_j$.

5) For $\alpha \in H_{k_i}(a_i, b)$, $\beta \in H_{k_j}(b, a_j)$, $b \in \ell^\sigma$, set $\alpha \cdot \beta = \langle \overline{\alpha, \beta} \rangle$.

A) Now prove that the definition of the composition $\cdot$ does not depend on the choice of the element from $\langle \alpha, \beta \rangle^*$. It is sufficient to prove the following proposition: Let $\langle \alpha_n, \beta_n \rangle \in S_n$, $\langle \gamma_n, \sigma_n \rangle \in S_n$, $\langle \alpha_n, \beta_n \rangle R_n \langle \gamma_n, \sigma_n \rangle$, $\tau_n \in H_{k_n}(a_n, a_n)$, $n = 1, 2$. Let $\{i, j\} = \{1, 2\}$, $\mu \in H_{k_i}(b, a_i)$, $\nu \in H_{k_j}(a_j, c)$, $b, c \in \ell^\sigma$. Then

a) $(\mu \overset{i}{\cdot} \alpha_i) \overset{j}{\cdot} \beta_i = (\mu \overset{i}{\cdot} \gamma_i) \overset{j}{\cdot} \sigma_i$;

b) $\alpha_i \overset{i}{\cdot} (\beta_i \overset{j}{\cdot} \nu) = \gamma_i \overset{i}{\cdot} (\sigma_i \overset{j}{\cdot} \nu)$;

c) $\langle \tau_i \overset{i}{\cdot} \alpha_i, \beta_i \overset{j}{\cdot} \tau_j \rangle R_i \langle \tau_i \overset{i}{\cdot} \gamma_i, \sigma_i \overset{j}{\cdot} \tau_j \rangle$;

d) $\alpha_i \overset{i}{\cdot} (\beta_i \overset{j}{\cdot} \alpha_j) \overset{i}{\cdot} \beta_j = \gamma_i \overset{i}{\cdot} (\sigma_i \overset{j}{\cdot} \gamma_j) \overset{i}{\cdot} \sigma_j$.

Thus let $\alpha_i \overset{i}{\cdot} \rho_i = \gamma_i$, $\beta_i = \rho_i \overset{j}{\cdot} \sigma_i$, $\rho_i \in \ell^m$ (in the proof of a),b) the indices i for $\alpha, \beta, \gamma, \sigma, \rho$ are left out):

a) $(\mu \overset{i}{\cdot} \alpha) \overset{j}{\cdot} \beta = (\mu \overset{i}{\cdot} \alpha) \overset{j}{\cdot} (\rho \overset{j}{\cdot} \sigma) = [(\mu \overset{i}{\cdot} \alpha) \overset{i}{\cdot} \rho] \overset{j}{\cdot} \sigma =$
$= [\mu \overset{i}{\cdot} (\alpha \overset{i}{\cdot} \rho)] \overset{j}{\cdot} \sigma = (\mu \overset{i}{\cdot} \gamma) \overset{j}{\cdot} \sigma$;

b) $\alpha \overset{i}{\cdot} (\beta \overset{j}{\cdot} \nu) = \alpha \overset{i}{\cdot} (\rho \overset{j}{\cdot} \sigma \overset{j}{\cdot} \nu) = \alpha \overset{i}{\cdot} [\rho \overset{i}{\cdot} (\sigma \overset{j}{\cdot} \nu)] = \gamma \overset{i}{\cdot} (\sigma \overset{j}{\cdot} \nu)$;

c) is evident;

d) $\alpha_i \overset{i}{\cdot} (\beta_i \overset{j}{\cdot} \alpha_j) \overset{i}{\cdot} \beta_j = \alpha_i \overset{i}{\cdot} [(\rho_i \overset{j}{\cdot} \sigma_i) \overset{j}{\cdot} \alpha_j] \overset{i}{\cdot} (\rho_j \overset{i}{\cdot} \sigma_j) = \alpha_i \overset{i}{\cdot}$
$\overset{i}{\cdot} [\rho_i \overset{i}{\cdot} (\sigma_i \overset{j}{\cdot} \alpha_j)] \overset{i}{\cdot} (\rho_j \overset{i}{\cdot} \sigma_j) = \gamma_i \overset{i}{\cdot} (\sigma_i \overset{j}{\cdot} \alpha_j) \overset{i}{\cdot} \rho_j \overset{i}{\cdot} \sigma_j =$
$= \gamma_i \overset{i}{\cdot} [(\sigma_i \overset{j}{\cdot} \alpha_j) \overset{j}{\cdot} \rho_j] \overset{i}{\cdot} \sigma_j = \gamma_i \overset{i}{\cdot} (\sigma_i \overset{j}{\cdot} \gamma_j) \overset{i}{\cdot} \sigma_j$.

B) Next, prove the associativity of the composition $\cdot$:

α) First verify associativity for elements of $k_1^m \cup k_2^m$:

1) If $\mu, \nu, \sigma \in k_n^m$ $(n = 1, 2)$ then evidently
$(\mu \cdot \nu) \cdot \sigma = \mu \cdot (\nu \cdot \sigma)$.

2) Let $\{i, j\} = \{1, 2\}$, $\alpha \in H_{k_i}(a_i, b)$, $\beta \in H_{k_j}(b, a_j)$, $b \in \ell^\sigma$.
Let $\mu \in H_{k_i}(c, a_i)$, $\nu \in H_{k_j}(a_j, d)$.
Then
$\mu \cdot (\alpha \cdot \beta) = (\mu \cdot \alpha) \cdot \beta$, $(\alpha \cdot \beta) \cdot \nu = \alpha \cdot (\beta \cdot \nu)$ follows immediately from the definition of the composition $\cdot$.

3) Let $\alpha \in H_{k_i}(a_i, b)$, $\rho \in H_\ell(b, c)$, $\beta \in H_{k_j}(c, a_j)$, $b, c \in \ell^\sigma$.
Then $(\alpha \cdot \rho) \cdot \beta = \overline{\langle \alpha \overset{i}{\cdot} \rho, \beta \rangle}$, $\alpha \cdot (\rho \cdot \beta) = \overline{\langle \alpha, \rho \overset{j}{\cdot} \beta \rangle}$. But
$\langle \alpha \overset{i}{\cdot} \rho, \beta \rangle \; R_i \; \langle \alpha, \rho \overset{j}{\cdot} \beta \rangle$.

β) Now prove associativity in the remaining cases:
Let $\{i, j\} = \{1, 2\}$, let $\mu \in H_k(a_i, a_j)$. Then there exist $\alpha \in k_i^m$, $\beta \in k_j^m$ such that $\mu = \alpha \cdot \beta$.

1) Let $\nu_i \in H_{k_i}(b_i, a_i)$, $\sigma_i \in H_{k_i}(c_i, b_i)$, $\nu_j \in H_{k_j}(a_j, b_j)$, $\sigma_j \in H_{k_j}(b_j, c_j)$, $b_i, c_i \in k_i^\sigma$, $b_j, c_j \in k_j^\sigma$. Then

$$\sigma_i \cdot (\nu_i \cdot \mu) = \sigma_i \cdot [\nu_i \cdot (\alpha \cdot \beta)] = \sigma_i \cdot [(\nu_i \cdot \alpha) \cdot \beta] = [\sigma_i \cdot (\nu_i \cdot \alpha)] \cdot \beta = (\sigma_i \cdot \nu_i) \cdot \mu \; ;$$

$$(\mu \cdot \nu_j) \cdot \sigma_j = [(\alpha \cdot \beta) \cdot \nu_j] \cdot \sigma_j = [\alpha \cdot (\beta \cdot \nu_j)] \cdot \sigma_j = \alpha \cdot [(\beta \cdot \nu_j) \cdot \sigma_j] = \alpha \cdot [\beta \cdot (\nu_j \cdot \sigma_j)] = \mu \cdot (\nu_j \cdot \sigma_j) .$$

2) Let $\nu_i \in H_{k_i}(b_i, a_i)$, $\sigma_i \in H_{k_j}(a_j, b_i)$, $\nu_j \in H_{k_j}(a_j, b_j)$, $\sigma_j \in H_{k_i}(a_i, b_j)$, $b_i, b_j \in \ell^{\sigma}$. Then

$(\sigma_i \cdot \nu_i) \cdot \mu = \sigma_i \cdot (\nu_i \cdot \alpha) \cdot \beta = \sigma_i \cdot [(\nu_i \cdot \alpha) \cdot \beta] = \sigma_i \cdot (\nu_i \cdot \mu)$;

$\mu \cdot (\nu_j \cdot \sigma_j) = \alpha \cdot (\beta \cdot \nu_j) \cdot \sigma_j = [\alpha \cdot (\beta \cdot \nu_j)] \cdot \sigma_j = (\mu \cdot \nu_j) \cdot \sigma_j$.

3) Let $\nu \in H_{k_i}(b, a_i)$, $\sigma \in H_{k_j}(a_j, c)$, $b \in k_i^{\sigma}$, $c \in k_j^{\sigma}$. Then $(\nu \cdot \mu) \cdot \sigma = [\nu \cdot (\alpha \cdot \beta)] \cdot \sigma = [(\nu \cdot \alpha) \cdot \beta] \cdot \sigma =$

$= (\nu \cdot \alpha) \cdot (\beta \cdot \sigma) = \nu \cdot [\alpha \cdot (\beta \cdot \sigma)] = \nu \cdot (\mu \cdot \sigma)$.

