Josef Kolomý

Determination of eigenvalues and eigenfunctions of bounded self-adjoint operators (Preliminary communication)

Commentationes Mathematicae Universitatis Carolinae, Vol. 6 (1965), No. 3, 387--392

Commentationes Mathematicae Universitatis Carolinae
6,3 (1965)

DETERMINATION OF EIGENVALUES AND EIGENFUNCTIONS OF BOUNDED SELF-ADJOINT OPERATORS

Josef KOLOMÝ, Praha

(Preliminary communication)

1. The problem of determining the eigenvalues and eigenfunctions of self-adjoint bounded operators has been developed by many authors. L.V. Kantorovič [1] used the method of steepest descent to determine the largest eigenvalue, and the corresponding eigenfunction, of completely continuous self-adjoint positive definite operators in Hilbert space. Later M.A. Krasnoselskij [2] suggested ten methods for calculation of eigenvalues in n-dimensional spaces, but without proofs. These and Kostarčuk's [3] methods are simpler in comparison with [1]. The fifth method from [2] was investigated by B.P. Pugačev [4] under the assumption that the linear bounded operator is self-adjoint and positive definite. Wang Jin-ru [5] improved the fourth method from [2] and performed a comparison of some these gradient methods. Another method was proposed by W. Karush [6].

In this note we shall deal with two methods which were described in [7],[8]. We assume throughout that H is a real Hilbert space. The basic idea of these methods is the following. Let us consider the equation

(1) $$Ax - \lambda x = 0 ,$$

where A is linear bounded operator in H , λ is a real parameter. Suppose that A is a positive self-adjoint operator in H (A is said to be positive if $(Ax,x) > 0$ for every $x \in H$, $x \neq 0$). We solve (1) by an iterative process

$$(2) \qquad x_{n+1} = \frac{1}{\lambda_{n+1}} Ax_n ,$$

where the parameters λ_n (n = 1,2,...) are to be determined from the condition that the functional $\| Ax - \tau x \|^2$ for the given element $x = x_n \in H$ is to assume its minimum on the set $\mathcal{R}$ ($\tau \in \mathcal{R}$) of all real numbers. Let us denote that value τ (dependent on n) by λ_{n+1} . Then we obtain that

$$(3) \qquad \lambda_{n+1} = \frac{(Ax_n , x_n)}{\| x_n \|^2}$$

and

$$(4) \qquad x_{n+1} = \frac{\| x_n \|^2}{(Ax_n , x_n)} Ax_n , \; x_0 \neq 0 , \; x_0 \in H \; (n = 0,1,2,...).$$

The second method was proposed by I.A. Birger [9] but without any assumptions or convergence proofs. His method is as follows: Let

$$(5) \qquad y_{n+1} = \mu_{n+1} Ay_n , \; \mu_{n+1} = \frac{(Ay_n , y_n)}{\| Ay_n \|^2} , \; y_0 \neq 0 , \; y_0 \in H .$$

<u>Theorem 1</u> ([7],[8]). Let A be a non-negative [$(Ax,x) \geq 0$ for every $x \in H$] completely continuous self-adjoint operator in H , let N be the null set of A and let $x_0 \in H \ominus N$, $y_0 \in H \ominus N$ be not orthogonal to the eigenspace $H_{\lambda_1^*}$ corresponding to the first eigenvalue λ_1^*

of A . Then $\{\lambda_n\}$ is monotone increasing and converges to λ_1^* . The sequence $\{\mu_n\}$ is monotone decreasing and converges to $(\lambda_1^*)^{-1}$. Both sequences $\{x_n\}$, $\{y_n\}$ converge in $H \ominus N$ to one of the eigenfunctions corresponding to λ_1^* .

These methods were generalized by I. Marek [10] for linear bounded operators in Banach space, which have a dominant eigenvalue. Simultaneously with [1], the method (5) was investigated by H.F. Bückner [11]. The purpose of this note is to show that the sequences $\{\lambda_n\}$, $\{\mu_n\}$ also converge in the case when the greatest point of the spectrum $\sigma(A)$ of A is not an eigenvalue of A , to remove the condition that λ_1^* be an isolated point of $\sigma(A)$ and to give some estimates. The proofs are omitted and will be published later, together with further theorems.

2. Suppose that A is linear self-adjoint positive operator in H . Let $\tilde{\lambda}_1$ be the greatest element and m the smallest element of the spectrum $\sigma(A)$. The spectrum $\sigma(A)$ lies in the segment $\langle m, \tilde{\lambda}_1 \rangle$, where $m = \inf_{\|x\|=1} (Ax,x)$, $\tilde{\lambda}_1 = \sup_{\|x\|=1} (Ax,x)$, $m \geq 0$. (The class of self-adjoint positive definite operators $(m > 0)$ is encluded in the class considered here.) Let $\{E_\lambda\}$ be the spectral family of A .