4) Let $\nu \in H_k(a_j, a_i)$. Then there exist $\gamma \in k_j^m$, $\delta \in k_i^m$ such that $\nu = \gamma \cdot \delta$.

a) Let $\sigma \in H_{k_j}(a_j, b)$, $\tau \in H_{k_j}(c, a_j)$, $b, c \in k_j^{\sigma}$. Then

$(\nu \cdot \mu) \cdot \sigma = [\gamma \cdot (\delta \cdot \alpha) \cdot \beta] \cdot \sigma = [\gamma \cdot \{(\delta \cdot \alpha) \cdot \beta\}] \cdot \sigma =$

$= \gamma \cdot [\{(\delta \cdot \alpha) \cdot \beta\} \cdot \sigma] = \gamma \cdot [(\delta \cdot \alpha) \cdot (\beta \cdot \sigma)] =$

$= \gamma \cdot [\delta \cdot \{\alpha \cdot (\beta \cdot \sigma)\}] = (\gamma \cdot \delta) \cdot \{\alpha \cdot (\beta \cdot \sigma)\} = \nu \cdot (\mu \cdot \sigma)$

and analogously

$\tau \cdot (\nu \cdot \mu) = \tau \cdot [\gamma \cdot (\delta \cdot \alpha) \cdot \beta] = (\tau \cdot \nu) \cdot \mu$.

b) Let $\sigma \in H_k(a_i, a_j)$. Then there exist $\xi \in k_i^m$, $\eta \in k_j^m$ such that $\sigma = \xi \cdot \eta$ and

$\sigma \cdot (\nu \cdot \mu) = (\xi \cdot \eta) \cdot [(\nu \cdot \alpha) \cdot \beta] = \xi \cdot \{\eta [(\nu \cdot \alpha) \cdot \beta]\} =$

$= \xi \cdot \{[\eta \cdot (\nu \cdot \alpha)] \cdot \beta\} = \xi \cdot \{[(\eta \cdot \nu) \cdot \alpha] \cdot \beta\} =$

$= \xi \cdot \{(\eta \cdot \nu) \cdot (\alpha \cdot \beta)\} = [\xi \cdot (\eta \cdot \nu)] \cdot \mu = (\sigma \cdot \nu) \cdot \mu$.

C) Now let $\psi_1 : k_1 \to H$, $\psi_2 : k_2 \to H$ be functors such that $(\mu)\psi_1 = (\mu)\psi_2$ whenever $\mu \in \ell$; we proceed to extend them. If $\{i, j\} = \{1, 2\}$, $\langle \overline{\alpha, \beta} \rangle \in H_k(a_i, a_j)$ then, of course, put $(\langle \overline{\alpha, \beta} \rangle)\psi = (\alpha)\psi_i \cdot (\beta)\psi_j$. If $\langle \overline{\alpha, \beta} \rangle = \langle \overline{\gamma, \delta} \rangle$, then evidently $(\alpha)\psi_i \cdot (\beta)\psi_j = (\gamma)\psi_i \cdot (\delta)\psi_j$. If $\overline{\psi} : k \to H$

is another functor such that $(\mu)\bar{\psi} = (\mu)\psi_n$ whenever $\mu \in k_n$ $(n = 1, 2)$, then evidently $\psi = \bar{\psi}$.

Definition. Let a set A be ordered by $<$, let $\mathcal{A} = \{k_\alpha; \alpha \in A\}$ be a system of small categories. If k_α is a full subcategory of $k_{\alpha'}$ whenever $\alpha, \alpha' \in A$, $\alpha < \alpha'$, then $\mathcal{A}$ will be called monotone. In this case denote by $\underset{\alpha \in A}{S} k_\alpha$ the category k such that $k^\sigma = \bigcup_{\alpha \in A} k_\alpha^\sigma$ and that all k_α are full subcategories of k.

Lemma 2. Let h_1, h_2 be small categories, ℓ a full subcategory of both, $h_1^\sigma \cap h_2^\sigma = \ell^\sigma$ and the set $h_1^\sigma - \ell^\sigma$ contains exactly one element. Then there exists the sum $\langle h, \iota_1, \iota_2 \rangle$ of h_1 and h_2 with the amalgamated subcategory ℓ such that $_1$, $_2$ are identical full iso-functors.

Proof: For any ordinal α denote by T_α the set of all ordinals less than α. Let τ be an ordinal such that $\tau > card\,(h_2^\sigma - \ell^\sigma)$ (we may suppose $card\,(h_2^\sigma - \ell^\sigma) > 0$). Let f be a one-to-one mapping of $h_2^\sigma - \ell^\sigma$ into the set of all isolated ordinals in T_τ. For $\alpha \in T_\tau$ denote by $h_{2,\alpha}$ the full subcategory of h_2 such that $h_{2,\alpha}^\sigma = \ell^\sigma \cup (T_\alpha) f^{-1}$. Using lemma 1 and transfinite induction, one obtains a monotone system $\{k_\alpha; \alpha \in T_\tau\}$ of small categories such that $k_0 = h_1$, $k_\alpha = \underset{\beta < \alpha}{S} k_\beta$ whenever α is non-isolated and if $\alpha = \beta + 1$, then k_α is the sum of k_β and $h_{2,\alpha}$ with the amalgamated subcategory $h_{2,\beta}$. Of course put $h = \underset{\alpha \in T_\tau}{S} k_\alpha$; then evidently h has the required properties.

Proposition 1. Let $\mathscr{k}_1$, $\mathscr{k}_2$ be small categories, $\mathscr{l}$ a full subcategory of both, $\mathscr{k}_1^\sigma \cap \mathscr{k}_2^\sigma = \mathscr{l}^\sigma$. Then there exists the sum $\langle \mathscr{k}, \iota_1, \iota_2 \rangle$ of $\mathscr{k}_1$ and $\mathscr{k}_2$ with the amalgamated subcategory $\mathscr{l}$ such that ι_1, ι_2 are identical full iso-functors.

Proof: The proof of proposition 1 is analoguous to that of lemma 2. It is only necessary to use the lemma 2 instead of lemma 1.

Lemma 3. Let $\mathscr{k}$, $\mathscr{h}$ be small categories; let $\mathscr{l}$ be a full subcategory of $\mathscr{h}$, let $\mathscr{h}^\sigma - \mathscr{l}^\sigma$ contain exactly one element; let $\mathscr{l}$ be a subcategory of $\mathscr{k}$, $\mathscr{k}^\sigma = \mathscr{l}^\sigma$. Then there exists the sum $\langle K, \iota, \varphi \rangle$ of $\mathscr{k}$ and $\mathscr{h}$ with the amalgamated subcategory $\mathscr{l}$; ι is the identical full iso-functor, φ is identical on $\mathscr{l}$ and on $\mathscr{h}^\sigma$.

Proof: Set $\mathscr{h}^\sigma - \mathscr{l}^\sigma = \{a\}$. We shall construct the category K with the required properties. On putting $K^\sigma = \mathscr{h}^\sigma$, $\mathscr{k}$ is to be a full subcategory of K. It remains to define the sets $H_K(a,a)$, $H_K(b,a)$, $H_K(a,b)$ for all $b \in \mathscr{k}^\sigma$, and also the composition $\cdot$ in K. Denote by $\overset{\mathscr{k}}{\cdot}$ (or $\overset{\mathscr{h}}{\cdot}$) the composition in $\mathscr{k}$ (or in $\mathscr{h}$ respectively). Let $b \in \mathscr{k}^\sigma$; put $A_b = \bigcup_{c \in \mathscr{k}^\sigma} \{H_{\mathscr{h}}(a,c) \times H_{\mathscr{k}}(c,b)\}$. Let R_b be the following relation on A_b: $\langle \alpha, \mu \rangle R_b \langle \alpha', \mu' \rangle$ if and only if there exists a $\varkappa \in \mathscr{l}^m$ such that $\alpha = \alpha' \overset{\mathscr{h}}{\cdot} \varkappa$, $\mu' = \varkappa \overset{\mathscr{k}}{\cdot} \mu$. Denote by R_b^* the smallest equivalence relation containing R_b.

Put $A_b^* = A_b / R_b$; if $\langle \alpha, \mu \rangle \in A_b$, denote by $\langle \alpha, \mu \rangle^*$ the element of A_b^* which contains $\langle \alpha, \mu \rangle$;

set $\langle\overline{\alpha,\mu}\rangle = \langle a, \langle \alpha,\mu\rangle^*, b\rangle$, and $H_K(a,b) = \{a\} \times A_b^* \times \{b\}$. Set $B_b = \bigcup_{c\in k^o}\{H_k(b,c) \times H_k(c,a)\}$. Let S_b be the following relation on B_b: $\langle\mu,\alpha\rangle\ S_b\ \langle\mu',\alpha'\rangle$ if and only if there exists a $\varkappa \in \ell^m$ such that $\mu' = \mu \overset{k}{\cdot} \varkappa$, $\alpha = \varkappa \overset{h}{\cdot} \alpha'$. Let S_b^* be the smallest equivalence relation containing S_b. But $B_b^* = B_b/_{S_b^*}$; if $\langle\mu,\alpha\rangle \in B_b$, denote by $\langle\mu,\alpha\rangle^*$ the element of B_b^* which contains $\langle\mu,\alpha\rangle$; set $\langle\overline{\mu,\alpha}\rangle = \langle b, \langle\mu,\alpha\rangle^*, a\rangle$ and $H_K(b,a) = \{b\}\times B_b^* \times \{a\}$.
Set $C = [\bigcup_{c,d\in k^o}\{H_h(a,c)\times H_k(c,d)\times H_h(d,a)\}] \cup [\bigcup_{c,d\in h^o}\{H_h(a,c)\times H_h(c,d)\times H_h(d,a)\}]$.
Let P be the following relation on C: $\langle\alpha,\beta,\gamma\rangle\ P\ \langle\alpha',\beta',\gamma'\rangle$ if and only if either $\beta,\beta' \in h^m$, $\alpha \overset{h}{\cdot} \beta \overset{h}{\cdot} \gamma = \alpha' \overset{h}{\cdot} \beta' \overset{h}{\cdot} \gamma'$ or there exist $\rho, \sigma \in \ell^m$, $\tilde\beta \in k^m$ such that $\alpha' = \alpha \overset{h}{\cdot} \rho$, $\beta' = \tilde\beta \overset{k}{\cdot} \sigma$, $\beta = \rho \overset{k}{\cdot} \tilde\beta$, $\gamma = \sigma \overset{h}{\cdot} \gamma'$. Let P^* be the smallest equivalence relation containing C. Put $C^* = C/_{P^*}$; for $\langle\alpha,\beta,\gamma\rangle \in C$ denote by $\langle\alpha,\beta,\gamma\rangle^*$ the element of C^* which contains $\langle\alpha,\beta,\gamma\rangle$; set $\langle\overline{\alpha,\beta,\gamma}\rangle = \langle a, \langle\alpha,\beta,\gamma\rangle^*, a\rangle$ and $H_K(a,a) = \{a\}\times C^* \times \{a\}$.

A) Definition of the composition $\cdot$ in K: Let
$b_1, b_2, b_3 \in k^o$, $\beta_1 \in H_k(b_1,b_2)$, $\beta_2 \in H_k(b_2,b_3)$, $\xi \in H_k(b_2,b_1)$, $\gamma_1 = \langle\overline{\mu_1,\alpha_1}\rangle \in H_K(b_1,a)$, $\gamma_2 = \langle\overline{\alpha_2,\mu_2}\rangle \in H_K(a,b_2)$, $\sigma = \langle\overline{\rho,\lambda}\rangle \in H_K(a,b_1)$, $\tau = \langle\overline{\xi,\eta,\vartheta}\rangle \in H_K(a,a)$, $\tau_o = \langle\overline{\xi_o,\eta_o,\vartheta_o}\rangle \in H_K(a,a)$.
Put

a) $\beta_1 \cdot \beta_2 = \beta_1 \overset{k}{\cdot} \beta_2$;

b) $\xi \cdot \gamma_1 = \langle\overline{\xi \overset{k}{\cdot} \mu_1, \alpha_1}\rangle$;

c) $\gamma_2 \cdot \xi = \langle\overline{\alpha_2, \mu_2 \overset{k}{\cdot} \xi}\rangle$;

d) $\gamma_1 \cdot \gamma_2 = \mu_1 \overset{k}{\cdot} (\alpha_1 \overset{h}{\cdot} \alpha_2) \overset{k}{\cdot} \mu_2$;

e) $\gamma_1 \cdot \tau = \langle\overline{\mu_1, \alpha_1 \overset{h}{\cdot} \xi \overset{h}{\cdot} \eta \overset{h}{\cdot} \vartheta}\rangle$ whenever $\eta \in h^m$, and
$\gamma_1 \cdot \tau = \langle\overline{\mu_1 \overset{k}{\cdot} (\alpha_1 \overset{h}{\cdot} \xi) \overset{k}{\cdot} \eta, \vartheta}\rangle$ otherwise x);

f) $\tau \cdot \gamma_2 = \langle\overline{\xi \overset{h}{\cdot} \eta \overset{h}{\cdot} \vartheta \overset{h}{\cdot} \alpha_2, \mu_2}\rangle$ if $\eta \in h^m$, and
$\tau \cdot \gamma_2 = \langle\overline{\xi, \eta \overset{k}{\cdot} (\vartheta \overset{h}{\cdot} \alpha_2) \overset{k}{\cdot} \mu_2}\rangle$ otherwise x);

x) Evidently, if $\eta \in \ell^m$ (or $\eta_o \in \ell^m$ respectively), then the two definitions coincide. Similarly in h).

g) $\delta \cdot \gamma_1 = \langle \overline{\rho, \lambda \overset{k}{\cdot} \mu_1, \alpha_1} \rangle$;

h) $\tau \cdot \tau_0 = \langle \overline{\xi, \eta, \vartheta \overset{h}{\cdot} \xi_0 \overset{h}{\cdot} \eta_0 \overset{h}{\cdot} \vartheta_0} \rangle$ whenever $\eta_0 \in h^m$,

$\tau \cdot \tau_0 = \langle \overline{\xi \overset{h}{\cdot} \eta \overset{h}{\cdot} \vartheta \overset{h}{\cdot} \xi_0, \eta_0, \vartheta_0} \rangle$ whenever $\eta \in h^m$,

and $\tau \cdot \tau_0 = \langle \xi, \eta \overset{k}{\cdot} (\vartheta \overset{h}{\cdot} \xi_0) \overset{k}{\cdot} \eta_0, \vartheta_0 \rangle$ otherwise.

B) Now prove that the definition of the composition $\cdot$ given in A) is independent of the choice of elements from $\langle \cdot , \cdot \rangle^*$ or $\langle \cdot , \cdot , \cdot \rangle^*$: The cases a) b) c) are evident.

d) Let $\langle \mu_1, \alpha_1 \rangle S_{b_1} \langle \mu_1', \alpha_1' \rangle$, i.e. $\mu_1' = \mu_1 \overset{k}{\cdot} \varkappa_1$, $\alpha_1 = \varkappa_1 \overset{h}{\cdot} \alpha_1'$ for some $\varkappa_1 \in \ell^m$. Then, since $\alpha_1' \overset{h}{\cdot} \alpha_2 \in \ell^m$, $\mu_1 \overset{k}{\cdot} (\alpha_1 \overset{h}{\cdot} \alpha_2) \overset{k}{\cdot} \mu_2 = \mu_1' \overset{k}{\cdot} (\alpha_1' \overset{h}{\cdot} \alpha_2) \overset{k}{\cdot} \mu_2$.

Let $\langle \mu_2, \alpha_2 \rangle R_{b_2} \langle \mu_2', \alpha_2' \rangle$, i.e. $\alpha_2 = \alpha_2' \overset{h}{\cdot} \ {}_2$, $\mu_2' = {}_2 \overset{k}{\cdot} \mu_2$ for some $\varkappa_2 \in \ell^m$. Then, since $\alpha_1 \overset{h}{\cdot} \alpha_2' \in \ell^m$, $\mu_1 \overset{k}{\cdot} (\alpha_1 \overset{h}{\cdot} \alpha_2) \overset{k}{\cdot} \mu_2 = \mu_1 \overset{k}{\cdot} (\alpha_1 \overset{h}{\cdot} \alpha_2') \overset{k}{\cdot} \mu_2'$.

e) Let $\langle \mu_1, \alpha_1 \rangle S_{b_1} \langle \mu_1', \alpha_1' \rangle$, i.e. $\mu_1' = \mu_1 \overset{k}{\cdot} \ {}_1$, $\alpha_1 = {}_1 \overset{h}{\cdot} \alpha_1'$ for some $\varkappa_1 \in \ell^m$. Then evidently $\langle \overline{\mu_1, \alpha_1} \rangle \cdot \langle \overline{\xi, \eta, \vartheta} \rangle = \langle \overline{\mu_1', \alpha_1'} \rangle \cdot \langle \overline{\xi, \eta, \vartheta} \rangle$ whenever $\eta \in h^m$. If $\eta \notin h^m$, then $\alpha_1' \overset{h}{\cdot} \xi \in \ell^m$, and consequently $\langle \overline{\mu_1 \overset{k}{\cdot} (\alpha_1 \overset{h}{\cdot} \xi) \overset{k}{\cdot} \eta, \vartheta} \rangle =$
$= \langle \overline{\mu_1' \overset{k}{\cdot} (\alpha_1' \overset{h}{\cdot} \xi) \overset{k}{\cdot} \eta, \vartheta} \rangle$.

Let $\langle \xi, \eta, \vartheta \rangle P \langle \xi', \eta', \vartheta' \rangle$.

α) Let $\eta, \eta' \in h^m$, $\xi \overset{h}{\cdot} \eta \overset{h}{\cdot} \vartheta = \xi' \overset{h}{\cdot} \eta' \overset{h}{\cdot} \vartheta'$. Then evidently $\langle \overline{\mu_1, \alpha_1} \rangle \cdot \langle \overline{\xi, \eta, \vartheta} \rangle = \langle \overline{\mu_1, \alpha_1} \rangle \cdot \langle \overline{\xi', \eta', \vartheta'} \rangle$.

β) Let $\xi' = \xi \overset{h}{\cdot} \varepsilon$, $\eta' = \tilde{\eta} \overset{k}{\cdot} \sigma$, $\eta = \varepsilon \overset{k}{\cdot} \tilde{\eta}$, $\vartheta = \sigma \overset{h}{\cdot} \vartheta'$ for some $\varepsilon, \sigma \in \ell^m$, $\tilde{\eta} \in k^m$. Then

$\langle \overline{\mu_1, \alpha_1} \rangle \cdot \langle \overline{\xi, \eta, \vartheta} \rangle = \langle \overline{\mu_1 \overset{k}{\cdot} (\alpha_1 \overset{h}{\cdot} \xi) \overset{k}{\cdot} \eta, \vartheta} \rangle =$

$= \langle \overline{\mu_1 \overset{k}{\cdot} (\alpha_1 \overset{h}{\cdot} \xi) \overset{k}{\cdot} \varepsilon \overset{k}{\cdot} \tilde{\eta}, \ \sigma \overset{h}{\cdot} \vartheta'} \rangle = \langle \overline{\mu_1 \overset{k}{\cdot} (\alpha_1 \overset{h}{\cdot} \xi') \overset{k}{\cdot} \eta', \vartheta'} \rangle =$

$= \langle \overline{\mu_1, \alpha_1} \rangle \cdot \langle \overline{\xi', \eta', \vartheta'} \rangle$.

f) is analoguous to e).

g) Let $\langle \rho, \lambda \rangle R_{b_1} \langle \rho', \lambda' \rangle$, i.e. $\rho = \rho' \overset{h}{\cdot} \pi$, $\lambda' = \pi \overset{k}{\cdot} \lambda$ for some $\pi \in \ell^m$. Then $\langle \overline{\rho, \lambda} \rangle \cdot \langle \overline{\mu_1, \alpha_1} \rangle = \langle \overline{\rho, \lambda \overset{k}{\cdot} \mu_1, \alpha_1} \rangle = \langle \overline{\rho', \lambda' \overset{k}{\cdot} \mu_1, \alpha_1} \rangle$
Let $\langle \mu_1, \alpha_1 \rangle S_{b_1} \langle \mu_1', \alpha_1' \rangle$, i.e. $\mu_1' = \mu_1 \overset{k}{\cdot} \varkappa_1$, $\alpha_1 = \varkappa_1 \overset{h}{\cdot} \alpha_1'$. Then evidently $\langle \overline{\rho, \lambda} \rangle \cdot \langle \overline{\mu_1, \alpha_1} \rangle =$
$= \langle \overline{\rho, \lambda} \rangle \cdot \langle \overline{\mu_1', \alpha_1'} \rangle$.