<u>Theorem 2</u>. Let A be a self-adjoint positive operator in H . Suppose $E_\lambda x_0 \neq x_0$ for $\lambda < \tilde{\lambda}_1$, (or that $E_\lambda y_0 \neq y_0$ for $\lambda < \tilde{\lambda}_1$,). Then $\{\lambda_n\}$ is monotone increasing and converges to $\tilde{\lambda}_1$ (and $\{\mu_n\}$ is monotone decreasing and converges to $\tilde{\lambda}_1^{-1}$).

Theorem 3. Under the assumptions of Theorem 2 let $\tilde{\lambda}_1$ not be an eigenvalue of A. Then both $\{x_n\}$, $\{y_n\}$ converge to $\emptyset$ weakly in H.

Theorem 4. Let A be a positive self-adjoint operator in H and suppose that $\tilde{\lambda}_1$ (not necessarily and isolated point of $\sigma(A)$) is an eigenvalue of A, $H_{\tilde{\lambda}_1}$ is the eigenspace corresponding to $\tilde{\lambda}_1$ and that the projection of x_0 ($x_0 \in H$) on $H_{\tilde{\lambda}_1}$ is $\xi_1^{(0)} e_1$, where $e_1 \in H_{\tilde{\lambda}_1}$, $\| e_1 \| = 1$, $\xi_1^{(0)} > 0$. Then

$\lim_{k \to \infty} \| x_k - N e_1 \| = 0$, where $N = \sup_{k = 1,2,\dots} \| x_k \| < +\infty$.

Now set $\cos(x,y) = \dfrac{(x,y)}{\|x\| \, \|y\|}$, $\sin(x,y) = \sqrt{1 - \cos^2(x,y)}$.

Then the following theorem holds.

Theorem 5. Let A be a positive self-adjoint operator in H, $E_\lambda x_0 \neq x_0$ for $\lambda < \tilde{\lambda}_1$, and suppose $\tilde{\lambda}_1$ is an isolated point of $\sigma(A)$. Then there exists a real q, $0 < q < 1$ such that for n_0 sufficiently large,

$$\tilde{\lambda}_1 - \frac{(Ax_{n_0+p}, x_{n_0+p})}{\| x_{n_0+p} \|^2} \leq q^{2p} \left(\tilde{\lambda}_1 - \frac{(Ax_{n_0}, x_{n_0})}{\| x_{n_0} \|^2} \right),$$

$$\| x_{n_0+p} - e_1 \| x_{n_0+p} \| \| \leq \sqrt{2}\, q^p [\| x_{n_0+p} \| (\| x_{n_0} \| - (x_{n_0}, e_1))]^{1/2}$$

$$\sin(x_{n_0+p}, e_1) \leq \frac{q^p}{(\tilde{\lambda}_1 - M)^{1/2}} (\tilde{\lambda}_1 - \lambda_{n_0+1})^{1/2},$$

where $m \leq \lambda \leq M < \tilde{\lambda}_1$, $(p = 1,2,\dots)$.

Theorem similar to theorems 4,5 also hold for second method (5). The methods (4),(5) seem to be very simple and convenient for computation. They can also be used for finding the extreme values m, $\tilde{\lambda}_1$ of the spectrum $\sigma(A)$.

R e f e r e n c e s :

[1] Л.В. КАНТОРОВИЧ, Функциональный анализ и прикладная математика. Усп.мат.наук 3,Ь.6, 1948,89-185.

[2] М.А. КРАСНОСЕЛЬСКИЙ, О некоторых приемах приближенного вычисления собственных значений и собственных векторов положительно определенной матрицы, Усп.мат.наук 11, Ь.2, 1856,151-158.

[3] В.Н. КОСТАРЧУК, Об одном методе решения систем линейных уравнений и отыскания собственных векторов матрицы, ДАН СССР,98,№ 4,1954, 531-534.

[4] В.П. ПУГАЧЕВ, О двух приемах приближенного вычисления собственных значений и собственных векторов, ДАН СССР,т.110,3,1956,334-337.

[5] WANG JIN-RU (WANG CHIN-JU), Gradient methods for finding eigenvalues and eigenvectors, Acta Math.Sinica Vol.14,No 4(1964),538-545.

[6] W. KARUSH, Determination of the extreme values of the spectrum of a bounded self-adjoint operator, Proc.Am.Math.Soc.Vol.2,No 6, 1951,980-989.

[7] J. KOLOMÝ, On convergence of the iterative methods, Comment.Math.Univ.Carol.1,3(1960),18-24.

[8] J. KOLOMÝ, On the solution of homogeneous functional equations in Hilbert space, Comment. Math.Univ.Carol.3,4(1962),36-47.

[9] И.А. БИРГЕР, Некоторые математические методы решения инженерных задач, Москва 1956.

[10] I. MAREK, On iterations of bounded linear operators and Kellogg's iterations in non self-adjoint eigenvalue problems, Czech Math.Journ.12(1962), 4,536-554.

[11] H.F. BÜCKNER, An iterative method for solving non-linear equations, PICC Symposium,Rome 1960(September),614-643.

(Received September 6,1965)