h) Let $\langle \xi, \eta, \vartheta \rangle P \langle \xi', \eta', \vartheta' \rangle$. Then evidently $\langle \overline{\xi, \eta, \vartheta} \rangle \cdot \langle \overline{\xi_0, \eta_0, \vartheta_0} \rangle = \langle \overline{\xi', \eta', \vartheta'} \rangle \cdot \langle \overline{\xi_0, \eta_0, \vartheta_0} \rangle$ and $\langle \overline{\xi_0, \eta_0, \vartheta_0} \rangle \cdot \langle \overline{\xi, \eta, \vartheta} \rangle = \langle \overline{\xi_0, \eta_0, \vartheta_0} \rangle \cdot \langle \overline{\xi', \eta', \vartheta'} \rangle$ whenever $\eta, \eta' \in h^m$, $\xi \overset{h}{\cdot} \eta \overset{h}{\cdot} \vartheta = \xi' \overset{h}{\cdot} \eta' \overset{h}{\cdot} \vartheta'$.
Let $\xi' = \xi \overset{h}{\cdot} \varepsilon$, $\eta' = \tilde{\eta} \overset{k}{\cdot} \sigma$, $\eta = \varepsilon \overset{k}{\cdot} \tilde{\eta}$, $\vartheta = \sigma \overset{h}{\cdot} \vartheta'$ for some $\varepsilon, \sigma \in \ell^m$, $\tilde{\eta} \in k^m$. Then $\langle \xi, \eta, \vartheta \overset{h}{\cdot} \xi_0 \overset{h}{\cdot} \eta_0 \overset{h}{\cdot} \vartheta_0 \rangle \ P^* \ \langle \xi', \eta', \vartheta' \overset{h}{\cdot} \xi_0 \overset{h}{\cdot} \eta_0 \overset{h}{\cdot} \vartheta_0 \rangle$ and $\langle \xi_0 \overset{h}{\cdot} \eta_0 \overset{h}{\cdot} \vartheta_0 \overset{h}{\cdot} \xi, \eta, \vartheta \rangle P^* \langle \xi_0 \overset{h}{\cdot} \eta_0 \overset{h}{\cdot} \vartheta_0 \overset{h}{\cdot} \xi', \eta', \vartheta' \rangle$ whenever $\eta_0 \in h^m$ and
$\langle \xi, \eta \overset{k}{\cdot} (\vartheta \overset{h}{\cdot} \xi_0) \overset{k}{\cdot} \eta_0, \vartheta_0 \rangle = \langle \xi, \varepsilon \overset{k}{\cdot} \tilde{\eta} \overset{k}{\cdot} (\sigma \overset{h}{\cdot} \vartheta' \overset{h}{\cdot} \xi_0) \overset{k}{\cdot} \eta_0, \vartheta_0 \rangle P^* \langle \xi', \eta' \overset{k}{\cdot} (\vartheta' \overset{h}{\cdot} \xi_0) \overset{k}{\cdot} \eta_0, \vartheta_0 \rangle$ and $\langle \xi_0, \eta_0 \overset{k}{\cdot} (\xi_0 \overset{h}{\cdot} \xi) \overset{k}{\cdot} \eta, \vartheta \rangle = \langle \xi_0, \eta_0 \overset{k}{\cdot} (\vartheta_0 \overset{h}{\cdot} \xi) \overset{k}{\cdot} \varepsilon \overset{k}{\cdot} \tilde{\eta}, \sigma \overset{h}{\cdot} \vartheta' \rangle P^* \langle \xi_0, \eta_0 \overset{k}{\cdot} (\vartheta_0 \overset{h}{\cdot} \xi') \overset{k}{\cdot} \eta', \vartheta' \rangle$ whenever $\eta_0 \in k^m$.
Consequently $\langle \overline{\xi, \eta, \vartheta} \rangle \cdot \langle \overline{\xi_0, \eta_0, \vartheta_0} \rangle = \langle \overline{\xi', \eta', \vartheta'} \rangle \cdot \langle \overline{\xi_0, \eta_0, \vartheta_0} \rangle$, $\langle \overline{\xi_0, \eta_0, \vartheta_0} \rangle \cdot \langle \overline{\xi, \eta, \vartheta} \rangle = \langle \overline{\xi_0, \eta_0, \vartheta_0} \rangle \cdot \langle \overline{\xi', \eta', \vartheta'} \rangle$.

C) Now prove that the composition $\cdot$ is associative:

I. For $\alpha, \beta, \gamma \in k^m$ evidently $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.

II. Let $\alpha, \beta \in k^m$, $\gamma = \langle \overline{\mu, \nu} \rangle \in H_K(b, a)$, $b \in k^\sigma$. Then

$$\alpha \cdot (\beta \cdot \gamma) = \langle \overline{\alpha \overset{k}{\cdot} \beta \overset{k}{\cdot} \mu, \nu} \rangle = (\alpha \cdot \beta) \cdot \gamma .$$

III. Let $\alpha \in k^m$, $\beta = \langle \overline{\mu, \nu} \rangle \in H_K(b, a)$, $b \in k^\sigma$.

1) Let $\gamma = \langle \overline{\rho, \sigma, \tau} \rangle$. If $\sigma \in h^m$, then $(\alpha \cdot \beta) \cdot \gamma = \langle \overline{\alpha \overset{k}{\cdot} \mu, \nu} \rangle \cdot \langle \overline{\rho, \sigma, \tau} \rangle = \langle \overline{\alpha \overset{k}{\cdot} \mu, \nu \overset{h}{\cdot} \rho \overset{h}{\cdot} \sigma \overset{h}{\cdot} \tau} \rangle = \alpha \cdot (\beta \cdot \gamma)$.
If $\sigma \notin h^m$, then $(\alpha \cdot \beta) \cdot \gamma = \langle \overline{(\alpha \overset{k}{\cdot} \mu) \overset{k}{\cdot} (\nu \overset{h}{\cdot} \rho) \overset{k}{\cdot} \sigma, \tau} \rangle = \alpha \cdot (\beta \cdot \gamma)$.

2) Let $\gamma = \langle \overline{\rho, \sigma} \rangle \in H_K(a, c)$, $c \in k^\sigma$. Then $(\alpha \cdot \beta) \cdot \gamma = \langle \overline{\alpha \overset{k}{\cdot} \mu, \nu} \rangle \cdot \langle \overline{\rho, \sigma} \rangle = (\alpha \overset{k}{\cdot} \mu) \overset{k}{\cdot} (\nu \overset{h}{\cdot} \rho) \overset{k}{\cdot} \sigma = \alpha \cdot (\beta \cdot \gamma)$.

IV. Let $\alpha = \langle \overline{\mu, \nu} \rangle \in H_K(b, a)$, $b \in k^\sigma$.

1) Let $\beta = \langle \overline{\rho, \sigma, \tau} \rangle$.

a) Let $\gamma = \langle \overline{\rho', \sigma', \tau'} \rangle$.

If $\sigma, \sigma' \in h^m$, then $(\alpha \cdot \beta) \cdot \gamma = \langle \overline{\mu, \nu \overset{h}{\cdot} \rho \overset{h}{\cdot} \sigma \overset{h}{\cdot} \tau} \rangle \cdot \langle \overline{\rho', \sigma', \tau'} \rangle = \langle \overline{\mu, \nu \overset{h}{\cdot} \rho \overset{h}{\cdot} \sigma \overset{h}{\cdot} \tau \overset{h}{\cdot} \rho' \overset{h}{\cdot} \sigma' \overset{h}{\cdot} \tau'} \rangle = \alpha \cdot (\beta \cdot \gamma)$;

If $\sigma \in h^m$, $\sigma' \notin h^m$, then $\alpha \cdot (\beta \cdot \gamma) = \langle \overline{\mu, \nu} \rangle \cdot \langle \overline{\rho \overset{h}{\cdot} \sigma \overset{h}{\cdot} \tau \overset{h}{\cdot} \rho', \sigma', \tau'} \rangle = \langle \overline{\mu \overset{k}{\cdot} (\nu \overset{h}{\cdot} \rho \overset{h}{\cdot} \sigma \overset{h}{\cdot} \tau \overset{h}{\cdot} \rho') \overset{k}{\cdot} \sigma', \tau'} \rangle = (\alpha \cdot \beta) \cdot \gamma$.

If $\sigma \notin h^m$, $\sigma' \notin h^m$, then $\alpha \cdot (\beta \cdot \gamma) = \langle \overline{\mu, \nu} \rangle \cdot \langle \overline{\rho, \sigma \overset{k}{\cdot} (\tau \overset{h}{\cdot} \rho') \overset{k}{\cdot} \sigma', \tau'} \rangle = \langle \overline{\mu \overset{k}{\cdot} (\nu \overset{h}{\cdot} \rho) \overset{k}{\cdot} \sigma \overset{k}{\cdot} (\tau \overset{h}{\cdot} \rho') \overset{k}{\cdot} \sigma', \tau'} \rangle = \langle \overline{\mu \overset{k}{\cdot} (\nu \overset{h}{\cdot} \rho) \overset{k}{\cdot} \sigma, \tau} \rangle \cdot \langle \overline{\rho', \sigma', \tau'} \rangle = (\alpha \cdot \beta) \cdot \gamma$.

If $\sigma \notin h^m$, $\sigma' \in h^m$, then $\alpha \cdot (\beta \cdot \gamma) = \langle \overline{\mu, \nu} \rangle \cdot \langle \overline{\rho, \sigma, \tau \overset{h}{\cdot} \rho' \overset{h}{\cdot} \sigma' \overset{h}{\cdot} \tau'} \rangle = \langle \overline{\mu \overset{k}{\cdot} (\nu \overset{h}{\cdot} \rho) \overset{k}{\cdot} \sigma, \tau \overset{h}{\cdot} \rho' \overset{h}{\cdot} \sigma' \overset{h}{\cdot} \tau'} \rangle = (\alpha \cdot \beta) \cdot \gamma$.

b) Let $\gamma = \langle \overline{\xi, \vartheta} \rangle \in H_K(a, c)$, $c \in k^\sigma$.

If $\sigma \in h^m$, then $(\alpha \cdot \beta) \cdot \gamma = \langle \overline{\mu, \nu \overset{h}{\cdot} \rho \overset{h}{\cdot} \sigma \overset{h}{\cdot} \tau} \rangle \cdot \langle \overline{\xi, \vartheta} \rangle = \mu \overset{k}{\cdot} (\nu \overset{h}{\cdot} \rho \overset{h}{\cdot} \sigma \overset{h}{\cdot} \tau \overset{h}{\cdot} \xi) \overset{k}{\cdot} \vartheta = \alpha \cdot (\beta \cdot \gamma)$.

If $\sigma \notin h^m$, then $(\alpha\cdot\beta)\cdot\gamma = \langle\overline{\mu \overset{k}{\cdot} (\nu \overset{h}{\cdot} \rho) \overset{k}{\cdot} \sigma, \tau}\rangle\cdot$
$\cdot\langle\overline{\xi,\vartheta}\rangle = \mu\overset{k}{\cdot}(\nu\overset{h}{\cdot}\rho)\overset{k}{\cdot}\sigma\overset{k}{\cdot}(\tau\overset{h}{\cdot}\xi)\overset{k}{\cdot}\vartheta = \langle\overline{\mu,\nu}\rangle\cdot$
$\cdot\langle\overline{\rho,\sigma\overset{k}{\cdot}(\tau\overset{h}{\cdot}\xi)\overset{k}{\cdot}\vartheta}\rangle = \alpha\cdot(\beta\cdot\gamma)$.

2) Let $\beta = \langle\overline{\xi,\vartheta}\rangle \in H_K(a,c)$, $c \in \mathcal{k}^{\sigma}$.

a) Let $\gamma \in H_K(c,d)$, $d \in \mathcal{k}^{\sigma}$. This case is dual to III.2).

b) Let $\gamma = \langle\overline{\pi,\varkappa}\rangle \in H_K(c,a)$. Then
$(\alpha\cdot\beta)\cdot\gamma = (\mu\overset{k}{\cdot}(\nu\overset{h}{\cdot}\xi)\overset{k}{\cdot}\vartheta)\cdot\langle\overline{\pi,\varkappa}\rangle =$
$= \langle\overline{\mu\overset{k}{\cdot}(\nu\overset{h}{\cdot}\xi)\overset{k}{\cdot}\vartheta\overset{k}{\cdot}\pi,\varkappa}\rangle = \langle\overline{\mu,\nu}\rangle\cdot\langle\overline{\xi,\vartheta\overset{k}{\cdot}\pi,\varkappa}\rangle =$
$= \alpha\cdot(\beta\cdot\gamma)$.

V. Let $\alpha = \langle\overline{\rho,\sigma,\tau}\rangle \in H_K(a,a)$.

1) Let $\beta = \langle\overline{\mu,\nu,\chi}\rangle$.

a) Let $\gamma = \langle\overline{\pi,\varphi,\psi}\rangle$.

If $\sigma,\nu,\varphi \in h^m$, then evidently $\alpha\cdot(\beta\cdot\gamma) = (\alpha\cdot\beta)\cdot\gamma$.

If $\sigma,\nu \in h^m$, $\varphi \notin h^m$, then $(\alpha\cdot\beta)\cdot\gamma =$
$= \langle\overline{\rho\overset{h}{\cdot}\sigma\overset{h}{\cdot}\tau\overset{h}{\cdot}\mu,\nu,\chi}\rangle\cdot\langle\overline{\pi,\varphi,\psi}\rangle =$
$= \langle\overline{\rho\overset{h}{\cdot}\sigma\overset{h}{\cdot}\tau\overset{h}{\cdot}\mu\overset{h}{\cdot}\nu\overset{h}{\cdot}\chi\overset{h}{\cdot}\pi,\varphi,\psi}\rangle = \alpha\cdot(\beta\cdot\gamma)$.

If $\sigma \in h^m, \nu \notin h^m, \varphi \in h^m$, then $(\alpha\cdot\beta)\cdot\gamma =$
$= \langle\overline{\rho\overset{h}{\cdot}\sigma\overset{h}{\cdot}\tau\overset{h}{\cdot}\mu,\nu,\chi\overset{h}{\cdot}\pi\overset{h}{\cdot}\varphi\overset{h}{\cdot}\psi}\rangle = \alpha\cdot(\beta\cdot\gamma)$.

If $\sigma \notin h^m$, $\nu,\varphi \in h^m$, then $(\alpha\cdot\beta)\cdot\gamma =$
$= \langle\overline{\rho,\sigma,\tau\overset{h}{\cdot}\mu\overset{h}{\cdot}\nu\overset{h}{\cdot}\chi\overset{h}{\cdot}\pi\overset{h}{\cdot}\varphi\overset{h}{\cdot}\psi}\rangle = \alpha\cdot(\beta\cdot\gamma)$.

If $\sigma \in h^m, \nu \notin h^m, \varphi \notin h^m$, then $(\alpha\cdot\beta)\cdot\gamma =$
$= \langle\overline{\rho\overset{h}{\cdot}\sigma\overset{h}{\cdot}\tau\overset{h}{\cdot}\mu,\nu,\chi}\rangle\cdot\langle\overline{\pi,\varphi,\psi}\rangle =$
$= \langle\overline{\rho\overset{h}{\cdot}\sigma\overset{h}{\cdot}\tau\overset{h}{\cdot}\mu,\nu\overset{k}{\cdot}(\chi\overset{h}{\cdot}\pi)\overset{k}{\cdot}\varphi,\psi}\rangle = \alpha\cdot(\beta\cdot\gamma)$.

If $\sigma \notin h^m$, $\nu \in h^m$, $\varphi \notin h^m$, then $(\alpha\cdot\beta)\cdot\gamma =$

$= \langle \overline{\rho, \sigma \overset{k}{\cdot} (\tau \overset{k}{\cdot} \mu \overset{k}{\cdot} \nu \overset{k}{\cdot} \chi \overset{k}{\cdot} \pi) \overset{k}{\cdot} \varphi, \psi} \rangle = \alpha \cdot (\beta \cdot \gamma).$

If $\sigma \notin h^m$, $\nu \notin h^m$, $\varphi \in h^m$, then $(\alpha \cdot \beta) \cdot \gamma =$

$= \langle \overline{\rho, \sigma \overset{k}{\cdot} (\tau \overset{k}{\cdot} (\mu) \overset{k}{\cdot} \nu, \chi \overset{k}{\cdot} \pi \overset{k}{\cdot} \varphi \overset{k}{\cdot} \psi} \rangle = \alpha \cdot (\beta \cdot \gamma)$.

If $\sigma \notin h^m$, $\nu \notin h^m$, $\varphi \notin h^m$, then $(\alpha . \beta) \cdot \gamma =$

$= \langle \overline{\rho \cdot \sigma \overset{k}{\cdot} (\tau \overset{k}{\cdot} (\mu) \overset{k}{\cdot} \nu, \chi} \rangle \cdot \langle \overline{\pi, \varphi, \psi} \rangle =$

$= \langle \overline{\rho, \sigma \overset{k}{\cdot} (\tau \overset{k}{\cdot} \mu) \overset{k}{\cdot} \nu \overset{k}{\cdot} (\chi \overset{k}{\cdot} \pi) \overset{k}{\cdot} \varphi, \psi} \rangle = \langle \overline{\rho, \sigma, \tau} \rangle \cdot$

$\cdot \langle \overline{\mu, \nu \overset{k}{\cdot} (\chi \overset{k}{\cdot} \pi) \overset{k}{\cdot} \varphi, \psi} \rangle = \alpha \cdot (\beta \cdot \gamma)$.

b) For $\gamma \in H_K(a, b)$, $b \in k^\sigma$ the case is dual to IV.1) a).

2) Let $\beta = \langle \overline{\mu, \nu} \rangle \in H_K(a, b)$, $b \in k^\sigma$.

a) For $\gamma \in H_K(b, c)$, $c \in k^\sigma$ the case is dual to III. 1.

b) Let $\gamma = \langle \overline{\pi, \varkappa} \rangle \in H_K(b, a)$.

If $\sigma \in h^m$, then $(\alpha . \beta) \cdot \gamma = \langle \overline{\rho \overset{k}{\cdot} \sigma \overset{k}{\cdot} \tau \overset{k}{\cdot} \mu, \nu} \rangle \cdot$
$\cdot \langle \overline{\pi, \quad} \rangle = \langle \overline{\rho \overset{k}{\cdot} \sigma \overset{k}{\cdot} \tau \overset{k}{\cdot} \mu, \nu \overset{k}{\cdot} \pi, \varkappa} \rangle = \alpha \cdot (\beta \cdot \gamma)$.

If $\sigma \notin h^m$, then $(\alpha \cdot \beta) \cdot \gamma = \langle \overline{\rho, \sigma \overset{k}{\cdot} (\tau \overset{k}{\cdot} \mu) \overset{k}{\cdot} \nu} \rangle \cdot$
$\cdot \langle \overline{\pi, \quad} \rangle = \langle \overline{\rho, \sigma \overset{k}{\cdot} (\tau \overset{k}{\cdot} \mu) \overset{k}{\cdot} \nu \overset{k}{\cdot} \pi, \varkappa} \rangle = \langle \overline{\rho, \sigma, \tau} \rangle \cdot$
$\cdot \langle \overline{\mu, \nu \overset{k}{\cdot} \pi, \varkappa} \rangle = \alpha \cdot (\beta \cdot \gamma)$.

VI. Let $\alpha = \langle \overline{\mu, \nu} \rangle \in H_K(a, b)$, $b \in k^\sigma$.

1) Let $\beta \in H_K(b, c)$, $c \in k^\sigma$.

a) For $\gamma \in H_K(c, d)$, $d \in k^\sigma$ the case is dual to II.

b) Let $\gamma = \langle \overline{\pi, \varkappa} \rangle \in H_K(c, a)$. Then
$(\alpha \cdot \beta) \cdot \gamma = \langle \overline{\mu, \nu \overset{k}{\cdot} \beta \overset{k}{\cdot} \pi, \varkappa} \rangle = \alpha \cdot (\beta \cdot \gamma)$.

2) Let $\beta \in H_K(b, a)$.

a) For $\gamma \in H_K(a, a)$ the case is dual to V.2) b).

b) For $\gamma \in H_K(a, c)$, $c \in k^\sigma$ the case is dual to IV.2) b).

This completes the proof of associativity.

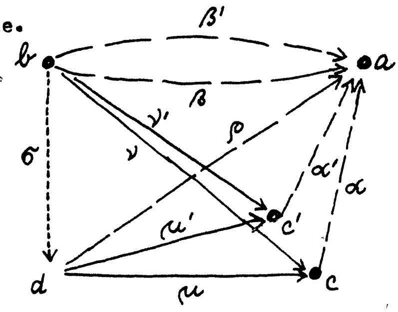
D) Evidently k is a full subcategory of K. Now we shall construct the functor $\varphi : h \to K$, identical on ℓ. For $\alpha \in \ell^m \cup \ell^\sigma$ put $(\alpha)\varphi = \alpha$. For $b \in \ell^\sigma$, $\alpha \in H_h(b, a)$ $\beta \in H_h(a, b)$ put $(\alpha)\varphi = \langle \overline{e_b, \alpha} \rangle$, $(\beta)\varphi = \langle \overline{\beta, e_b} \rangle$. For $\sigma \in H_h(a, a)$ put $(\sigma)\varphi = \langle \overline{e_a, \sigma, e_a} \rangle$. It is easy to see that φ is the required functor.

E) Now let H be an arbitrary category, and $\psi_k : k \to H$, $\psi_h : h \to H$ be functors such that $(\alpha)\psi_k = (\alpha)\psi_h$ for all $\alpha \in \ell^m \cup \ell^\sigma$. Evidently, if $b \in k^\sigma$, $\langle \mu, \alpha \rangle S_b \langle \mu', \alpha' \rangle$ (or $\langle \alpha, \mu \rangle R_b \langle \alpha', \mu' \rangle$) then $(\mu)\psi_k \cdot (\alpha)\psi_h = (\mu')\psi_k \cdot (\alpha')\psi_h$ (or $(\alpha)\psi_h \cdot (\mu)\psi_k = (\alpha')\psi_h \cdot (\mu')\psi_k$ respectively); and if $\langle \alpha, \beta, \gamma \rangle\, C\, \langle \alpha', \beta', \gamma' \rangle$, then $(\alpha)\psi_h \cdot (\beta)\psi_h \cdot (\gamma)\psi_h = (\alpha')\psi_h \cdot (\beta')\psi_h \cdot (\gamma')\psi_h$ whenever $\beta, \beta' \in h^m$, $(\alpha)\psi_h \cdot (\beta)\psi_k \cdot (\gamma)\psi_h = (\alpha')\psi_h \cdot (\beta')\psi_k \cdot (\gamma')\psi_h$ whenever $\beta, \beta' \in k^m$.
Consequently, put $(\alpha)\psi = (\alpha)\psi_k$ whenever $\alpha \in k^m$, $(\langle \mu, \alpha \rangle)\psi = (\mu)\psi_k \cdot (\alpha)\psi_h$ for $\langle \mu, \alpha \rangle \in H_K(b, a)$, $(\langle \alpha, \mu \rangle)\psi = (\alpha)\psi_h \cdot (\mu)\psi_k$ for $\langle \alpha, \mu \rangle \in H_K(a, b)$.
For $\langle \alpha, \beta, \gamma \rangle \in H_K(a, a)$ then put $(\langle \alpha, \beta, \gamma \rangle)\psi = (\alpha)\psi_h \cdot (\beta)\psi_h \cdot (\gamma)\psi_h$ whenever $\beta \in h^m$, $(\langle \alpha, \beta, \gamma \rangle)\psi = (\alpha)\psi_h \cdot (\beta)\psi_k \cdot (\gamma)\psi_h$ whenever $\beta \in k^m$.
Evidently ψ has the required properties. The unicity of ψ is evident.

<u>Note 4</u>: Evidently, if there exists a category H and functors $\psi_h : h \to H$, $\psi_k : k \to H$ where $(\alpha)\psi_h = (\alpha)\psi_k$

for all $\alpha \in \ell$ and such that ψ_h is an iso-functor, then necessarily φ is also an iso-functor.

However, in general φ need not be an iso-functor; thus, consider the following example.



Let h, k, ℓ be categories described thus (cf. the diagram):

$\ell^o = k^o = \{b, c, c', d\}$,

$h^o = \{b, c, c', d, a\}$,

$\mu, \mu', \nu, \nu' \in \ell^m$,

$\alpha, \alpha', \rho, \beta, \beta' \in h^m - \ell^m$

$\sigma \in k^m - \ell^m$ (the identities are not indicated);

$\mu \overset{h}{\cdot} \alpha = \rho = \mu' \overset{h}{\cdot} \alpha'$, $\nu \overset{h}{\cdot} \alpha = \beta$, $\nu' \overset{h}{\cdot} \alpha' = \beta'$, $\sigma \overset{k}{\cdot} \mu = \nu$, $\sigma \overset{k}{\cdot} \mu' = \nu'$.

Then evidently $\langle e_b, \beta \rangle S_b^* \langle \nu, \alpha \rangle$, $\langle \nu, \alpha \rangle S_b^* \langle \sigma, \rho \rangle S_b^*$ $S_b^* \langle \nu', \alpha' \rangle S_b^* \langle e_b, \beta' \rangle$, and consequently $(\beta)\varphi = (\beta')\varphi$.

Lemma 4: Let h' be a full subcategory of a small category h. Let $\varphi' : h' \longrightarrow H'$ be a functor onto H', identical on $(h')^o$. Then there exists a category H and a functor $\varphi : h \longrightarrow H$ such that H' is a full subcategory of H, φ' is the restriction of φ, $\varphi/_{h^o}$ is identical, and if $\psi : h \longrightarrow G$ is an arbitrary functor such that $(\alpha)\psi = (\beta)\psi$ whenever $\alpha, \beta \in (h')^m$ and $(\alpha)\varphi' = (\beta)\varphi'$, then there exists exactly one functor $\chi : H \longrightarrow G$ with $\varphi \cdot \chi = \psi$.

Proof: Let R be the following relation on h^m:

$\alpha R \beta \Longleftrightarrow \alpha, \beta \in (h')^m, (\alpha)\varphi' = (\beta)\varphi'$. Let R^* be the smallest equivalence on h^m which contains R and such that if $\alpha R^* \alpha'$, $\beta R^* \beta'$ and $\alpha \cdot \beta$ or $\alpha' \cdot \beta'$ are defined, then $\alpha \cdot \beta \; R^* \alpha' \cdot \beta'$. Let $\bar{H}$ be a category such that $\bar{H}^\sigma = h^\sigma$ and $H_{\bar{H}}(a, b) = \{a\} \times \times H_h(a, b)/_{R^*} \times \{b\}$ whenever $a, b \in h^\sigma$ (the definition of the composition is obvious). A certain category H, isomorphic with $\bar{H}$, has the properties required in lemma 4.

Note: $\langle \varphi, H \rangle$ will be termed the prolongation of H' to h through φ'.

Lemma 5: Let k, h be small categories; let l be a full subcategory of h and a subcategory of k, let $k^\sigma = l^\sigma$. Then there exists the sum $\langle K, \iota, \varphi \rangle$ of k and h with the amalgamated subcategory l such that ι is the identical full iso-functor, φ is identical on l and on h^σ.

Proof: If α is an ordinal, denote by T_α the set of all ordinals less than α. Let τ be an ordinal with $\operatorname{card} \tau > \operatorname{card} h^\sigma - l^\sigma$ (we may suppose $h^\sigma - l^\sigma > 0$). Let f be a one-to-one mapping of $h^\sigma - l^\sigma$ into the set of all isolated ordinals from T_τ. For $\alpha \in T_\tau$ denote by h_α the full subcategory of h such that $h_\alpha^\sigma = l^\sigma \cup (T_\alpha) f^{-1}$. Now apply transfinite induction. Let $\gamma \in T_\tau$ and assume that one has constructed the systems $\mathbb{K}_\gamma = \{k_\alpha ; \alpha \in T_\gamma\}$, $\mathfrak{S}_\gamma = \{\rho_\alpha ; \alpha \in T_\gamma\}$, $\mathfrak{X}_\gamma = \{\chi_\alpha ; \alpha \in T_\gamma\}$ with the following three properties

1_γ) $\mathbb{K}_\gamma$, $\mathbb{S}_\gamma$ are monotone systems of small categories, $k_0 = k$, $s_0 = l$; s_α is a subcategory of k_α; $s_\alpha^\sigma = k_\alpha^\sigma$; denote by $\iota_\alpha : s_\alpha \to k_\alpha$, $\alpha : k \to k_\alpha$ the inclusion-functors.

2_γ) χ_α is a functor of h_α onto s_α; $\chi_0 : l \to l$ is identical; every $\chi_\alpha / h_\alpha^\sigma$ is identical; if $\alpha < \beta < \gamma$, then $\chi_\beta / h_\alpha = \chi_\alpha$.

3_γ) If H is a category and if $\psi_k : k \to H$, $\psi_{h_\alpha} : h_\alpha \to H$ $(\alpha < \gamma)$ are functors such that $(\mu)\psi_k = (\mu)\psi_{h_\alpha}$ whenever $\mu \in l$, then there exists exactly one functor $\psi_\alpha : k_\alpha \to H$ with $\psi_k = \iota_\alpha \cdot \psi_\alpha$, $\psi_{h_\alpha} = \chi_\alpha \cdot \iota_\alpha \cdot \psi_\alpha$.

We are to construct k_γ, s_γ, χ_γ. The construction is evident whenever γ is a limit ordinal: put $k_\gamma = \underset{\alpha<\gamma}{S} k_\alpha$, $s_\gamma = \underset{\alpha<\gamma}{S} s_\alpha$, $\chi_\gamma = \bigcup_{\alpha<\gamma} \chi_\alpha$. Thus, assume $\gamma = \beta + 1$. Denote by $\langle \Phi, s^* \rangle$ the prolongation of s_β to h_γ through χ_β. Since $(s^*)^\sigma = (h_\gamma)^\sigma$, there is $k_\beta^\sigma \cap (s^*)^\sigma = s_\beta^\sigma$; the set $(s^*)^\sigma - s_\beta^\sigma$ contains at most one element. Denote by $\langle k_\gamma, \iota_\beta^\gamma, \pi \rangle$ the sum of k_β and s^* with the amalgamated subcategory s_β. Then $\iota_\beta^\gamma : k_\beta \to k_\gamma$ is the full inclusion-functor and π is identical on s_β and on $(s^*)^\sigma$. Set $\iota_\gamma = \iota_\beta \cdot \iota_\beta^\gamma$ and $s_\gamma = (s^*)\pi$; then s_γ is a subcategory of k_γ, denote by $\tau_\gamma : s_\gamma \to k_\gamma$ the inclusion-functor. Let $\tilde{\pi} : s^* \to s_\gamma$ be the functor such that $\pi = \tilde{\pi} \cdot \tau_\gamma$. Put $\chi_\gamma = \Phi \cdot \tilde{\pi}$; since $\Phi / h_\beta = \chi_\beta$ and $\tilde{\pi}$ is identical on s_β, there is $\chi_\gamma / h_\beta = \chi_\beta$. Thus there have been defined k_γ, s_γ, χ_γ such that $1_{\gamma+1}$) and $2_{\gamma+1}$) hold. It remains to prove that, to given functors $\psi_{k}: k \to H$, $\psi_{h_\gamma}: h_\gamma \to H$ for which $(\mu)\psi_{k} = (\mu)\psi_{h_\gamma}$ whenever $\mu \in l$, there exists exactly one functor $\psi_\gamma : k_\gamma \to H$ with $\psi_{k} = \iota_\gamma \cdot \psi_\gamma$, $\psi_{h_\gamma} = \chi_\gamma \cdot \bar{\iota}_\gamma \cdot \psi_\gamma$. If such ψ_{k}, ψ_{h_γ} are given put $\bar{\psi} = \vec{\iota} \cdot \psi_{h_\gamma}$, where $\vec{\iota}: h_\beta \to h_\gamma$ is the inclusion-functor. Then there exists exactly one functor $\psi_\beta : k_\beta \to H$ such that $\psi_{k} = \iota_\beta \cdot \psi_\beta$, $\bar{\psi} = \chi_\beta \cdot \bar{\iota}_\beta \cdot \psi_\beta$. But then $(\mu)\psi_{h_\gamma} = (\mu)\bar{\psi} = (\nu)\bar{\psi} = (\nu)\psi_{h_\gamma}$ whenever $\mu, \nu \in h_\beta^m$ and $(\mu)\chi_\beta = (\nu)\chi_\beta$; consequently there exists exactly one functor $\psi^* : s^* \to H$ such that $\psi_{h_\gamma} = \Phi \cdot \psi^*$. Now we must prove that $(\mu)\psi_\beta = (\mu)\psi^*$ whenever $\mu \in (s_\beta)^m$. Thus let $\mu \in (s_\beta)^m$; then $\mu = (\varkappa)\chi_\beta$ for some $\varkappa \in h_\beta^m$, and hence $(\mu)\psi^* = (\varkappa)\chi_\beta\psi^* = (\varkappa)\Phi\,\psi^* = (\varkappa)\psi_{h_\gamma} = (\varkappa)\bar{\psi} = (\varkappa)\chi_\beta \cdot \bar{\iota}_\beta \cdot \psi_\beta = (\mu)\psi_\beta$. Since $\langle k_\gamma, \iota_\beta^\gamma, \pi \rangle$ is the sum of k_β and s^* with the amalgamated subcategory s_β, there exists exactly one functor $\psi_\gamma : k_\gamma \to H$ such that $\psi_\beta = \iota_\beta^\gamma \cdot \psi_\gamma$, $\psi^* = \pi \cdot \psi_\gamma$; therefore $\psi_{k} = \iota_\beta \cdot \psi_\beta = \iota_\gamma \cdot \psi_\gamma$, $\psi_{h_\gamma} = \Phi \cdot \psi^* = \Phi \cdot \tilde{\pi} \cdot \bar{\iota}_\gamma \cdot \psi_\gamma = \chi_\gamma \cdot \bar{\iota}_\gamma \cdot \psi_\gamma$.

Using transfinite induction one defines systems $\mathbb{K}_\tau = \{k_\alpha ; \alpha \in T_\tau\}$, $S_\tau = \{s_\alpha ; \alpha \in T_\tau\}$, $X_\tau = \{\chi_\alpha ; \alpha \in T_\tau\}$ such that $1_\tau)$ $2_\tau)$ $3_\tau)$ are satisfied. Put $K = \underset{\alpha \in T_\tau}{S} k_\alpha$, denote by $\iota : k \to K$ the inclusion-functor. Put

$s = \underset{\alpha \in T_\tau}{S} s_\alpha$, $\chi = \bigcup_{\alpha \in T_\tau} \chi_\alpha$. Then $\chi : h \xrightarrow{onto} s$, s is a subcategory of K; denote by $\iota_s : s \to K$ the inclusion-functor, put $\varphi = \chi \cdot \iota_s$. Then evidently $\langle K, \iota, \varphi \rangle$ has the required properties.

__Proposition 2__: Let $\mathfrak{k}$, $\mathfrak{h}$ be small categories, let ℓ be a full subcategory of $\mathfrak{h}$ and a subcategory of $\mathfrak{k}$, $\mathfrak{k}^{\sigma} \cap \mathfrak{h}^{\sigma} = \ell^{\sigma}$. Then there exists the sum $\langle K, \quad , \varphi \rangle$ of $\mathfrak{k}$ and $\mathfrak{h}$ with the amalgamated subcategory ℓ; is full identical iso-functor, φ is identical on $\mathfrak{h}^{\sigma}$.

__Proof__: Denote by $\mathfrak{k}^*$ the full subcategory of $\mathfrak{k}$ such that $(\mathfrak{k}^*)^{\sigma} = \ell^{\sigma}$. Let $\langle K^*, \quad , \varphi^* \rangle$ be the sum of $\mathfrak{k}^*$ and $\mathfrak{h}$ with the amalgamated subcategory ℓ. Let $\langle K, \ _{\mathfrak{k}}, \ _{K^*} \rangle$ be the sum of $\mathfrak{k}$ and K^* with the amalgamated subcategory $\mathfrak{k}^*$. Then evidently $\langle K, \ _{\mathfrak{k}}, \varphi^* . \ _{K^*} \rangle$ has the required properties.

__Lemma 6__: Let $\mathfrak{k}_1$, $\mathfrak{k}_2$ be small categories, ℓ be a subcategory of both, $\mathfrak{k}_1^{\sigma} = \ell^{\sigma} = \mathfrak{k}_2^{\sigma}$, $\mathfrak{k}_1^{m} \cap \mathfrak{k}_2^{m} = \ell^{m}$. Then there exists the sum of $\mathfrak{k}_1$ and $\mathfrak{k}_2$ with the amalgamated subcategory ℓ.

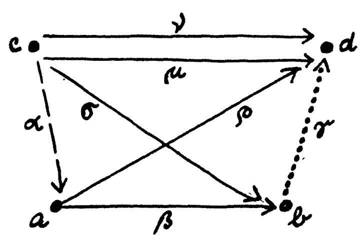
__Proof__: For $a, b \in \ell^{\sigma}$ denote by $\overline{H}(a, b)$ the set of all n-tuples $\langle \alpha_1, \dots, \alpha_n \rangle$ (where $n = 1, 2, \dots$) such that there exist $c_0, \dots, c_n \in \ell^{\sigma}$ with $c_0 = a$, $c_n = b$, $\alpha_s \in H_{\mathfrak{k}_1}(c_{s-1}, c_s) \cup H_{\mathfrak{k}_2}(c_{s-1}, c_s)$. Set $M = \bigcup_{a, b \in \ell^{\sigma}} \overline{H}(a, b)$. For $\langle \alpha_1, \dots, \alpha_n \rangle \in \overline{H}(a, b)$, $\langle \beta_1, \dots, \beta_m \rangle \in \overline{H}(b, c)$, put $\langle \alpha_1, \dots, \alpha_n \rangle \cdot \langle \beta_1, \dots, \beta_m \rangle = \langle \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \rangle$. Let now R be the following relation on M: $\langle \alpha_1, \dots, \alpha_n \rangle R \langle \beta_1, \dots, \beta_{n-1} \rangle$ if and only if there exists an $s_0 \in \{1, \dots, n-1\}$ such that $\alpha_s = \beta_s$ for $s = 1, 2, \dots, s_0 - 1$, $\alpha_{s_0} \cdot \alpha_{s_0+1} = \beta_{s_0}$ either in $\mathfrak{k}_1$ or in $\mathfrak{k}_2$ and $\alpha_s = \beta_{s-1}$ for $s = s_0 + 2, \dots, n$.

Let $\mathfrak{k}$ be the category such that $\mathfrak{k}^{\sigma} = \ell^{\sigma}$ and that $H_{\mathfrak{k}}(a, b) = \{a\} \times \overline{H}(a, b)/_{R^*} \times \{b\}$ for $a, b \in \mathfrak{k}^{\sigma}$.

The definition of the composition in k is evident. Now it is easy to define functors $\varphi_1 : k_1 \rightarrow k$, $\varphi_2 : k_2 \rightarrow k$ such that $\langle k, \varphi_1, \varphi_2 \rangle$ is the sum of k_1 and k_2 with the amalgamated subcategory ℓ.

Note: Let k_1, k_2 be small categories, ℓ be a subcategory of both, $k_1^\sigma = \ell^\sigma = k_2^\sigma$, $k_1^m \cap k_2^m = \ell^m$. Denote by $\langle k, \varphi_1, \varphi_2 \rangle$ the sum of k_1 and k_2 with the amalgamated subcategory ℓ. Then not necessarily $(k_1^m)\varphi_1 \cap$ $\cap (k_2^m)\varphi_2 = (\ell^m)\varphi_1$, and φ_1, φ_2 need not be iso-functors. Examples will be given.

Examples: Consider the diagram. Let $i \in \{1, 2, 3\}$. Let



${}^i k_1^\sigma = {}^i\ell^\sigma = {}^i k_2^\sigma = \{a, b, c, d\}$. Denote by $\langle {}^i k, {}^i\varphi_1, {}^i\varphi_2 \rangle$ the sum of ${}^i k_1$ and ${}^i k_2$ with the amalgamated subcategory ${}^i\ell$. Set $S =$ $= \{e_a, e_b, e_c, e_d\}$.

a) Let ${}^1k_1^m = S \cup \{\alpha, \beta, \rho, \sigma, \mu\}$, ${}^1k_2^m = S \cup \{\beta, \gamma, \rho, \sigma, \nu\}$, ${}^1\ell^m = S \cup \{\beta, \sigma, \rho\}$. Let $\alpha \overset{^1k_1}{\cdot} \beta = \sigma$, $\beta \overset{^1k_2}{\cdot} \gamma = \rho$, $\sigma \overset{^1k_2}{\cdot} \gamma = \nu$, $\alpha \overset{^1k_1}{\cdot} \rho = \mu$. Then necessarily $(\mu)\,{}^1\varphi_1 =$ $= (\nu)\,{}^1\varphi_2$, consequently $({}^1k_1^m)\,{}^1\varphi_1 \cap ({}^1k_2^m)\,{}^1\varphi_2 \supsetneqq ({}^1\ell^m)\,{}^1\varphi$.

b) Let ${}^2k_1^m = S \cup \{\alpha, \beta, \rho, \sigma, \mu, \nu\}$, ${}^2k_2^m = \{\beta, \gamma, \rho, \sigma, \nu\} \cup S$ ${}^2\ell^m = S \cup \{\beta, \sigma, \rho, \nu\}$. Let $\alpha \overset{^2k_1}{\cdot} \beta = \sigma$, $\beta \overset{^2k_2}{\cdot} \gamma = \rho$, $\sigma \overset{^2k_2}{\cdot} \gamma = \nu$,

$\alpha \overset{^2k_1}{\cdot} \rho = \mu$. Then necessarily $(\mu)\,{}^2\varphi_1 = (\nu)\,{}^2\varphi_1$, consequently ${}^2\varphi_1$ is not an iso-functor.

Similarly one may construct small categories ${}^3k_1$, ${}^3k_2$, ${}^3\ell$ such that ${}^3\varphi_2$ is not an iso-functor.

Now, put $k_1 = \bigcup_{i=1}^{3} {}^{i}k_1$, $k_2 = \bigcup_{i=1}^{3} {}^{i}k_2$, $\ell = \bigcup_{i=1}^{3} {}^{i}\ell$, denote by $\langle k, \varphi_1, \varphi_2 \rangle$ the sum of k_1 and k_2 with the amalgamated subcategory ℓ. Then neither φ_1 nor φ_2 is an iso-functor and $(k_1^m)\varphi_1 \cap (k_2^m)\varphi_2 \supsetneqq (\ell^m)\varphi_1$.

The categories k_1, k_2, ℓ may be turned in semigroups by the adjunction of one element θ such that $\varkappa \cdot \sigma = \theta$ whenever either $\varkappa, \sigma \in k_1^m$ and $\varkappa \overset{k_1}{\cdot} \sigma$ is not defined, or $\varkappa, \sigma \in k_2^m$ and $\varkappa \overset{k_2}{\cdot} \sigma$ is not defined.

__Proposition 3__: Let k_1, k_2 be small categories, ℓ a subcategory of both, $k_1^m \cap k_2^m = \ell^m$ $k_1^\sigma \cap k_2^\sigma = \ell^\sigma$. Then there exists the sum of k_1 and k_2 with the amalgamated subcategory ℓ.

__Proof__: Denote by $\overline{k}_1$ (or $\overline{k}_2$) the full subcategory of k_1 (or k_2) such that $\overline{k}_1^\sigma = \ell^\sigma$ (or $\overline{k}_2^\sigma = \ell^\sigma$ respectively). Denote by $\langle \overline{k}, \overline{\varphi}_1, \overline{\varphi}_2 \rangle$ the sum of $\overline{k}_1$ and $\overline{k}_2$ with the amalgamated subcategory ℓ. Let $i \in \{1, 2\}$. Since $\varphi_i / \overline{k}_i^\sigma$ is one-to-one, $s_i = (\overline{k}_i)\overline{\varphi}_i$ is a subcategory of $\overline{k}$; denote by $\iota_i : s_i \longrightarrow \overline{k}$ the inclusion-functor. Let $\widetilde{\varphi}_i : \overline{k}_i \xrightarrow{onto} s_i$ be a functor such that $\widetilde{\varphi}_i \cdot \iota_i = \overline{\varphi}_i$. Denote by $\langle \varphi_i^*, s_i^* \rangle$ the prolongation of s_i to k_i through $\widetilde{\varphi}_i$, and by $\tilde{\iota}_i : s_i \longrightarrow s_i^*$ the full inclusion-functor. Let $\langle \widetilde{k}_i, \iota_i, \varphi_i \rangle$ be the sum of $\overline{k}$ and s_i^* with the amalgamated subcategory s_i (use lemma 5). Denote by $\langle k, \overline{\iota}_1, \overline{\iota}_2 \rangle$ the sum of $\widetilde{k}_1$ and $\widetilde{k}_2$ with the amalgamated subcategory $\mathcal{K}$. Now we prove that $\langle \mathcal{k}, \varphi_1^* \cdot \varphi_1 \cdot \iota_1, \varphi_2^* \cdot \varphi_2 \cdot \iota_2 \rangle$ is the sum of $\mathcal{k}_1$ and $\mathcal{k}_2$ with the amalgamated subcategory ℓ. Evidently $\overline{\varphi_1} = \varphi_1^* \cdot \varphi_1 \cdot \iota_1 : \mathcal{k}_1 \to \mathcal{k}$, $\overline{\varphi_2} = \varphi_2^* \cdot \varphi_2 \cdot \iota_2 : \mathcal{k}_2 \to \mathcal{k}$ are functors such that $(\mu)\overline{\varphi_1} = (\mu)\overline{\varphi_2}$ whenever $\mu \in \ell$. Now let there be given functors $\psi_1 : \mathcal{k}_1 \to H$, $\psi_2 : \mathcal{k}_2 \to H$ such that $(\mu)\psi_1 = (\mu)\psi_2$ whenever $\mu \in \ell$. Let $\overline{\psi_1} : \mathcal{K}_1 \to H$, $\psi_2 : \mathcal{K}_2 \to H$ be such that $\psi_1/_{\mathcal{K}_1} = \overline{\psi_1}$, $\psi_2/_{\mathcal{K}_2} = \psi_2$. Then there exists exactly one functor $\overline{\psi} : \mathcal{K} \to H$ with $\overline{\varphi_1}\,\overline{\psi} = \psi_1$, $\overline{\varphi_2}\,\overline{\psi} = \psi_2$. For $i = 1, 2$ put $\widetilde{\psi}_i = \iota_i \cdot \overline{\psi}$, $\widetilde{\psi}_i : s_i \to H$. Since $\overline{\psi_i} = \widetilde{\varphi}_i \cdot \iota_i \cdot \overline{\psi}$, there is $(\mu)\psi_i = (\nu)\psi_i$ whenever $\mu, \nu \in \mathcal{K}_i$, $(\mu)\widetilde{\varphi}_i = (\nu)\widetilde{\varphi}_i$; consequently there exists exactly one functor $\psi_i^* : s_i^* \to H$ such that $\varphi_i^* \cdot \psi_i^* = \psi_i$. If $\mu \in s_i$, then $(\mu)\overline{\psi} = (\mu)\psi_i^*$. Indeed, $\mu = (\nu)\overline{\varphi_i}$ for some $\nu \in \mathcal{K}_i$ and then $(\mu)\overline{\psi} = (\nu)\overline{\varphi_i} \cdot \overline{\psi} = (\nu)\overline{\psi_i} = (\nu)\psi_i = (\nu)\varphi_i^* \cdot \psi_i^* = (\nu)\overline{\varphi_i} \cdot \psi_i^* = (\mu)\psi_i^*$. Since $\langle \widetilde{\mathcal{k}}_i, \iota_i, \varphi_i \rangle$ is the sum of $\mathcal{K}$ and s_i^* with the amalgamated subcategory s_i, and since $\overline{\psi} : \mathcal{K} \to H$, $\psi^* : s_i^* \to H$, there exists exactly one functor $\overline{\overline{\psi_i}} : \widetilde{\mathcal{k}}_i \to H$ such that $\iota_i \cdot \overline{\overline{\psi_i}} = \overline{\psi}$, $\varphi_i \cdot \overline{\overline{\psi_i}} = \psi_i^*$. Now, $\overline{\overline{\psi_1}} : \widetilde{\mathcal{k}}_1 \to H$, $\overline{\overline{\psi_2}} : \widetilde{\mathcal{k}}_2 \to H$, and $(\mu)\overline{\overline{\psi_1}} = (\mu)\iota_1 \cdot \overline{\overline{\psi_i}} = (\mu)\overline{\psi} = (\mu)\iota_2 \cdot \overline{\overline{\psi_2}} = (\mu)\overline{\overline{\psi_2}}$ whenever $\mu \in \mathcal{K}$; thus there exists exactly one functor $\overline{\overline{\overline{\psi}}} : \mathcal{k} \to H$ such that $\iota_1 \cdot \overline{\overline{\overline{\psi}}} = \overline{\overline{\psi_1}}$, $\iota_2 \cdot \overline{\overline{\overline{\psi}}} = \overline{\overline{\psi_2}}$. But then $\varphi_i^* \cdot \varphi_i \cdot \iota_i \cdot \overline{\overline{\overline{\psi}}} = \varphi_i^* \cdot \varphi_i \cdot \overline{\overline{\psi_i}} = \varphi_i^* \cdot \psi_i^* = \psi_i$ $(i = 1, 2)$. If $\overline{\overline{\overline{\psi}}} : \mathcal{k} \to H$ is a functor such that

$\varphi_i^* \cdot \varphi_i \cdot \overline{\iota}_i \cdot \overline{\overline{\psi}} = \psi_i \quad (i = 1, 2)$, then necessarily $\varphi_i \cdot \overline{\iota}_i \cdot \overline{\overline{\psi}} = \psi_i^* \quad (i = 1, 2)$. Hence $\overline{\iota}_i \cdot \overline{\overline{\psi}} = \overline{\psi_i}$ and therefore $\overline{\psi} = \overline{\overline{\psi}}$.

<u>Note</u>: It is easily seen that the theorem may be carried over for a set of small categories in the following manner:

Let $\{ k_\alpha \, ; \, \alpha \in A \}$ be a system of small categories, let ℓ be a subcategory of all $k_\alpha^o \cap k_{\alpha'}^o = \ell^o$, $k_\alpha^m \cap k_{\alpha'}^m = \ell^m$ whenever $\alpha, \alpha' \in A$, $\alpha \neq \alpha'$. Then there exists the sum $\langle k \, ; \, \{ \varphi_\alpha \, ; \, \alpha \in A \} \rangle$ of the system $\{ k_\alpha \, ; \, \alpha \in A \}$ with the amalgamated subcategory ℓ (defined analoguously as for the system of small categories in the definition given previously).

If there exists an $\alpha_0 \in A$ such that ℓ is a full subcategory of all k_α for $\alpha \in A - \{\alpha_0\}$, then φ_{α_0} is a full iso-functor and $(k_{\alpha_0}^m)\varphi_{\alpha_0} \cap \left(\bigcup_{\alpha \in A - \{\alpha_0\}} (k_\alpha^m)\varphi_\alpha \right) = (\ell^m)\varphi_{\alpha_0}$.

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